9 Q41
L8R88p
Ser. A
SI

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

(A.)

FOR THE YEAR MDCCCXCI.

VOL. 182.

AUG 31 1892
SMITHSONIAN DEPOSIT.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to Her Majesty.

MDCCCXCII.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

List of Institutions entitled to receive the Philosophical Transactions or Proceedings of the Royal Society.

Institutions marked A are entitled to receive Philosophical Transactions, Series A, and Proceedings.
" " B " " " " Series B, and Proceedings.
" " AB " " " " Series A and B, and Proceedings.
" " *p* " " Proceedings only.

America (Central).
Mexico.
p. Sociedad Científica "Antonio Alzate."

America (North). (See United States.)

America (South).
Buenos Ayres.
p. Museo de la Plata.
AB. Museo Nacional.
Caracas.
B. University Library.
Cordova.
AB. Academia Nacional de Ciencias.
Demerara.
p. Royal Agricultural and Commercial Society, British Guiana.
Rio de Janeiro.
p. Observatorio.

Australia.
Adelaide.
p. Royal Society of South Australia.
Brisbane.
p. Royal Society of Queensland.
Melbourne.
p. Observatory.
p. Royal Society of Victoria.
AB. University Library.
Sydney.
p. Geological Survey.
p. Linnean Society of New South Wales.
AB. Royal Society of New South Wales.
AB. University Library.

Austria.
Agram.
p. Societas Historico-Naturalis Croatica.
Brünn.
AB. Naturforschender Verein.

Austria (continued).
Gratz.
AB. Naturwissenschaftlicher Verein für Steiermark.
Hermannstadt.
p. Siebenbürgischer Verein für die Naturwissenschaften.
Innsbruck.
AB. Das Ferdinandeum.
p. Naturwissenschaftlich - Medicinischer Verein.
Klausenburg.
AB. Az Erdélyi Muzeum. Das Siebenbürgische Museum.
Prague.
AB. Königliche Böhmische Gesellschaft der Wissenschaften.
Schemnitz.
p. K. Ungarische Berg- und Först-Akademie.
Trieste.
B. Museo di Storia Naturale.
p. Società Adriatica di Scienze Naturali.
Vienna.
p. Anthropologische Gesellschaft.
AB. Kaiserliche Akademie der Wissenschaften.
p. K.K. Geographische Gesellschaft.
AB. K.K. Geologische Reichsanstalt.
B. K.K. Zoologisch-Botanische Gesellschaft.
B. Naturhistorisches Hof-Museum.
p. Œsterreichische Gesellschaft für Meteorologie.

Belgium.
Brussels.
B. Académie Royale de Médecine.
AB. Académie Royale des Sciences.
B. Musée Royal d'Histoire Naturelle de Belgique.
p. Observatoire Royal.
p. Société Malacologique de Belgique.

Belgium (continued).
Ghent.
AB. University.
Liége.
AB. Société des Sciences.
p. Société Géologique de Belgique.
Louvain.
p. Laboratoire de Microscopie et de Biologie Cellulaire.
AB. L'Université.

Canada.
Hamilton.
p. Hamilton Association.
Montreal.
AB. McGill University.
p. Natural History Society.
Ottawa.
AB. Geological Survey of Canada.
AB. Royal Society of Canada.
Toronto.
p. Astronomical and Physical Society.
p. Canadian Institute.
AB. University.

Cape of Good Hope.
A. Observatory.
AB. South African Library.

Ceylon.
Colombo.
B. Museum.

China.
Shanghai.
p. China Branch of the Royal Asiatic Society.

Denmark.
Copenhagen.
AB. Kongelige Danske Videnskabernes Selskab.

England and Wales.
Aberystwith.
AB. University College.
Bangor.
AB. University College of North Wales.
Birmingham.
AB. Free Central Library.
AB. Mason College.
p. Philosophical Society.
Bolton.
p. Public Library.
Bristol.
p. Merchant Venturers' School.
Cambridge.
AB. Philosophical Society.
p. Union Society.
Cooper's Hill.
AB. Royal Indian Engineering College.

England and Wales (continued).
Dudley.
p. Dudley and Midland Geological and Scientific Society.
Essex.
p. Essex Field Club.
Greenwich.
A. Royal Observatory.
Kew.
B. Royal Gardens.
Leeds.
p. Philosophical Society.
AB. Yorkshire College.
Liverpool.
AB. Free Public Library.
p. Literary and Philosophical Society.
A. Observatory.
AB. University College.
London.
AB. Admiralty.
p. Anthropological Institute.
B. British Museum (Nat. Hist.).
AB. Chemical Society.
p. "Electrician," Editor of the.
B. Entomological Society.
AB. Geological Society.
AB. Geological Survey of Great Britain.
p. Geologists' Association.
AB. Guildhall Library.
A. Institution of Civil Engineers.
A. Institution of Mechanical Engineers.
A. Institution of Naval Architects.
p. Iron and Steel Institute.
AB. King's College.
B. Linnean Society.
AB. London Institution.
p. London Library.
A. Mathematical Society.
p. Meteorological Office.
p. Odontological Society.
p. Pharmaceutical Society.
p. Physical Society.
p. Quekett Microscopical Club.
p. Royal Agricultural Society.
p. Royal Asiatic Society.
A. Royal Astronomical Society.
B. Royal College of Physicians.
B. Royal College of Surgeons.
p. Royal Engineers (for Libraries abroad, six copies).
AB. Royal Engineers. Head Quarters Library.
p. Royal Geographical Society.
p. Royal Horticultural Society.
p. Royal Institute of British Architects.

England and Wales (continued).

London (continued).

AB. Royal Institution of Great Britain.
B. Royal Medical and Chirurgical Society.
p. Royal Meteorological Society.
p. Royal Microscopical Society.
p. Royal Statistical Society.
AB. Royal United Service Institution.
AB. Society of Arts.
p. Society of Biblical Archæology.
p. Society of Chemical Industry (London Section).
p. Standard Weights and Measures Department.
AB. The Queen's Library.
AB. The War Office.
AB. University College.
p. Victoria Institute.
B. Zoological Society.

Manchester.

AB. Free Library.
AB. Literary and Philosophical Society.
p. Geological Society.
AB. Owens College.

Netley.

p. Royal Victoria Hospital.

Newcastle.

AB. Free Library.
p. North of England Institute of Mining and Mechanical Engineers.
p. Society of Chemical Industry (Newcastle Section).

Norwich.

p. Norfolk and Norwich Literary Institution.

Oxford.

p. Ashmolean Society.
AB. Radcliffe Library.
A. Radcliffe Observatory.

Penzance.

p. Geological Society of Cornwall.

Plymouth.

B. Marine Biological Association.
p. Plymouth Institution.

Richmond.

A. "Kew" Observatory.

Salford.

p. Royal Museum and Library.

Stonyhurst.

p. The College.

Swansea.

AB. Royal Institution.

Woolwich.

AB. Royal Artillery Library.

Finland.

Helsingfors.

p. Societas pro Fauna et Flora Fennica.
AB. Société des Sciences.

France.

Bordeaux.

p. Académie des Sciences.
p. Faculté des Sciences.
p. Société de Médecine et de Chirurgie.
p. Société des Sciences Physiques et Naturelles.

Cherbourg.

p. Société des Sciences Naturelles.

Dijon.

p. Académie des Sciences.

Lille.

p. Faculté des Sciences.

Lyons.

AB. Académie des Sciences, Belles-Lettres et Arts.

Marseilles.

p. Faculté des Sciences.

Montpellier.

AB. Académie des Sciences et Lettres.
B. Faculté de Médecine.

Paris.

AB. Académie des Sciences de l'Institut.
p. Association Française pour l'Avancement des Sciences.
p. Bureau des Longitudes.
p. Bureau International des Poids et Mesures.
p. Commission des Annales des Ponts et Chaussées.
p. Conservatoire des Arts et Métiers.
p. Cosmos (M. L'Abbé Valette).
AB. Dépôt de la Marine.
AB. École des Mines.
AB. École Normale Supérieure.
AB. École Polytechnique.
AB. Faculté des Sciences de la Sorbonne.
AB. Jardin des Plantes.
A. L'Observatoire.
p. Revue Internationale de l'Électricité.
p. Revue Scientifique (Mons. H. de Varigny).
p. Société de Biologie.
AB. Société d'Encouragement pour l'Industrie Nationale.
AB. Société de Géographie.
p. Société de Physique.
B. Société Entomologique.
AB. Société Géologique.
p. Société Mathématique.
p. Société Météorologique de France.

Toulouse.

AB. Académie des Sciences.
A. Faculté des Sciences.

Germany.

Berlin.
- A. Deutsche Chemische Gesellschaft.
- A. Die Sternwarte.
- *p.* Gesellschaft für Erdkunde.
- AB. Königliche Preussische Akademie der Wissenschaften.
- A. Physikalische Gesellschaft.

Bonn.
- AB. Universität.

Bremen.
- *p.* Naturwissenschaftlicher Verein.

Breslau.
- *p.* Schlesische Gesellschaft für Vaterländische Kultur.

Brunswick.
- *p.* Verein für Naturwissenschaft.

Carlsruhe. See Karlsruhe.

Danzig.
- AB. Naturforschende Gesellschaft.

Dresden.
- *p.* Verein für Erdkunde.

Emden.
- *p.* Naturforschende Gesellschaft.

Erlangen.
- AB. Physikalisch-Medicinische Societät.

Frankfurt-am-Main.
- AB. Senckenbergische Naturforschende Gesellschaft.
- *p.* Zoologische Gesellschaft.

Frankfurt-am-Oder.
- *p.* Naturwissenschaftlicher Verein.

Freiburg-im-Breisgau.
- AB. Universität.

Giessen.
- AB. Grossherzogliche Universität.

Görlitz.
- *p.* Naturforschende Gesellschaft.

Göttingen.
- AB. Königliche Gesellschaft der Wissenschaften.

Halle.
- AB. Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.
- *p.* Naturwissenschaftlicher Verein für Sachsen und Thüringen.

Hamburg.
- *p.* Naturhistorisches Museum.
- AB. Naturwissenschaftlicher Verein.

Heidelberg.
- *p.* Naturhistorisch-Medizinische Gesellschaft.
- AB. Universität.

Germany (continued).

Jena.
- AB. Medicinisch-Naturwissenschaftliche Gesellschaft.

Karlsruhe.
- A. Grossherzogliche Sternwarte.

Kiel.
- A. Sternwarte.
- AB. Universität.

Königsberg.
- AB. Königliche Physikalisch - Ökonomische Gesellschaft.

Leipsic.
- *p.* Annalen der Physik und Chemie.
- A. Astronomische Gesellschaft.
- AB. Königliche Sächsische Gesellschaft der Wissenschaften.

Magdeburg.
- *p.* Naturwissenschaftlicher Verein.

Marburg.
- AB. Universität.

Munich.
- AB. Königliche Bayerische Akademie der Wissenschaften.
- *p.* Zeitschrift für Biologie.

Münster.
- AB. Königliche Theologische und Philosophische Akademie.

Potsdam.
- A. Astrophysikalisches Observatorium.

Rostock.
- AB. Universität.

Strasburg.
- AB. Universität.

Tübingen.
- AB. Universität.

Würzburg.
- AB. Physikalisch-Medicinische Gesellschaft.

Holland. (See NETHERLANDS.)

Hungary.

Pesth.
- *p.* Königl. Ungarische Geologische Anstalt.
- AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.

India.

Bombay.
- AB. Elphinstone College.

Calcutta.
- AB. Asiatic Society of Bengal.
- AB. Geological Museum.
- *p.* Great Trigonometrical Survey of India.
- AB. Indian Museum.
- *p.* The Meteorological Reporter to the Government of India.

India (continued).
Madras.
B. Central Museum.
A. Observatory.
Roorkee.
p. Roorkee College.

Ireland.
Armagh.
A. Observatory.
Belfast.
AB. Queen's College.
Cork.
p. Philosophical Society.
AB. Queen's College.
Dublin.
A. Observatory.
AB. National Library of Ireland.
B. Royal College of Surgeons in Ireland.
AB. Royal Dublin Society.
AB. Royal Irish Academy.
Galway.
AB. Queen's College.

Italy.
Bologna.
AB. Accademia delle Scienze dell' Istituto.
Florence.
p. Biblioteca Nazionale Centrale.
AB. Museo Botanico.
p. Reale Istituto di Studi Superiori.
Genoa.
p. Società Lingustica di Scienze Naturali e Geografiche.
Milan.
AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.
AB. Società Italiana di Scienze Naturali.
Naples.
p. Società di Naturalisti.
AB. Società Reale, Accademia delle Scienze.
B. Stazione Zoologica (Dr. DOHRN).
Padua.
p. University.
Pisa.
p. Società Toscana di Scienze Naturali.
Rome.
p. Accademia Pontificia de' Nuovi Lincei.
A. Reale Ufficio Centrale di Meteorologia e di Geodinamica, Collegio Romano.
AB. Reale Accademia dei Lincei.
p. R. Comitato Geologico d' Italia.
AB. Società Italiana delle Scienze.
Siena.
p. Reale Accademia dei Fisiocritici.

Italy (continued).
Turin.
p. Laboratorio di Fisiologia.
AB. Reale Accademia delle Scienze.
Venice.
p. Ateneo Veneto.
AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

Japan.
Tokiô.
AB. Imperial University.
Yokohama.
p. Asiatic Society of Japan.

Java.
Batavia.
AB. Bataviaasch Genootschap van Kunsten en Wetenschappen.
Buitenzorg.
p. Jardin Botanique.

Luxembourg.
Luxembourg.
p. Société des Sciences Naturelles.

Malta.
p. Public Library.

Mauritius.
p. Royal Society of Arts and Sciences.

Netherlands.
Amsterdam.
AB. Koninklijke Akademie van Wetenschappen.
p. K. Zoologisch Genootschap 'Natura Artis Magistra.'
Delft.
p. École Polytechnique.
Haarlem.
AB. Hollandsche Maatschappij der Wetenschappen.
p. Musée Teyler.
Leyden.
AB. University.
Rotterdam.
AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.
Utrecht.
AB. Provinciaal Genootschap van Kunsten en Wetenschappen.

New Zealand.
Wellington.
AB. New Zealand Institute.

Norway.
Bergen.
AB. Bergenske Museum.
Christiania.
AB. Kongelige Norske Frederiks Universitet.

Norway (continued).
Tromsoe.
p. Museum.
Trondhjem.
AB. Kongelige Norske Videnskabers Selskab.
Nova Scotia.
Halifax.
p. Nova Scotian Institute of Science.
Windsor.
p. King's College Library.
Portugal.
Coimbra.
AB. Universidade.
Lisbon.
AB. Academia Real das Sciencias.
p. Secção dos Trabalhos Geologicos de Portugal.
Russia.
Dorpat.
AB. Université.
Kazan.
AB. Imperatorsky Kazansky Universitet.
Kharkoff.
p. Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow.
Kieff.
p. Société des Naturalistes.
Moscow.
AB. Le Musée Public.
B. Société Impériale des Naturalistes.
Odessa.
p. Société des Naturalistes de la Nouvelle-Russie.
Pulkowa.
A. Nikolai Haupt-Sternwarte.
St. Petersburg.
AB. Académie Impériale des Sciences.
AB. Comité Géologique.
p. Compass Observatory.
A. L'Observatoire Physique Central.
Scotland.
Aberdeen.
AB. University.
Edinburgh.
p. Geological Society.
p. Royal College of Physicians (Research Laboratory).
p. Royal Medical Society.
A. Royal Observatory.
p. Royal Physical Society.
p. Royal Scottish Society of Arts.
AB. Royal Society.
Glasgow.
AB. Mitchell Free Library.
p. Philosophical Society.
Servia.
Belgrade.
p. Académie Royale de Serbie.
Sicily.
Acireale.
p. Società Italiana dei Microscopisti.
Catania.
AB. Accademia Gioenia di Scienze Naturali.
Palermo.
A. Circolo Matematico.
AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
A. Reale Osservatorio.
Spain.
Cadiz.
A. Observatorio de San Fernando.
Madrid.
p. Comisión del Mapa Geológico de Espãna.
AB. Real Academia de Ciencias.
Sweden.
Gottenburg.
AB. Kongl. Vetenskaps och Vitterhets Samhälle.
Lund.
AB. Universitet.
Stockholm.
A. Acta Mathematica.
AB. Kongliga Svenska Vetenskaps-Akademie.
AB. Sveriges Geologiska Undersökning.
Upsala.
AB. Universitet.
Switzerland.
Basel.
p. Naturforschende Gesellschaft.
Bern.
AB. Allg. Schweizerische Gesellschaft.
p. Naturforschende Gesellschaft.
Geneva.
AB. Société de Physique et d'Histoire Naturelle.
AB. Institut National Genevois.
Lausanne.
p. Société Vaudoise des Sciences Naturelles.
Neuchâtel.
p. Astronomische Mittheilungen (Professor R. Wolf).
p. Société des Sciences Naturelles.
Zürich.
AB. Das Schweizerische Polytechnikum.
p. Naturforschende Gesellschaft.
Tasmania.
Hobart.
p. Royal Society of Tasmania.

United States.

Albany.
- AB. New York State Library.

Annapolis.
- AB. Naval Academy.

Baltimore.
- AB. Johns Hopkins University.

Berkeley.
- *p.* University of California.

Boston.
- AB. American Academy of Sciences.
- B. Boston Society of Natural History.
- A. Technological Institute.

Brooklyn.
- AB. Brooklyn Library.

Cambridge.
- AB. Harvard University.
- B. Museum of Comparative Zoology.

Chapel Hill (N.C.).
- *p.* Elisha Mitchell Scientific Society.

Charleston.
- *p.* Elliott Society of Science and Art of South Carolina.

Chicago.
- AB. Academy of Sciences.

Cincinnatti.
- *p.* Journal of Comparative Neurology.

Davenport (Iowa).
- *p.* Academy of Natural Sciences.

Madison.
- *p.* Wisconsin Academy of Sciences.

Mount Hamilton (California).
- A. Lick Observatory.

New Haven (Conn.).
- AB. American Journal of Science.
- AB. Connecticut Academy of Arts and Sciences.

United States (continued).

New York.
- *p.* American Geographical Society.
- *p.* American Museum of Natural History.
- *p.* New York Academy of Sciences.
- *p.* New York Medical Journal.
- *p.* School of Mines, Columbia College.

Philadelphia.
- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p.* Franklin Institute.
- *p.* Wagner Free Institute of Science.

Rochester (N.Y.).
- *p.* Academy of Science.

St. Louis.
- *p.* Academy of Science.

Salem (Mass.).
- *p.* American Association for the Advancement of Science.
- *p.* Essex Institute.
- AB. Peabody Academy of Science.

San Francisco.
- AB. California Academy of Sciences.

Washington.
- *p.* Department of Agriculture.
- A. Office of the Chief Signal Officer.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- *p.* United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.

West Point (N.Y.)
- AB. United States Military Academy.

Adjudication of the Medals of the Royal Society for the year 1891, by the President and Council.

The Copley Medal to Stanislao Cannizzaro, For.Mem.R.S., for his Contributions to Chemical Philosophy, especially for his Application of Avogadro's Theory.

A Royal Medal to Charles Lapworth, F.R.S., for his Researches among the Older Rocks of Britain.

A Royal Medal to Arthur William Rücker, F.R.S., for his Researches on Liquid Films, and his Contributions to our Knowledge of Terrestrial Magnetism.

The Davy Medal to Victor Meyer, for his Researches on the Determination of Vapour Densities at High Temperatures.

The Bakerian Lecture, "On Tidal Prediction," was delivered by Professor G. H. Darwin, F.R.S.

The Croonian Lecture, "On the Mammalian Nervous System, its Functions, and their Localisation determined by an Electrical Method," was delivered by Francis Gotch, and Professor Victor Horsley, F.R.S.

CONTENTS.

(A)

VOL. 182.

I. *On the Determination of the Specific Resistance of Mercury in Absolute Measure. By* J. Viriamu Jones, *Principal and Professor of Physics in the University College of South Wales and Monmouthshire, Cardiff. Communicated by Professor* Clifton, *F.R.S.* . . . . . page 1

II. *On the Determination of some Boiling and Freezing Points by means of the Platinum Thermometer. By* E. H. Griffiths, *M.A., Sidney Sussex College, Cambridge. Communicated by* R. T. Glazebrook, *F.R.S.* 43

III. *On the Specific Heats of Gases at Constant Volume.*—Part I. *Air, Carbon Dioxide, and Hydrogen. By* J. Joly, *M.A., B.E., Assistant to the Professor of Civil Engineering, Trinity College, Dublin. Communicated by Professor* Fitzgerald, *M.A., F.R.S., F.T.C.D.* 73

IV. *On the Determination of the Boiling-Point of Sulphur, and on a Method of Standardising Platinum Resistance Thermometers by reference to it.—Experiments made at the Cavendish Laboratory, Cambridge. By* H. L. Callendar, *M.A., Fellow of Trinity College, Cambridge, and* E. H. Griffiths, *M.A., Sidney Sussex College, Cambridge. Communicated by Professor* J. J. Thomson, *F.R.S.* 119

V. Bakerian Lecture.—*On Tidal Prediction. By* G. H. Darwin, *F.R.S., Plumian Professor and Fellow of Trinity College, Cambridge* 159

VI. *On Ellipsoidal Harmonics. By* W. D. Niven, *F.R.S.* . . . . . page 231

VII. *The Conditions of Chemical Change between Nitric Acid and certain Metals. By* V. H. Veley, *M.A., the University Museum, Oxford. Communicated by Professor* Odling, *F.R.S.* 279

VIII. *The Variations of Electromotive Force of Cells consisting of certain Metals, Platinum, and Nitric Acid. By* G. J. Burch, *B.A., and* V. H. Veley, *M.A., the University Museum, Oxford. Communicated by Professor* Odling, *F.R.S.* 319

IX. *The Influence of Stress and Strain on the Physical Properties of Matter.*—Part III. *Magnetic Induction* (continued).—*The Internal Friction of Iron, Nickel, and Cobalt, studied by means of Magnetic Cycles of very minute range. By* Herbert Tomlinson, *B.A., F.R.S.* 341

X. *On the Reflection and Refraction of Light at the Surface of a Magnetized Medium. By* A. B. Basset, *M.A., F.R.S.* 371

XI. *On the Causes which Produce the Phenomena of New Stars. By* J. Norman Lockyer, *F.R.S.* 397

XII. *On* Stokes's *Current Function. By* R. A. Sampson, *B.A., Fellow of St. John's College, Cambridge. Communicated by Professor* Greenhill, *F.R.S.* . 449

XIII. *Quadrant Electrometers. By* W. E. Ayrton, J. Perry, *and* W. E. Sumpner 519

XIV. *On a Determination of the Mean Density of the Earth and the Gravitation Constant by means of the Common Balance. By* J. H. Poynting, *D.Sc., F.R.S., Professor of Physics, Mason College, Birmingham* 565

Index 657

LIST OF ILLUSTRATIONS.

Plates 1 to 3.—Professor J. V. Jones on the Determination of the Specific Resistance of Mercury in Absolute Measure.

Plates 4 and 5.—Mr. E. H. Griffiths on the Determination of some Boiling and Freezing Points by means of the Platinum Thermometer.

Plate 6.—Mr. J. Joly on the Specific Heats of Gases at Constant Volume.—Part I. Air, Carbon Dioxide, and Hydrogen.

Plates 7 and 8.—Messrs. H. L. Callendar and E. H. Griffiths on the Determination of the Boiling-Point of Sulphur, and on a Method of Standardising Platinum Resistance Thermometers by reference to it.

Plates 9 to 12.—Messrs. W. E. Ayrton, J. Perry, and W. E. Sumpner on Quadrant Electrometers.

Plates 13 to 19.—Professor J. H. Poynting on a Determination of the Mean Density of the Earth and the Gravitation Constant by means of the Common Balance.

PHILOSOPHICAL TRANSACTIONS.

I. *On the Determination of the Specific Resistance of Mercury in Absolute Measure.*

By J. Viriamu Jones, *Principal and Professor of Physics in the University College of South Wales and Monmouthshire, Cardiff.*

Communicated by Professor Clifton, *F.R.S.*

Received August 8,—Read November 20, 1890.

[Plates 1–3.]

§ 1. "On the whole I am of opinion that if it is desirable at the present time to construct apparatus on the most favourable scale, so as to reach the highest attainable accuracy, the modification of Lorenz's method last described is the one that offers the best prospect of success. Before this is done however, it appears to me important that the value now three times obtained in the Cavendish Laboratory by distinct methods should be approximately verified (or disproved) by other physicists. To distinguish between this value and those obtained for instance by Kohlrausch, by Lorenz, or by the First B.A. Committee, should not require the construction of unusually costly apparatus. Until the larger question is disposed of it appears premature to discuss the details of arrangements from which the highest degree of precision is to be expected."

The above passage, which concludes a paper communicated by Lord Rayleigh to the 'Philosophical Magazine,'* a little before the Electrical Congress at Paris, at which the legal ohm was defined to be the resistance of a column of mercury of 1 sq. mm. section and 1060 mm. long, seems not to have met with adequate response in this country. So far as experiments in English Laboratories are concerned the

* 'Phil. Mag.,' November, 1882, "Comparison of Methods for the Determination of Resistances in Absolute Measure."

determination of the ohm remains where Lord RAYLEIGH left it, except for the contribution made by GLAZEBROOK and FITZPATRICK in their re-measurement of the Specific Resistance of Mercury in terms of the B.A. Unit, which is one of the elements in the determination of the Specific Resistance of Mercury in Absolute Measure by Lord RAYLEIGH'S adaptation of LORENZ'S method.

The measurements to be described in this paper were undertaken in the hope of paving the way for the final determination referred to by LORD RAYLEIGH; and with a view of submitting to the test of experiment certain modifications of LORENZ'S method, which had occurred to the author as likely to lead to increased accuracy and certainty. Therefore, though the results are given it is asked that the work should be regarded rather as justifying a suggested method of measurement than as pretending to be a satisfactory or final determination of the physical constant sought.

Probably the experiments are sufficient to make good the statement that, if the apparatus be constructed on a scale a little larger and with a certain perfecting of detail, a single set of observations will give a result accurate to 1 part in 10,000; and the author is of opinion that an even greater accuracy may be obtained by the method if due regard be paid to the maintenance of definite temperatures in all parts of the apparatus.

§ 2. In LORENZ'S method a metallic disc is made to rotate in the mean plane of a co-axial standard coil. Wires touching the centre and circumference of the disc are led to the ends of the resistance to be measured; and the same current is passed through this resistance and the standard coil. The connections being rightly made we may by varying either the rate of rotation of the disc or the resistance to be measured, so arrange matters as to have no change of current in the circuit of the disc and wires joining it to the ends of the resistance, when the direction of the current through the resistance and the standard coil is changed. When this arrangement is effected there is a balance between the electromotive force due to the motion of the disc in the magnetic field of the current in the standard coil, and the difference of potential at the ends of the resistance, due to the current traversing it. If this adjustment be made we will say that the apparatus is in an equilibrium position.

If M = the coefficient of mutual induction of the standard coil and the circumference of the disc,

n = the rate of rotation of the disc (number of revolutions per second),

R = the resistance,

γ = the current through the standard coil and the resistance,

then in an equilibrium position

$$\mathrm{M}n\gamma = \mathrm{R}\gamma \quad \text{or} \quad \mathrm{R} = \mathrm{M}n.$$

The variations in the method introduced in the present investigation are mainly the following:—

(i.) The use of a standard coil with one layer of wire on it.

(ii.) The elimination by a system of differential measurements of the errors that have so far attended the use of a mercury column as the measured resistance.

§ 3. *The Standard Coil.*—The coil C (Plate 1, figs. 1 and 2), is of double silk covered copper wire of about ·02 inch diameter, resting in a screw thread of ·025 inch pitch, cut on a hollow cylinder of brass of approximately 10 inches and 10·5 inches internal and external radius. The axial length of the cylinder is something over five inches and 185 turns of wire are wound on it. Precise details of its dimensions are given later. The lathe on which the cylinder was turned was made for the College Workshop by Sir J. Whitworth and Co. with special care, and it is an exceptionally massive and accurate tool.

The brass cylinder has three lugs L, L, L, by which it was bolted to the face plate of the lathe, the outside faces of the lugs having been carefully scraped to fit the plate. In use the cylinder is bolted in precisely similar fashion to a strong triangular brass frame, supported on three levelling screws (as shown in Plate 1, figs. 1 and 2); so that it is used in a position as like as possible to that in which it was turned.

The first operation was to fit the lugs roughly to the face plate by filing, and a roughing cut was taken over the entire surface of the cylinder, so as to get rid of as much as possible of the internal strain due to unequal cooling of the casting. It was then unbolted; the lugs were carefully scraped to fit the face plate, and the final turning proceeded with. Very fine cuts were taken in finishing, and it was hoped by these precautions that we should secure a true right circular cylinder. Nevertheless our subsequent measurements show that the section of the cylinder at right angles to its axis is slightly oval. It appears, indeed, to be a task of no ordinary difficulty to turn a cylinder of large size that shall remain true to the thousandth of an inch. The screw thread of 40 turns to the inch was next cut on the cylinder to a depth of about ·012 inch.

It is necessary for purposes of adjustment to know the mean plane of the screw thread. The next operation was, therefore, directed to find this plane, and to mark the circle of its intersection with the inner surface of the cylinder.

A microscope and a **V**-tool were fixed to the slide rest, the microscope being at right angles to the lathe bed, and the tool being adjusted so that the image of its point was at the central mark of the graduated glass plate in the eye-piece of the microscope. The microscope and tool were placed in such a position that when the slide rest moved forward the microscope remained outside and the tool passed inside the cylinder. A wheel of 9·75 inches radius divided into 360 parts on its circumference was attached to the guide screw, which is of pitch ·25 inch. A tenth of a division on the wheel could be estimated with tolerable accuracy.

A generating line was then drawn on the outside of the cylinder, and the microscope was focussed so as to bring the ridges of the screw thread on this line into the field of view. Next the guide screw was turned so as to bring first the right hand and then the left hand ridge of the first thread on to the central mark in the field of

view. The mean of the readings of the wheel in these two positions was taken to be the reading for the centre of the hollow.

The wheel and guide screw were then turned until first the right hand and then the left hand ridges of the last thread on the cylinder were in like manner brought on to the central mark, the mean of these readings being again taken as the reading for the centre of the hollow. The number of turns and fractions of a turn made in order to bring the microscope from the first position to the second position gives the axial length between the first and last hollows on the generating line. By adding half this number to the reading for the first hollow, the reading for the mean plane was obtained. The wheel and guide screw were then moved to this position, and a circle was cut inside the cylinder. This circular cut is used in adjusting the mean plane of the coil to the plane of the disc.

Small holes were then radially bored through the brass of the cylinder from the points where the generating line along which the measurements had been taken intersected the first and last hollows, and these holes were bushed with well paraffined ebonite.

The next step was to wind the wire in the screw thread. The wire was led from the bobbin over a pulley fixed some feet above it, under a movable pulley with a weight attached, over another fixed pulley, and finally under a pulley fixed to the slide rest which delivered it into the screw thread on the brass cylinder. The object of the movable pulley with the weight attached was to keep the tension in the wire uniform during the winding.

One end of the wire was passed through the left hand small hole bushed with ebonite above referred to, and secured ; the guide screw was then thrown in gear and the cylinder as it rotated drew the wire uniformly into place. When the winding was complete the wire was clamped about three inches from the right hand hole until the cut end was passed through and drawn tight. Melted paraffin was then run into the holes in the bushes to secure the ends of the wire. Blocks of ebonite were afterwards attached to the brass cylinder at the ends of the generating line before-mentioned ; and binding screws were fixed in them, to which the ends of the wire were soldered.

Tests, the details of which are given in § 21, show that the coil is well insulated.

The advantage in constructing a standard coil with a single layer of wire is that every part of it is visible, and that nothing is done to alter the position of the wire after measurements are made. If a coil consists of many layers, it is not quite easy to say where the lower layers place themselves after measurement under the pressure of the superincumbent layers.

Lord Rayleigh* found, in the case of certain coils used by him, that the mean radius calculated from measurements made in winding was greater by 1 part in 2000 than the mean radius calculated from measurements made in unwinding, a result clearly due to the displacement of the lower layers (' Phil. Trans.,' 1882) (*cf.* Glaze-

* ' Phil. Trans.,' 1882, pp. 673–4.

BROOK and SARGANT, 'Phil. Trans.,' 1883, p. 267). We should therefore expect that if the mean radius of a coil of many layers is calculated from measurements made in winding it will be reckoned too large.

In the face of this, it is interesting to note that in the measurements made by Lord RAYLEIGH and Mrs. SIDGWICK by the method of LORENZ, the series in which the coils are so placed as to make the induction coefficient "nearly independent of a knowledge of the mean radius," gives a result rather larger than the other series, which is what might be expected if the mean radius is overestimated. But the smallness of the variation (about 1 part in 10,000) seems to show that the error of mean radius must be very small for the pair of coils (wound by Professor CHRYSTAL) used by Lord RAYLEIGH in these experiments.

In general, we should expect, as far as this cause goes, that values of the specific resistance in determining which coils of many ayers are used, would come out too low, since an overestimation of the mean radius means an underestimation of the coefficient of mutual induction.

The apparent disadvantage in a coil of a single layer is that the number of turns will be fewer, and, therefore, the coefficient of mutual induction of the coil and disc smaller than in the case of a coil of many layers. For some purposes this might be a grave objection; but it is not so for the work in hand. Although the coefficient of mutual induction of the coil and disc is only a little more than one-twentieth of that of Lord RAYLEIGH'S coils and disc in the first and second series of his measurements by the method of LORENZ, yet it seems ample for the purpose.

In order, however, to get a sufficient number of turns on the coil of a single layer, its axial length is made too long to allow of our calculating the coefficient of mutual induction of the coil and disc by Lord RAYLEIGH'S formula of approximation. It, therefore, became necessary to calculate a new formula, which was obtained by the direct integration of the expression $\iint(ds\,ds'/r)\cos\epsilon$ for a circle and coaxial helix, and communicated to the Physical Society (*vide* 'Phil. Mag.,' Jan., 1889). By Lord RAYLEIGH'S formula the coefficient in these experiments would be given as $n \times 90{\cdot}0383$, its value by the formula for circle and helix being $n \times 89{\cdot}7717$. The difference amply confirms the need of the new formula for a coil of such axial length as that used.

§ 4. *The Use of a Mercury Column by the Differential Method.*—The smallness of the resistance to be measured has usually been regarded as one of the difficulties of LORENZ'S method. If, on the one hand, mercury has been used, there has been the difficulty of calculating the specific resistance from the measured resistance of the mercury filling a glass tube of a large sectional area; while, on the other hand, where solid conductors have been used there has been the difficulty of comparing the resistances measured with ordinary standards. Lord RAYLEIGH, by a shunt method, overcame the latter difficulty. But, notwithstanding the success which attended the use of this method, it has seemed better to the author to revert to LORENZ'S use of mercury.

The physical constant to be measured is the specific resistance of mercury. If this can, by direct measurement, be determined with accuracy, it seems inconvenient and undesirable first to determine the resistance of the B.A. unit in absolute measure, and then to determine the specific resistance of mercury in B.A. units. If, consistently with accuracy, the artificial B.A. unit can be dropped out of our experiments as well as out of the result, the measurements being made direct on mercury, the simplicity would seem to be a recommendation.

The difficulty in the way of using mercury in a glass tube presented itself thus: if the wires from the disc (the terminal portions of which may be called the electrodes) are led to the ends of the tube, the equipotential surfaces touched by them are not plane; and if they are let into the tube at some distance from the ends, it is difficult to see how the distance between them is to be determined with the requisite accuracy.

Both these difficulties disappear if, instead of placing the mercury in a glass tube, it is placed in a long trough, and if, instead of measuring the distance between the electrodes, one electrode is kept fixed while measurement is made of the distance moved through by the other between two equilibrium positions corresponding to two different rates of rotation of the disc. The latter measurement it is easy to make with accuracy, for the movable electrode may be rigidly attached to the movable headstock of a Whitworth measuring machine, or some other measuring bank placed parallel to the length of the trough. The two equilibrium positions may be taken near the middle of the trough, so as to avoid danger of curvature in the equipotential surfaces passing through the electrode in its two positions.

Let n_1, n_2 be the rates of rotation of the disc, and let l be the distance between the corresponding equilibrium positions of the movable electrode.

Then

$$\mathrm{M}(n_1 - n_2) = \frac{l}{\mathrm{A}}\rho,$$

where $\rho =$ the specific resistance of mercury, and $\mathrm{A} =$ the area of section of the mercury column.

But we are met by a new difficulty, the determination of the section of the mercury column. The capillary depression at the sides of the trough would make it a serious task to determine the section by direct measurement to the required degree of accuracy. Fortunately this difficulty may be overcome by a further differential method, viz., by making observations with the mercury at two different heights in the trough.

Let

$b =$ the breadth of the trough,

$h_2 - h_1 =$ the difference of height of the mercury surface in its two positions,

and let

$\mathrm{A} =$ the section of the mercury column when the mercury surface is at the lower position.

Then we have, denoting by dashed letters the new values of the rates of rotation and the distance between the corresponding equilibrium positions,

$$M(n_1 - n_2) = \frac{l}{A}\rho,$$

and

$$M(n_1' - n_2') = \frac{l'}{A + b(h_2 - h_1)}\rho,$$

whence, eliminating A, we have

$$\rho = \frac{Mb(h_2 - h_1)}{\dfrac{l'}{n_1' - n_2'} - \dfrac{l}{n_1 - n_2}}.$$

It is assumed in the above calculation that the sides of the trough are plane, parallel, and vertical; and that the temperature remains constant.

Hence the determination of the specific resistance involves the determination of:—

(i.) Four equilibrium positions, two at each depth, with the rates of rotation of the disc to which they correspond.
(ii.) The breadth of the trough.
(iii.) The difference of level of the mercury surface at the two depths.
(iv.) The coefficient of mutual induction of the coil and disc.

§ 5. The general arrangement of the apparatus used in the experiments to be described is shown in Plate 2, fig. 5.

M. Electromotor.
g. Flexible coupling.
$\beta, \beta, \beta, \beta$. Bearings.
W. Fly-wheel.
Dr. Cylinder, with rows of black and white teeth, for measurement of rate of rotation.
F. Tuning-fork box.
C. Standard coil.
D. Disc.
E. Battery of secondary cells.
m, m. Whitworth measuring machine.
Y. Loose headstock on measuring machine.
Z. Fixed ,, ,, ,,
Tr. Mercury trough.
p_1, p_2. Iron plates to which the battery wires are soldered.
T_1, T_2. Thermometers giving the temperature of the mercury in the trough.
σ. Spherometer.
O. Movable electrode.
P. Fixed electrode.

X, X, X, X. Box surrounding trough and measuring machine.
K_1, K_2, K_3. Keys.
U_1. Coil for adjusting the induction, so as to obtain the required throw of the galvanometer needle when the current is reversed at K_2.
U_2. Coil for adjusting, so that reversal of the battery current shall produce no effect on the galvanometer needle when the galvanometer circuit is broken.
G. Galvanometer.
S. Lamp and scale.

§ 6. *The Mercury Trough.*—The trough used was cut in paraffin wax, and was prepared in the following manner. A strong casting of iron was made with its sides strengthened by outside ribs, as shown in Plate 2, figs. 6, 7, and 8. Its internal dimensions are 51·25 inches in length, by about 3 inches in breadth, by about 4 inches in depth. To fill it with paraffin wax, the casting was placed with its central part over a stove, and BUNSEN burners were inserted under the projecting ends. Paraffin wax, cut in small pieces, was then put in the trough, and as this melted, more wax was added until the trough was quite full of molten paraffin. The fire and burners were then removed, and the whole left to cool slowly. After a solid coat had formed on the surface, a hole was made through it to the liquid paraffin in the interior, and through this hole a further quantity of melted wax was poured during the progress of solidification. This process was continued until the iron case was filled with solid paraffin wax.

The case so filled was then fastened to the bed of the lathe with its length parallel to the run of the bed and the free paraffin surface upwards. A cutter attached to the slide rest rotating about 2,000 turns per minute round a vertical axis was used to cut out the required channel in the paraffin wax, the channel being approximately 43·5 inches long by 1·5 inches broad by 3 inches deep. When the channel was cut for the first time a certain number of air holes were found in the base and lateral faces. Experiments on a smaller trough had, however, shown that these might be satisfactorily removed by depositing paraffin wax in a thin layer on the sides and base of the channel, and re-cutting. This was successfully accomplished for the large trough, and a uniform surface was secured. The channel was finished by carrying a scraper from end to end which took a very thin cut off both faces and base. The result of the scraping was a very smooth and highly finished surface.

It was thought that the breadth of the trough might be obtained by measuring the width of the scraper. But they were not found to be equal when the breadth of the trough was subsequently measured. The difference amounts to ·006 or ·007 inch, the width of the scraper being the greater. This difference may be due in part to the fact that the measurement of the trough was made at a different temperature (about 15°·5 C.) to that at which it was cut. Unfortunately the latter temperature was not registered.

This method of obtaining a channel for the mercury was adopted in the hope that

so cut in the lathe, it might present a breadth constant to about the ten-thousandth of an inch. This hope was not realised. The results of calibrating the trough over the ten inches of it, used in the course of the experiments, are given in § 28. Probably more satisfactory results might be obtained by building up the trough of worked glass or scraped marble.

The objections to the use of paraffin wax are :—

(i.) The softness of the wax makes it necessary to proceed with great care in measuring its breadth lest damage should be done to the surface. This difficulty is satisfactorily surmounted by the form of internal callipers adopted in the calibration.

(ii.) The coefficient of expansion of paraffin wax is very large. It was found by making measurements with the Whitworth Measuring Machine on a bar of paraffin wax, 12 inches long, that the linear coefficient of expansion is as much as ·0003. Distortion would therefore be the result of varying the temperature, since the wax is enclosed in an iron casing. The danger of inaccuracy due to this cause was met by measuring the breadth of the trough at a temperature as nearly as possible the same as that at which the electrical observations were made, care being taken to maintain the trough at a uniform temperature during the observations. As will be seen from Table VI. the extreme temperature readings do not differ from one another by more than ·5° C. But, notwithstanding this care, the author is of opinion that the difference in the results of different sets of observations depends in the main upon variations in the trough due to varying temperature.

§ 7. The disc, axle, and bearings are of phosphor bronze. The details are shown in Plate 1, figs. 1, 2, 3, 4.

The ratio of the radii of disc and coil is ·61635, very nearly that used in the measurements of Lord Rayleigh.

The disc is insulated from the axle by well paraffined ebonite (Plate 1, fig. 3); otherwise there would be a short-circuiting of part of the radius of the disc through the bearings and bed.

It was at first intended to drive the disc by rope gearing from the electromotor, but in the course of preliminary experiments, though the rope was 40 yards long, and the joining of the ends took place over a length of quite six feet, there was a sudden variation in speed, visible at the time-measuring tuning fork, and producing a distinct movement of the galvanometer needle, whenever the joint passed over the pulley. The rope gearing was therefore abandoned, and the motor coupled direct, as in the figure. Unfortunately, by an oversight, the part of the axle, $\xi\,\xi$ (Plate 2, fig. 5), joining the motor to the previously prepared part of the apparatus was made of steel instead of phosphor bronze, and the presence of the steel has some slight effect (about 4·3 parts in 10,000) on the coefficient of mutual induction of the coil and disc. No permanent magnetic field in the steel affects the result ; but the greater permeability of the steel increases the coefficient of mutual induction.

The disc was ground true in place by means of a small emery wheel driven by an electromotor, the disc at the same time being rotated.

§ 8. The brushes are of phosphor bronze wire, of diameter ·116 inch. The brush resting on the circumference of the disc at first gave great trouble. It was found impossible with an ordinary brush made of a number of layers of thin phosphor bronze sheet, and controlled by a spring, to obtain sufficiently constant readings at the galvanometer, however well the circumference of the disc and the brush were initially amalgamated. Trials were made in the hope of improvement with amalgamated copper and amalgamated lead, and with the substitution of a dead weight pressure for the spring. But no satisfactory result could be obtained.

It was noticed, however, that after amalgamation the readings were fairly steady for a short interval; and it seemed likely that if mercury could be continuously supplied to the surface of contact between the brush and the disc, the electromotive force at their contact would be rendered much more constant. This led on to the idea of a brush consisting of a single wire, perforated by a channel, through which a constant flow of mercury might be maintained from a cistern of adjustable height and a brush of this description was finally adopted, the details of which are shown at at Q, Plate 1, figs. 3 and 4.

A piece of copper, an inch long and ·375 inch square, has a hole, ·25 inch diameter, drilled to a depth of ·75 inch, and at right angles to this a second hole of sectional area slightly larger than that of the phosphor bronze wire brush. In the wire brush a channel ·03 inch diameter is drilled to the depth of about an inch, and the wire is brazed into the copper piece, as shown in the drawings. A small hole is then made with a fine pointed drill, connecting the cavity in the copper piece with the channel in the wire. The brush being in place, an india-rubber tube is passed over the free end of the copper piece, which carries to the brush a constant supply of mercury from the cistern. When the apparatus is in use, the height of the cistern is adjusted so that a continuous stream of mercury is dragged out of the brush by the rotating disc.

The extent to which this new form of brush meets the difficulty of varying electromotive force at the contact is best estimated by the actual galvanometer readings taken in the course of the experiments which are detailed in Tables I.–V.* It certainly presents very great advantages over any other form of brush tried.

The central brush Q′ is made in similar fashion, but it is not necessary to have so rapid a flow of mercury. It is sufficient to make the cavity in the copper piece of the mercury cistern and to replenish it at intervals. The details of its position relatively to the disc are shown in fig. 3. It enters the disc to a depth of ·06 inch.

§ 9. The motor used is a Siemens Dynamo machine, to which a steel fly-wheel has been fitted, as shown at W in Plate 2, fig. 5, the current being obtained from secondary cells. The current passes through resistance coils, and may also be varied continuously through a small range by a slide resistance of platinoid wire after the larger adjustment has been made. A shunt worked by a lever provides means of taking out or putting in a small resistance suddenly, so as to allow the observer taking note of the speed to

* Of these tables only Table I. is printed. The others are very similar.

counteract small variations due to alteration in the lubrication of the bearings and the friction of the brushes on the commutator of the motor. The bearings β, β, β, β, have a length about six times their diameter, and all the bearings are fitted with sight feed lubricators.

The proper lubrication of the bearings seems to be of the greatest importance. In making any final determination it would be necessary to obtain a more constant rate of rotation than has yet been realised, and to maintain it, as far as possible, without interference from the observer. A better result may, perhaps, be achieved if a forced system of lubrication is adopted. The motor brushes would also need further attention.

§ 10. The measurement of the rate of rotation is made with the help of a tuning fork by the method of Macleod and Clarke ('Phil. Trans.,' 1880). At Dr, Plate 2, fig. 5, there is attached to the axle a concentric brass cylinder (made in two pieces, subsequently bolted together round the axle), on the blackened surface of which are painted eleven rows of white teeth, the teeth in any row being equidistant, and the number of teeth in the rows being 7, 8, 9, 10, 11, 12, 13, 15, 17, 19, 21, respectively. In front of this cylinder, or drum, is placed a Koenig tuning fork of pitch approximately 64, the prongs of which have screwed into their ends pieces of brass with narrow central slits about an inch long, parallel to the length of the fork. At the centre of each slit a cross-wire is placed at right angles to its length. The fork is enclosed in a box and a thermometer divided to tenths of a degree is placed between the prongs. The tuning fork is vertical, so that the displacement of any point of a slit is at right angles to that of a tooth.

Fig. 11.

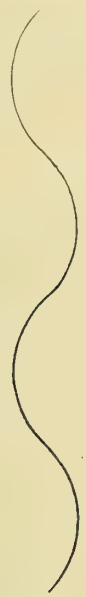


Fig. 12.

When the disc is rotating and the fork in vibration, for an observer looking through the fork slit at the drum, any point of the slit is illuminated, while a white space is passing the line joining the observer's eye to the point of the slit looked at, and

unilluminated while a dark space is passing that line. If the interval between the passage of successive white teeth is equal to the periodic time of the fork, any point of the slit looked at will appear fixed in one position, but the position will be different for each point, and the slit as a whole will be seen as an undulating line (fig. 11), stationary if the synchronism is exact, and if not varying so as to give the appearance of a wave moving one way or the other according as the interval between the passage of two teeth is less or greater than the periodic time of the fork.

In practice it is better, however, to adjust so that the interval between the passage of two teeth is half the periodic time of the fork. In this case, if the synchronism is exact, any point of the slit is seen as two points in the two positions it occupies when illumination takes place, and the whole slit appears as in fig. 12. If the synchronism is not exact, the chain figure appears to move in one direction or the other according as the interval between the passage of successive teeth is greater or less than half the periodic time of the fork.

The advantage of using the second figure rather than the first, is that it is easier by reference to the cross wire at the centre of the slit to tell when the figure is stationary and the synchronism exact.

The box surrounding the fork is blackened ; and two holes, one in front, and one behind, give the observer a line of vision.

The observations are made with the naked eye.

For the chain figure we have, when synchronism is exact,

$$n = \frac{2P}{N}$$

where n = the number of revolutions of the disc per second,

N = number of teeth in the row used,

P = vibration frequency of the tuning fork.

When a first adjustment has been made and the figure is nearly stationary, the direction of its motion indicates whether the rate of rotation is too quick or too slow, and the requisite change of resistance to obtain synchronism is made by the observer.

The fork was not maintained in vibration electrically, but bowed. It would be more convenient in observing, to use an electrically maintained fork ; but preliminary measurement of the pitch of a fork and of a reed, maintained electrically in vibration, tended to throw doubt on them as time-keepers (*vide* ' Phil. Mag.,' April, 1889, " On the Use of Lissajous' Figures to Determine a Rate of Rotation, and of a Morse Receiver to Measure the Periodic Time of a Reed or Tuning Fork.")

By the above method, the rate of rotation of the disc is controlled and referred to the fork during the observations.

It remains to indicate how the pitch of the fork was determined. This was effected by an inverse process. If, while synchronism is maintained between the

rotating apparatus and the fork by keeping the figure stationary, the number of revolutions in a given time is determined, the pitch of the fork may be calculated from the formula

$$P = \frac{nN}{2}.$$

In the present experiments the number of revolutions of the cylinder in a given time was determined by a telegraphic method.

An eccentric ring, fixed on the axle at J (Plate 1, fig. 2), communicates a vibratory motion in a vertical plane to a small lever horizontal in its mean position. To the end of the lever is attached a piece of ebonite, through which passes an arched piece of platinum wire, under the ends of which are two mercury cups (η η); and the mercury cups are so adjusted that, for about half the revolution period, the ends of the platinum wire dip in the mercury, and for the other half are lifted above it. The circuit of one pen of a BAIN'S Electro-Chemical Telegraph Receiver is completed when the ends of the platinum wire are in the mercury, and broken when they are out. A series of dashes and spaces is therefore made on the telegraph tape when the disc is rotated, a dash and a space corresponding to one revolution. Side by side with the record of the revolutions, a second pen marks the record of the laboratory standard clock. In the clock, the teeth of the escapement wheel lift a lever once a second, thereby breaking contact in the circuit of the second pen of the receiver. Lest inequality in the cutting of the teeth of the escapement wheel should introduce an error, the record of an integral multiple of a minute has been taken in the determinations. To determine the vibration frequency of the fork, it is only necessary to maintain the figure chosen stationary for three or four minutes while the clock and rotating apparatus make their records side by side upon the tape. Counting the tape, we obtain the rate of rotation of the cylinder, and hence, by the formula, the vibration frequency of the fork.

The results of two determinations of the vibration frequency of the fork by this method are given in § 27.

§ 11. The measuring machine was made by Sir J. WHITWORTH and Co., and is similar to the ten-thousandth machines ordinarily made by them, except that its bed is of unusual length. It will receive and measure lengths up to 40 inches. To the loose headstock Y (Plate 3, figs. 17, 18) is attached an arm Ar, which projects at right angles to the length of the bed, and in it the movable electrode O is fixed. A screw to move the loose headstock runs along the centre of the graduated bed, and carries at the end, remote from the fixed headstock, a wheel, H (Plate 2, fig. 5), divided on its circumference into 500 parts. The pitch of the screw is ·5 inch, and therefore one division on the wheel, H, corresponds to ·001 inch. The readings for the position of the movable electrode were, during the experiments, taken on this screw and wheel. The screw is, however, not of guaranteed accuracy, and therefore the distance between the equilibrium positions has been subsequently measured by reference to the standard

bars. In Table VI. the corrected values of l are obtained in this way, the uncorrected values being the differences of the readings on the long screw.

§ 12. The wires of the battery circuit are indicated in Plate 2, fig. 5. The current enters and leaves the mercury in the trough by iron plates, $p_1 p_2$, placed vertically at the ends of it. The plates are 5 inches long, 1·75 inch broad, and ·25 inch thick. Some trouble was found in keeping the plates when left in contact with the mercury free of rust; they required frequent cleaning.

The battery E consisted of 12 secondary cells coupled in series; and the resistance of the circuit is such that the current used is something less than ·5 ampère.

The wires from the battery are led to the key K_1, and thence to the commutator K_2. The latter is of the form described by the author at the British Association meeting at Aberdeen (*vide* 'Electrician,' vol. 15, p. 370). A wire from the commutator goes through the standard coil to the iron plate, p_1, at one end of the trough; and from the plate, p_2, at the other end, a wire passes through the coil U_2 to the commutator. The object of the coil U_2 will be subsequently explained.

The wires of the disc and galvanometer circuit are also shown in Plate 2, fig. 5. The brushes are connected by flexible pieces of copper tape to the ends of the stiff cable of the circuit. The cable is seven strands, No. 16 B.W.G. copper wire, thickly insulated with vulcanised rubber. The cable from the central brush passes to the movable electrode O in the trough; from the fixed electrode P it passes through the coil U_1 (the object of which is explained hereafter, and which consists of only two or three turns) to the galvanometer key, K_3, and thence to the copper tape attached to the outside brush.

Great care was taken in insulating the battery and galvanometer circuits.

The central brush was placed to earth through a gas pipe.

The electrodes are of platinum wire ·04 inch in diameter, and descend vertically into the mercury, the lower ends being bent so as to be horizontal for about ·25 inch. Before they were placed in position the end of each was pointed, and the whole wire was thickly covered with paraffin wax. The wax was then shaved away at the end until the platinum point only just emerged so as to secure as nearly as possible a definite point of contact between the electrode and the mercury. The height of the electrode points above the bottom of the trough was about ·5 inch.

§ 13. *Spherometer.*—The spherometer is shown drawn quarter full size in Plate 3, figs. 9 and 10. It consists of a steel screw V, of pitch $\frac{1}{14}$-inch, supported on a stiff iron stand, provided with three levelling screws. The large head h h, on the top is divided into 360 divisions, and a scale s s, is provided to mark complete turns.

The end of the screw is turned down to a very fine point and highly polished.

A mercury cup d, is provided, by means of which a wire coming from a battery through a galvanometer could be placed in electrical contact with the screw, the other wire from the battery being soldered to the iron plate p_1 (Plate 2, fig. 5), and thereby placed in electrical communication with the mercury in the trough.

The position of the mercury surface in the trough is determined by moving the spherometer screw downwards until the passage of a current through the galvanometer indicates that the circuit is complete.

It is of the utmost importance that the point of the screw should be kept clean, otherwise successive observations may differ from one another as much as ·001 inch; but, with clean mercury, if the point is carefully wiped with filter paper before making the observations, half-a-dozen observations may be without difficulty obtained, none of which differ from the mean by more than ·00005 inch. One division on the head is equal to $\frac{1}{5040}$ inch, and the head is large enough to allow of an accurate estimation of tenths of a division.

To preserve the point, sparking at the mercury surface should be reduced as much as possible. With this view, a large resistance is inserted in the circuit, and, between successive observations, contact is broken at the galvanometer before the screw is raised.

Details of the calibration of the spherometer screw are given in § 29. The mercury used has been several times treated with nitric acid and distilled.

§ 14. *Thermometers.*—The thermometers are three in number, one at each end of the trough, and one between the prongs of the fork. They were made by Negretti and Zambra, and are graduated to tenths of a degree. The two trough thermometers were sent to Kew for correction, and it is the corrected temperatures that are inserted in Tables I.–V.*

§ 15. It has already been pointed out that the trough must suffer distortion if the temperature varies, the breadth towards the central part becoming less as the temperature increases. After the experiments were concluded, a measurement was taken at 19·2° C., and the breadth of the trough was found to be 1·51362 inch, the breadth at 15·5° C. being 1·51681 inch. This gives a decrease of breadth of ·000862 per degree or

$$b_t = b_{15\cdot5}\,(1 - \cdot000568\,\overline{t - 15\cdot5}).$$

If, then, the temperature of the paraffin wax could be determined with accuracy, a correction might be made in each set of observations for that temperature. But the wax is a poor conductor of heat, and it takes a long time to assume the temperature of the medium in which it is placed, and different portions of it are likely to be at different temperatures, unless the temperature of the surrounding space has remained for a very long time constant. An attempt was, therefore, made to keep a temperature as constant as possible during the course of the observations.

The trough and measuring machine were enclosed in a wooden box X, Plate 2, fig. 5, covered with several layers of felt paper, and felt curtains were slung round it from a beam above. The thermometers in the trough could be read through windows in the box by lifting the curtain; and everything was so arranged that a set of observations could be taken without opening the box further than to remove for a short time one of the glass windows. The windows were eight inches high by two inches broad.

* *Vide* note, p. 10.

The room outside was kept as nearly as could be at 15°·5 C., which was taken as standard temperature throughout the observations. The extreme variation of temperature in the box did not amount to more than half a degree during the month in which the observations were taken.

The motor, bearings, coil, trough, and measuring machine all rest on slate slabs placed on brick pillars built on the four feet of concrete forming the floor of the laboratory. It is of great importance that the trough supports should be such as to free it from all vibration; otherwise waves on the mercury render the spherometer readings uncertain.

§ 16. *The Galvanometer.*—The galvanometer used was a Thomson reflecting galvanometer, made by ELLIOTT, of resistance ·968 ohm. The suspension, however, was altered. The vertical brass-piece ordinarily bearing the compensating magnet was replaced by a glass tube surmounted by brass fittings giving the usual suspension adjustments. The suspending fibre was a quartz thread about thirteen inches long, one of a number kindly sent to the author by Mr. BOYS. As suspending fibres for galvanometer needles, it is difficult to praise these fibres too highly. They will give a constant zero in a field many times weaker than the weakest in which it would be possible to use silk fibres. With the galvanometer in question the period of oscillation with a quartz fibre could be made as much as twenty-one or twenty-two seconds, while still maintaining a zero as constant as silk gave with a period of about four seconds.

In the course of the experiments the oscillation period was about eight seconds.

The distance of the scale S (Plate 2, fig. 5) from the galvanometer was forty inches. The divisions of the scale are forty to the inch; and the readings were taken through a large magnifying glass.

Adjustment of the Instruments and Method of Observation.

§ 17. The first operation was to place the frame supporting the axle of the rotating disc in such a position that the axis of rotation should be approximately at right angles to the magnetic meridian, and the plane of the disc, therefore, in the plane of the meridian. The object was to get rid, as far as possible, of any induction current in the disc due to a magnetic field, other than that of the current through the standard coil. In the event it proved that this current was so small as to need no special compensating apparatus during the observations. When no current was passing through the standard coil, the deflection obtained by rotating the disc did not amount to more than could easily be counteracted by a slight movement of the compensating magnet placed on the table at some two feet distance from the galvanometer.

Attention was next directed to the four bearings β, β, β, β, Plate 2, fig. 5, so as to ensure smooth and steady rotation. They were carefully scraped to correct any slight

inequalities introduced by bolting them down to the brick pillars, and a sight feed lubricator giving about three drops per minute was fitted to each.

The shaft of the electromotor was coupled to the axle by a flexible coupling *g*, Plate 2, fig. 5 ; and the disc was then ground true in place as mentioned in § 7.

§ 18. The adjustment of the standard coil so that it should be co-axial with the disc, and that its mean plane should coincide with the mean plane of the disc, was made with the help of the circular line inside the coil cylinder, the cutting of which has been already described in § 3. To effect this an arm, *f f*, Plate 3, figs. 13 and 14, of adjustable length, had been prepared to fit the disc. The face at *e* was scraped plane so that when the arm was firmly clamped in the position shown in figs. 13 and 14, the plane of the face as nearly as possible coincided with the mean plane of the disc. Great care was taken to effect this by continually reversing the arm during the scraping and testing with the help of a straight edge and feeling piece.

It will be subsequently shown that the effect of any possible small error in the accuracy with which this was accomplished is negligible.

The adjustment of the standard coil consists in placing the coil so that, when the arm on the disc is adjusted to proper length, the edge of its face *e* travels, when the disc is turned, over the circle cut on the inside surface of the coil cylinder in the mean plane of the coil (*vide* § 3). The coil is first moved into position as nearly as may be by eye, and the final adjustments are made by turning the levelling screws *l*, *l*, *l*, Plate 1, figs. 1 and 2, and by taps on the coil frame, which move either the levelling screws in the cups over the bases of the cups, or the cups as a whole over the slate slab.

It has already been observed that the coil cylinder is slightly oval, and hence the arm could only be made to touch it at two points at the ends of the shortest diameter. It will be seen from § 24 that the difference between the longest and shortest radius is about ·008 inch.

§ 19. To adjust the mercury trough its base must be levelled and its lateral faces made vertical and parallel to the length of the bed of the measuring machine. The trough was not supported so as to give the requisite number of independent movements to effect these adjustments. It was therefore brought into position by successive small movements, the levelling of the base in the direction of its length being tested by a delicate spirit level, and the verticality of the side faces by the same level with the help of an accurate steel square, the stock of which was pressed up against the face. The parallelism of the faces to the run of the measuring machine was tried by fixing in the electrode arm, Ar, Plate 3, fig. 18, a turned cylinder and inserting a wooden wedge between the cylinder and the face. If the reading on the wedge is the same when the cylinder is moved from one end of the trough to the other, the required parallelism is obtained. Further by making the reading when the wedge was inserted between the cylinder and the one face the same as when it was inserted between the cylinder and the other, the axis of the cylindrical hole was brought into the median plane of the

trough. The electrode subsequently inserted was approximately axial to the hole, and therefore nearly in this median plane.

The spherometer was adjusted so that the divided head plate was level both when the instrument was used and when it was calibrated (*vide* § 29).

§ 20. The apparatus being in place and the connecting wires added, it was found that while the galvanometer circuit was broken, a slight deflection of the galvanometer needle accompanied a change in the direction of the current through the battery circuit. A compensating coil U_2 (Plate 2, fig. 5), consisting of nearly three turns of the wire of the battery circuit, satisfactorily counterbalanced this disturbance of the magnetic field at the needle.

§ 21. Tests for insulation were next made with an E.M.F. of 24 volts.

The insulation resistance between the wire of the coil and the coil cylinder was found to be more than 50 megohms, that between the disc and axle more than 100 megohms, and that of the completed circuits (the cups under the levelling screws of the coil being on thin paraffined ebonite plates) more than 70 megohms.

§ 22. The observations consist fundamentally in the determination of equilibrium positions of the movable electrode, together with the rates of rotation to which they correspond (*vide* § 4). In determining an equilibrium position, a definite rate of rotation was first obtained by the observer at the tuning-fork and maintained by him, as accurately as possible, throughout the determination, an assurance of stationariness being received from him by the observer at the galvanometer when the readings were being taken.

It was found best, following Lord Rayleigh ('Phil. Trans.,' 1883, "Experiments by the Method of Lorenz to determine the Value of the B.A. Unit of Resistance"), to take two sets of galvanometer readings for each equilibrium position, one set giving the change of galvanometer reading corresponding to reversal of the battery current for a position slightly to one side of the equilibrium position; the other, the change of galvanometer readings corresponding to reversal for a second position slightly on the other side of the equilibrium position.

The galvanometer readings corresponding to the two positions of the commutator being called E and W, passage through an equilibrium position is indicated by a change in the algebraic sign of W − E. When W − E has been found for two positions slightly differing from one another, and including between them the equilibrium position, the latter may be deduced by simple interpolation.

Since the readings E and W are not quite fixed owing to small changes in the speed of the disc and slight variations at the brush contacts, it is necessary to take a number of reversals and to find W − E as the mean of the values obtained by combining the first value of E with the first of W, the second of E with the first of W, the second of E with the second of W, &c. The more quickly the reversals can be made to succeed one another the better will be the results, for with rapid reversals,

variations in the position of the needle corresponding to no current through the standard coil are less likely to supervene. It is, therefore, best not to wait for the needle to come, even approximately, to rest after the disturbance due to the induction current on reversal, but to take the readings for the extreme positions in an oscillation, and, having previously found the co-efficient of damping, to calculate the position of rest from these two readings.

The coil U_1 (Plate 2, fig. 5) is used to adjust the induction so that the swing of the needle at reversal may be of suitable magnitude for this purpose.

It is advantageous to use the same portion of the trough for the observations at both depths of mercury; any slight irregularities in the base of the trough are then without effect. The rates of rotation and the depths requisite to effect this were, therefore, determined by preliminary experiments. It was found that if the depths are approximately ·75 and 1·25 inch that by the use of the rows of 8 and 13 teeth on the drum at the smaller depth, and of 13 and 21 teeth at the greater depth, the required result was secured.

The thermometers in the trough were read immediately before and immediately after each pair of sets of galvanometer readings determining an equilibrium position, and the thermometer between the prongs of the tuning-fork before and after each set.

The measurements on the spherometer were made directly before and after the addition of the mercury required to raise the level in the trough from the first to the second height. The temperatures of the trough thermometers when each measurement was taken were observed and recorded. Each spherometer measurement recorded in Tables I.–V.* is the mean of from four to six observations, the extreme observations differing from each other by not more than ·00003 inch.

Details of Measurement and Calculation.

§ 23. The diameter of the disc was measured in the Whitworth measuring machine. Measurements were made in four directions, approximately equally inclined to each other, with the following results :—

Diameter.	Measurement.
	inches.
0°–180°	12·98954
45–225	12·98979
90–270	12·98949
135–315	12·98951
	Mean 12·98958

The temperature at which the measurements were made was 12° C.

* *Vide* note, p. 10.

Hence the radius of the disc at 12° C. = 6·49479 inches, and correcting for temperature we have

the radius at 15°·5 C. = 6·49493 inches.

§ 24. The diameter of the coil was determined in similar fashion, but on account of the silk covering of the wire, the measurement could not be carried to the same degree of accuracy as in the case of the disc. For this reason, and also because of the ellipticity of the coil cylinder the measurements were much more numerous, and were made along 18 diameters. The cylinder while still in the lathe had been divided along its edge into intervals of 10°.

Diameter.	Measurement.
	inches.
0–180	21·0898
10–190	21·0929
20–200	21·0951
30–210	21·0933
40–220	21·0960
50–230	21·0998
60–240	21·1017
70–250	21·1026
80–260	21·1044
90–270	21·1038
100–280	21·1056
110–290	21·1041
120–300	21·1014
130–310	21·0979
140–320	21·0945
150–330	21·0924
160–340	21·0900
170–350	21·0910
	Mean 21·09757
	t = 17° C.

The mean of a large number of measurements of the diameter of the wire was ·02191 inch.

The measurements were taken at points equidistant, or nearly so, along a length of about 10 yards of wire.

Subtracting the diameter of the wire from the measured outside diameter of the coil we obtain 21·07566 as the mean diameter from the axis of the wire on one side to the axis of the wire on the other: and hence,

the mean radius at 17° C. = 10·53783, and correcting for temperature,
the mean radius at 15°·5 C. = 10·53774.

Calculation of the Induction Coefficient.

§ 25. It has been shown by the author* that the coefficient of mutual induction of a circle and co-axial helix, the circle being in the mean plane of the helix, and the helix consisting of a whole number of turns (n), is given by the formula :—

$$\mathrm{M} = -8n\varpi\frac{\mathrm{A}a}{\mathrm{A}+a}\left\{\mathrm{P}_0 - \tfrac{1}{2}\cdot\tfrac{1}{3}\left(\frac{x}{\mathrm{A}+a}\right)^2\mathrm{P}_1 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}\left(\frac{x}{\mathrm{A}+a}\right)^4\mathrm{P}_2 - \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{7}\left(\frac{x}{\mathrm{A}+a}\right)^6\mathrm{P}_3 + \&c.\right\}$$

$$= -8n\varpi\frac{\mathrm{A}a}{\mathrm{A}+a}\Sigma(-1)^m\frac{1.3.5\ldots\overline{2m-1}}{2.4.6\ldots 2m}\frac{1}{2m+1}\left(\frac{x}{\mathrm{A}+a}\right)^{2m}\mathrm{P}_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I.})$$

where A = the radius of a circular section of the cylinder determined by the helix,

a = the radius of the circle,
$2x$ = the axial length of the helix,

and

$$\mathrm{P}_m = \int_0^{\varpi/2}\frac{\cos 2\theta\, d\theta}{(1-c^2\sin^2\theta)^{(2m+1)/2}},$$

c being equal to $\dfrac{2\sqrt{(\mathrm{A}a)}}{\mathrm{A}+a}$.

As the formula has not previously been used for the calculation of the induction coefficient, it may be of utility to indicate the method of calculation that has been found convenient. For our present purpose it will be sufficient if six terms of the series are found.

Let

$$\mathrm{Q}_m = \int_0^{\varpi/2}\frac{d\theta}{(1-c^2\sin^2\theta)^{(2m+1)/2}};$$

then

$$\mathrm{P}_m = \left(1-\frac{2}{c^2}\right)\mathrm{Q}_m + \frac{2}{c^2}\mathrm{Q}_{m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II.})$$

and

$$\mathrm{Q}_m = \mathrm{Q}_{m-1} + \frac{c}{2m-1}\frac{d}{dc}\mathrm{Q}_{m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{III.})$$

Observing that

$$\mathrm{Q}_0 = \mathrm{F}$$

and

$$\mathrm{Q}_{-1} = \mathrm{E},$$

* "On the Calculation of the Coefficient of Mutual Induction of a Circle and a Coaxal Helix," communicated to the Physical Society in November, 1888 ('Phil. Mag.', Jan., 1889).

where F and E are the complete elliptic integrals to modulus c, and using the relation

$$\frac{dF}{dc} = \frac{E}{c(1-c^2)} - \frac{F}{c}$$

we have

$$Q_1 = \frac{E}{1-c^2}.$$

The calculation may now be conducted systematically as follows:—

(i.) Calculate

$$Q_1, \quad \frac{dQ_1}{dc}, \quad \frac{d^2Q_1}{dc^2}, \quad \frac{d^3Q_1}{dc^3}, \quad \&c.$$

or denoting differentiation with regard to c by a dot

$$Q_1, \ \dot{Q}_1, \ \ddot{Q}_1, \ \dddot{Q}_1, \ \&c.$$

These may be successively calculated from the equation

$$(1-c^2)\, Q_1 = E,$$

and its derivatives

$$c\,(1-c^2)\,\dot{Q}_1 - 2c^2Q_1 = E - F,$$

$$c\,(1-c^2)\,\ddot{Q}_1 + \dot{Q}_1\,(1-5c^2) - 3cQ_1 = 0,$$

$$c\,(1-c^2)\,\dddot{Q}_1 + 2(1-4c^2)\,\ddot{Q}_1 - 13c\dot{Q}_1 - 3Q_1 = 0,$$

$$c\,(1-c^2)\,\ddddot{Q}_1 + (3-11c^2)\,\dddot{Q}_1 - 29c\ddot{Q}_1 - 16\dot{Q}_1 = 0.$$

(ii.) Calculate

$$Q_2, \ \dot{Q}_2, \ \ddot{Q}_2, \ \dddot{Q}_2$$

from the equation

$$Q_2 = Q_1 + \frac{c}{3}\dot{Q}_1,$$

and its derivatives

$$\dot{Q}_2 = \tfrac{4}{3}\dot{Q}_1 + \frac{c}{3}\ddot{Q}_1,$$

$$\ddot{Q}_2 = \tfrac{5}{3}\ddot{Q}_1 + \frac{c}{3}\dddot{Q}_1,$$

$$\dddot{Q}_2 = \tfrac{6}{3}\dddot{Q}_1 + \frac{c}{3}\ddddot{Q}_1.$$

(iii.) Calculate

$$Q_3, \ \dot{Q}_3, \ \ddot{Q}_3$$

from the equation

$$Q_3 = Q_2 + \frac{c}{5}\dot{Q}_2,$$

and its derivatives

$$\dot{Q}_3 = \tfrac{6}{5}\,\dot{Q}_2 + \frac{c}{5}\,\ddot{Q}_2,$$

$$\ddot{Q}_3 = \tfrac{7}{5}\,\ddot{Q}_2 + \frac{c}{5}\,\dddot{Q}_2.$$

(iv.) Calculate Q_4, $\dot{Q}_4$ from the equation

$$Q_4 = Q_3 + \frac{c}{7}\,\dot{Q}_3$$

and its derivative

$$\dot{Q}_4 = \frac{8}{7}\,\dot{Q}_3 + \frac{c}{7}\,\ddot{Q}_3\,.$$

(v.) Calculate Q_5 from the equation

$$Q_5 = Q_4 + \frac{c}{9}\,\dot{Q}_4\,.$$

(vi.) Calculate P_0, P_1, P_2, P_3, P_4, P_5 from the values of E, F, Q_1, Q_2, Q_3, Q_4, Q_5, by successive applications of Equation II.

For the coil and the circumference of the disc we have :—

$$\begin{aligned}
A &= 10{\cdot}53774 \\
a &= 6{\cdot}49493 \\
\log c &= \bar{1}{\cdot}9874084 \\
E &= 1{\cdot}0665493 \\
F &= 2{\cdot}8509606 \\
Q_1 &= 1{\cdot}8931552 \times 10 \\
\dot{Q}_1 &= 6{\cdot}2026914 \times 10^2 \\
\ddot{Q}_1 &= 4{\cdot}3150869 \times 10^4 \\
\dddot{Q}_1 &= 4{\cdot}5196380 \times 10^6 \\
\ddddot{Q}_1 &= 6{\cdot}3189387 \times 10^8 \\
\hline
Q_2 &= 2{\cdot}1977948 \times 10^2 \\
\dot{Q}_2 &= 1{\cdot}4799606 \times 10^4 \\
\ddot{Q}_2 &= 1{\cdot}5354118 \times 10^6 \\
\dddot{Q}_2 &= 2{\cdot}1365132 \times 10^8 \\
\hline
Q_3 &= 3{\cdot}0951221 \times 10^3 \\
\dot{Q}_3 &= 3{\cdot}1606635 \times 10^5 \\
\ddot{Q}_3 &= 4{\cdot}3658744 \times 10^7 \\
\hline
Q_4 &= 4{\cdot}6957152 \times 10^4 \\
\dot{Q}_4 &= 6{\cdot}4199487 \times 10^6 \\
\hline
Q_5 &= 7{\cdot}3990015 \times 10^5 \\
\hline
\end{aligned}$$

$$P_0 = - \cdot 93092203$$
$$P_1 = - 1 \cdot 5149680 \times 10$$
$$P_2 = - 2 \cdot 0589783 \times 10^2$$
$$P_3 = - 2 \cdot 9988831 \times 10^3$$
$$P_4 = - 4 \cdot 6004090 \times 10^4$$
$$P_5 = - 7 \cdot 2872416 \times 10^5$$

$$\tfrac{1}{2} \cdot \tfrac{1}{3} \left(\frac{x}{A+a}\right)^2 P_1 = - \cdot 04654257$$

$$\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} \left(\frac{x}{A+a}\right)^4 P_2 = - \cdot 00524698$$

$$\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{1}{7} \left(\frac{x}{A+a}\right)^6 P_3 = - \cdot 00083851$$

$$\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{7}{8} \cdot \tfrac{1}{9} \left(\frac{x}{A+a}\right)^8 P_4 = - \cdot 00016136$$

$$\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{7}{8} \cdot \tfrac{9}{10} \cdot \tfrac{1}{11} \left(\frac{x}{A+a}\right)^{10} P_5 = - \cdot 00003469$$

$$\therefore \Sigma (-1)^m \frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \frac{1}{2m+1} \left(\frac{x}{A+a}\right)^{2m} P_m = - \cdot 88891460$$

and $M = 89 \cdot 7717\, n$.

Two small corrections are needed in order to obtain from the above value of M the actual coefficient of mutual induction used in the experiments, viz. the correction for the central brush, and the correction for the iron in the field.

The radius of the central brush = ·05847 inch: and the coefficient of mutual induction of its circumference and the standard coil = ·00626 n. Subtracting this from the coefficient of mutual induction of the circumference of the disc and the coil we have $M = 89 \cdot 7654\, n$.

The number of turns n on the coil is 185,

$$\therefore M = 16606 \cdot 60.$$

The iron in the field consists of the steel axle $\xi\,\xi$ (Plate 2, fig. 5), the steel fly-wheel, and the iron of the motor. The fly-wheel and motor iron will, on account of their distance, hardly affect the result to a greater extent than the probable error of the mean of the five sets of observations (Tables I.–V.). But the effect of the steel in the part of the axle $\xi\,\xi$ is appreciable. It was estimated and allowed for as follows. A steel bar of the same dimensions was prepared which could be placed in a position on the other side of the coil and disc symmetrical with regard to their mean plane

to that occupied by the axle $\xi\,\xi$. A series of observations were then made of the equilibrium positions of the movable electrode with the steel bar in and out of place, the rate of rotation and the level of the mercury in the trough being constant, and proper corrections being introduced for temperature in the manner indicated later in § 31.

Let

M = the coefficient of mutual induction with the steel bar away.

$M + \Delta M$ = the coefficient of mutual induction with the steel bar in place.

L = the initial distance between the two electrodes.

ΔL = the distance moved through by the movable electrode to its new equilibrium position when the steel bar is introduced into the field.

Then

$$\frac{\Delta M}{M} = \frac{\Delta L}{L}.$$

In the observations made

$$\begin{aligned} L &= 26{\cdot}8615, \\ \Delta L &= \quad {\cdot}01157, \\ \therefore \Delta M &= \quad {\cdot}0004307\ M, \\ &= \quad 7{\cdot}153. \end{aligned}$$

This may be taken as a sufficiently close approximation to the correction that must be added to the calculated coefficient of mutual induction in consequence of the steel axle $\xi\,\xi$. Making this correction, we have as the final value of the coefficient used in the experiments

$$M = 16613{\cdot}75.$$

A calculation was made to find the effect of the finite thickness of the wire of the coil on the coefficient of induction. In the formula it is assumed that the current is concentrated in the axis of the wire. Calculation showed that the mean of the induction coefficients, obtained on the suppositions that the current is concentrated on the outside and inside of the wire respectively, does not differ from the induction coefficient calculated on the hypothesis that the current is concentrated in the axis of the wire by more than, approximately, 1 part in 300,000. In the experiments we have the current uniformly distributed over the section of the wire, and the error introduced by supposing the current concentrated in the axis is less than the error calculated in the extreme case taken, and for present purposes quite negligible.

§ 26. It seems proper here to consider the effect on the coefficient of mutual induction of a possible non-coincidence of the mean planes of the coil and disc.

In making the adjustment (§ 18), the edge of the face e of the arm $f\,f$, Plate 3, figs. 13 and 14, is made to travel over the circular line cut as described in § 3 on the

inside of the coil cylinder. If the face e does not coincide with the mean plane of the disc, and if the plane of the line on the coil cylinder does not coincide with the mean plane of the coil, the adjustment does not bring the mean planes into coincidence, but into positions such that the distance between the mean planes is equal to the algebraical sum of the distance between the plane of the circle described by the edge of e and the mean plane of the disc, and the distance between the plane of the circular line in the coil cylinder and the mean plane of the coil. Having regard to the precautions taken in drawing the line and preparing the arm, the sum of these errors cannot be as much as ·01 inch.

If the distance between the mean planes is assumed to be half the pitch of the helix, the coefficient of mutual induction may be calculated as follows :—we may consider the coefficient in this case to be made up of two parts, viz., the coefficient of mutual induction of the circumference of the disc and of the portion of the coil of axial length $= x + x/n$ on one side of it, and the coefficient of mutual induction of the circumference of the disc and the portion of the coil of axial length $= x - x/n$ on the other side of it. Calling the first of these M_1, and the second M_2, we have

$$M_1 = -4n\varpi \frac{A\alpha}{A+\alpha} \Sigma(-1)^m \frac{1.3.5\ldots\overline{2m-1}}{2.4.6\ldots\ 2m} \frac{1}{2m+1} P_m\left(\frac{x}{A+\alpha}\right)^{2m}\left(1+\frac{1}{n}\right)^{2m+1},$$

$$M_2 = -4n\varpi \frac{A\alpha}{A+\alpha} \Sigma(-1)^m \frac{1.3.5\ldots\overline{2m-1}}{2.4.6\ldots\ 2m} \frac{1}{2m+1} P_m\left(\frac{x}{A+\alpha}\right)^{2m}\left(1-\frac{1}{n}\right)^{2m+1},$$

$\therefore M_1 + M_2$

$$= -4n\varpi \frac{A\alpha}{A+\alpha} \Sigma(-1)^m \frac{1.3.5\ldots\overline{2m-1}}{2.4.6\ldots\ 2m} \frac{1}{2m+1} P_m\left(\frac{x}{A+\alpha}\right)^{2m}\left\{\left(1+\frac{1}{n}\right)^{2m+1}+\left(1-\frac{1}{n}\right)^{2m+1}\right\}$$

$$= -8n\varpi \frac{A\alpha}{A+\alpha}\left[P_0 - \tfrac{1}{2}\cdot\tfrac{1}{3}\left(\frac{x}{A+\alpha}\right)^2 P_1\left(1+\frac{3}{n^2}\right) + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}\left(\frac{x}{A+\alpha}\right)^4 P_2\left(1+\frac{10}{n^2}+\frac{5}{n^4}\right)\right.$$
$$\left. - \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{7}\left(\frac{x}{A+\alpha}\right)^6 P_3\left(1+\frac{21}{n^2}+\frac{70}{n^4}+\frac{7}{n^6}\right) + \&c.\right]$$

$$= -8n\varpi \frac{A\alpha}{A+\alpha}(-\cdot88891460 + \cdot00000408 - \cdot00000154 + \cdot00000051$$
$$- \cdot00000017 + \cdot00000006)$$

$$= -8n\varpi \frac{A\alpha}{A+\alpha}(-\cdot88891460 + \cdot00000294)$$

$$= M(1 - \cdot00000331).$$

Or if the distance between the mean planes amounts to one half the pitch of the helix, *i.e.*, to ·0125 inch, the coefficient of mutual induction is diminished only by about one part in 300,000.

Hence there can be no appreciable error due to this cause.

Determination of the Vibration Frequency of the Tuning Fork.

§ 27. The details of two measurements of the vibration frequency of the fork by the method described in § 10 are given in the following tables. To the degree of accuracy required the clock needed no correction.

Temperature of fork.	Seconds.	Tape reading.	Seconds.	Tape reading.	Revolutions in three minutes.
Initial 14·98	0	1·72	180	2917·10	2915·38
	1	17·97	181	2933·30	2915·33
	2	34·09	182	2949·42	2915·33
	3	50·41	183	2965·95	2915·54
	4	66·44	184	2981·82	2915·38
Final 15·01	5	82·78	185	2998·26	2915·48
Mean 14·995	..	..	..	Mean	2915·407

Hence number of revolutions per minute . = 971·802
And „ „ „ „ second . = 16·1967
And P = 64·7868, or, corrected* to 15°·5, P = 64·7832

Temperature of fork.	Seconds.	Tape reading.	Seconds.	Tape reading.	Revolutions in three minutes.
Initial 15·06	0	2·0	180	2917·37	2915·37
	1	17·96	181	2933·33	2915·37
	2	34·17	182	2949·50	2915·33
	3	49·92	183	2965·25	2915·33
	4	66·54	184	2981·91	2915·37
Final 15·08	5	82·47	185	2997·86	2915·39
Mean 15·07	..	..	..	Mean	2915·360

Hence number of revolutions per minute . = 971·787
And „ „ „ „ second . = 16·19645
And P = 64·7858, or corrected* to 15°·5, P = 64·7828

Mean $P_{15°·5}$ = 64·7830.

§ 28. *Calibration of the Trough.*—The form of the callipers used to measure the breadth of the trough is shown in elevation and plan in Plate 3, figs. 15 and 16.

Two side pieces of brass *r*, *r*, with their faces carefully scraped plane and parallel, are suspended by silk threads from the wooden framework of the apparatus. Between

* The correcting coefficient used is that found by Macleod and Clarke ('Phil. Trans.,' 1880), viz., — ·00011.

them are a pair of wedges, w_1, w_2, with their faces scraped plane, and such that, when they are put together, as in the figure, their outside faces are parallel. The wedge w_1 is graduated into twentieths of an inch, and an index mark is placed on the wedge w_2. When the apparatus is in the trough, the centre of the brass pieces r, r, is at the mean height of the mercury surface during the measurements recorded in Tables I.–V. The method of procedure was to use the wedge with the handle attached as a feeling piece, and to push it in until when very slightly lifted it was held in place. Measurements were made every half-inch along the portion of the trough used. At each point four readings were taken which are recorded in the table below. Only in one case do the four readings at any point differ from one another by more than ·2 division (= ·0001 inch approximately). The following table contains the results of the trough measurements :—

Position in trough.	Wedge readings.				Mean.
Inches.					
24·5	33·4	33·4	33·4	33·5	33·42
24·0	34·5	34·6	34·5	34·4	34·50
23·5	35·0	35·0	35·0	35·0	35·00
23·0	34·9	34·9	35·0	34·9	34·92
22·5	35·3	35·3	35·1	35·3	35·25
22·0	34·4	34·4	34·5	34·3	34·40
21·5	34·3	34·2	34·3	34·3	34·28
21·0	33·6	33·7	33·7	33·8	33·70
20·5	33·9	33·9	33·9	34·0	33·92
20·0	33·7	33·7	33·6	33·6	33·65
19·5	33·2	33·2	33·25	33·2	33·21
19·0	33·7	33·7	33·7	33·8	33·72
18·5	34·9	34·9	34·7	34·7	34·80
18·0	35·0	35·0	35·0	35·0	35·00
17·5	35·8	35·7	35·9	35·7	35·78
17·0	35·8	35·9	35·9	36·0	35·90
16·5	35·1	35·2	35·3	35·3	35·22
16·0	33·5	33·6	33·7	33·7	33·62
15·5	32·2	32·0	32·2	32·5	32·22
15·0	34·3	34·4	34·3	34·5	34·38
Mean at 15°·5 C. 34·344					

After the observations in the trough were completed the whole apparatus was transferred to the measuring machine, and the breadth corresponding to given readings of the wedges determined. The breadth corresponding to the mean reading in the trough was then calculated by simple interpolation.

Wedges transferred to measuring machine.	
Wedge readings.	Thickness given by measuring machine.
41·2 36·9	1·5100″ 1·5080
4·3	·002
A decrease in thickness of ·002″ corresponds to a movement of 4·3 divisions on the wedges. ∴ The thickness corresponding to the reading 34·344 of the wedges = 1·51681 inch.	

§ 29. *Calibration of Spherometer Screw.*—For this purpose the wedges and one of the side pieces r were used with a Whitworth Surface Plate. The steel point of the spherometer screw was protected by screwing on an auxiliary brass cover with rounded end.

The surface plate was first levelled. The spherometer was then placed on an independent stand on one side of the plate, so that the screw was over the plate; and the stand was so adjusted for height that when the spherometer head h (fig. 9) was levelled as in § 19, the reading on the head for contact with the wedges when they are placed on the surface plate under the screw point is near the readings for the mercury surface in its lower position given in Tables I.–V.

The reading for the surface of the side piece placed on the wedges will then be near the readings for the mercury surface in its higher position, for in each set of observations the difference of level is nearly ·5 inch, and the breadth of the side piece is about the same.

In order to determine the reading for contact between the auxiliary screw point and either the surface of the side piece on the wedges, or the surface of the wedges alone, the spherometer screw head was moved gradually downwards until when the pieces were made to slide over the surface plate under the screw point a slight mark on the surface resulted from the passage.

A series of sets of readings were in the first place taken for contact between the screw point and the surface of the side piece placed on the wedges, the thickness of the wedges being diminished between successive sets by an amount corresponding to five divisions on the graduated wedge. The part of the series required is incorporated in the following table:—

Wedges on Surface Plate and Side piece on Wedges.

Wedge readings.	Spherometer readings.								Thickness of wedges by measuring machine.	Thickness of side piece by measuring machine.	Height of spherometer point above surface plate.
	I.		II.		III.		Mean.				
	Turns.	Divisions.	Turns.	Divisions.	Turns.	Divisions.	Turns.	Divisions.			
									Inch.	Inch.	Inch.
25	29	329·5	29	329·0	29	329·5	29	329·3	·48739	·51242	·99981
20	29	340·5	29	340·0	29	341·0	29	340·5	·48510	·51242	·99752
15	29	352·0	29	352·5	29	352·0	29	352·2	·48294	·51242	·99536

The height of the screw point above the surface plate corresponding to the reading of the spherometer scale is in each case equal to the sum of the thicknesses of the wedges and the side piece, which were separately determined in the measuring machine. These thicknesses are recorded in the sixth and seventh columns in the above table, and their sum, placed in the last column, gives in each case the height of the screw point above the surface plate.

A similar set of measurements was then made on the wedges alone, the side piece being removed. The results are given in the accompanying table :—

Wedges on Surface Plate.

Wedge readings.	Spherometer readings.								Thickness of wedges by measuring machine.	Height of spherometer point above surface plate.
	I.		II.		III.		Mean.			
	Turns.	Divisions.	Turns.	Divisions.	Turns.	Divisions.	Turns.	Divisions.		
									Inch.	Inch.
55	36	325·0	36	325·5	36	325·0	36	325·2	·50104	·50104
50	36	337·5	36	338·0	36	338·0	36	337·8	·49883	·49883
45	36	349·0	36	349·0	36	349·0	36	349·0	·49657	·49657
40	37	0	37	0	37	0	37	0	·49423	·49423

The above tables are extracted from a more extended set of measurements. They contain the measurements required in interpreting the readings of the spherometer contained in Tables I.–V.

The following tables were then constructed by simple interpolation from the above results, each spherometer reading having the corresponding height of the screw point above the surface plate attached to it :—

Spherometer readings.		Height (calculated).	Spherometer readings.		Height (calculated).
Turns.	Divisions.		Turns.	Divisions.	
29	330	·999667	36	331	·500023
29	331	·999462	36	332	·499847
29	332	·999258	36	333	·499672
29	333	·999053	36	334	·499497
29	334	·998849	36	335	·499321
29	335	·998644	36	336	·499146
29	336	·998440	36	337	·498970
29	337	·998235	36	338	·498790
29	338	·998031	36	339	·498588
29	339	·997826	36	340	·498386
29	340	·997622	36	341	·498184
29	341	·997428	36	342	·497982
29	342	·997243	36	343	·497781
29	343	·997059	36	344	·497579
29	344	·996874	36	345	·497377
29	345	·996689	36	346	·497175
29	346	·996505	36	347	·496973
29	347	·996320	36	348	·496772
29	348	·996136	36	349	·496570
29	349	·995951	36	350	·496357
29	350	·995766	36	351	·496145
29	351	·995582	36	352	·495932
29	352	·995397	36	353	·495719
			36	354	·495507
36	325	·501075	36	355	·495294
36	326	·500900	36	356	·495081
36	327	·500724	36	357	·494868
36	328	·500549	36	358	·494656
36	329	·500374	36	359	·494443
36	330	·500198	37	0	·494230

In calculating the difference of level of the mercury surfaces from the readings given in Tables I.–V., it is only necessary to take the difference of the two corresponding heights in the spherometer calibration Table given above.

§ 30. A record of the observations is given in Tables I.–V. Each table contains the observations requisite for a complete determination which involves finding four equilibrium positions. The upper part of each table is devoted to the observations giving the two equilibrium positions with the surface of the mercury in the trough at the lower level, the lower part to those giving the two equilibrium positions with the mercury surface at the higher level, and in the middle is placed the record of the spherometer readings, from which the difference of level may be calculated. Each equilibrium position is obtained by simple interpolation from two values of W — E (*vide* § 22), and the positions c_1, c_2, of the movable electrode corresponding to them. The galvanometer readings, from which these two values of W — E are calculated, together with other necessary data, are placed in each case above the line recording the equilibrium position deduced from them.

Above each set of galvanometer readings, the following corresponding observations are given :—

(i.) The reading for the movable electrode, c.

(ii.) t_1, t_2, the temperatures of the thermometers T_1, T_2 (Plate 2, fig. 5), taken before or after the galvanometer readings according as the set is marked α or β.

(iii.) θ, the temperature of the thermometer between the prongs of the tuning fork, taken before and after the galvanometer readings.

(iv.) N, the number of teeth in the row on the drum, Dr (Plate 2, fig. 5), seen through the fork slit.

Each set is distinguished by a Roman numeral indicating the table to which it belongs, an Arabic numeral indicating the equilibrium position (1, 2, 3, or 4) it helps to determine, and the letter α or β according as it is the first or second set belonging to that equilibrium position.

In any set of galvanometer readings, the observed readings are contained in the first four columns, the first two columns under the letter E being the readings for an oscillation when the commutator is in the position E, and the third and fourth columns, under the letter W, being the readings for an oscillation when the commutator is in the position W. For instance, in the first set in Table I (I 1α) the readings called out by the observer at the galvanometer followed one another, thus :— − 42, + 23·5, Reversal; + 26·5, − 13, Reversal; − 28·5, + 15·5, Reversal; + 26, − 14, Reversal; − 28·5, + 15, Reversal; + 27·5, − 16, Reversal, &c.

The fifth column under E contains the positions of rest of the galvanometer needle, calculated from the readings on the same line in the first two columns, and the coefficient of damping. The numbers in the sixth column under W are similarly calculated from those in the third and fourth columns.

The value of the coefficient of damping was 0·6757.

TABLE I.—Set No. 1. (3rd April, 1890.)

Number of experiment.	c_1.	t_1.	t_2.	θ.	N.	Number of experiment.	c_2.	t_1.	t_2.	θ.	N.
I (1) α	15 +160	15·45	15·5	15·0 15·1	8	I (1) β	15 +180	15·45	15·5	15·1 15 2	8

E.		W.		E.	W.	W.–E.	E.		W.		E.	W.	W.–E.
−42	+23·5	+26·5	−13	−2·9	+2·9	+5·8	−38	+26	+20	−21	+ ·2	−4·5	−4·7
−28·5	+15·5	+26	−14	−2·2	+2·1	+5·1 +4·3	−26·5	+17·5	+24	−22·5	− ·2	−3·8	−4·3 −3·6
−28·5	+15	+27·5	−16	−2·5	+1·5	+4·6 +4·0	−26	+19·5	+24·5	−19·5	+1·2	−1·8	−5·0 −3·0
−27	+13·5	+27	−16	−2·8	+1·3	+4·3 +4·1	−26	+19·5	+26	−23	+1·2	−3·2	−3·0 −4·4
−28·5	+15·5	+28	−16·5	−2·2	+1·4	+3·5 +3·6	−24	+15	+29	−27	− ·7	−4·4	−2·5 −3·7
−28·5	+15	+26	−16	−2·5	+ ·9	+3·9 +3·4	−27	+15·5	+24	−25·5	−1·6	−5·5	−2·8 −3·9
−30	+13·5	+27	−19	−4·0	− ·5	+4·9 +3·5	−33	+18·5	+41·5	−37	−2·3	−5·3	−3·2 −3·0
−31·5	+17	+23·5	−13·5	−2·6	+1·4	+2·1 +4·0	−24	+12	+25	−25	−2·5	−4·8	−2·8 −2·3
−31	+19	+33·5	−19	−1·2	+2·2	+2·6 +3·4	−25·5	+15	+24	−23·5	−1·3	−4·3	−3·5 −3·0
−26·5	+16	+30	−13·5	−1·1	+4·0	+3·3 +5·1	−28·5	+16·5	+31·5	−29	−1·6	−4·6	−2·7 −3·0
−34·5	+18	+36	−22	−3·2	+1·4	+7·2 +4·6	−26	+15·5	+26	−28	−1·2	−6·2	−3·4 −5·0
−49	+28·5	+17	−10	−2·7	+ ·9	+4·1 +3·6	−32	+18·5	+28	−27	−1·9	−4·8	−4·3 −2·9
					Mean	+4·13						Mean	−3·48

Calculated reading for equilibrium position 15″·17085.

TABLE I.—Set No. 1. (3rd April, 1890)—continued.

Number of experiment.	c_3.	t_1.	t_2.	θ.	N.	Number of experiment.	c_4.	t_1.	t_2.	θ.	N.
I (2) α.	24 +425	15·46	15·5	15·4 15·5	13	I (2) β.	24 +445	15·46	15·5	15·5 15·5	13

E.		W.		E.	W.	W.–E.	E.		W.		E.	W.	W.–E.
−22	+5	+ 9	−11	−5·9	−2·9	+3·0	−18	+10	+11·5	−17	−1·3	−5·5	−4·2
−22	+5·5	+15	−12·5	−5·6	−1·4	+2·7 +4·2	−14	+ 7·5	+11	−15·5	−1·2	−4·8	−4·3 −3·6
−21	+6·5	+16·5	−11	−4·6	− ·1	+3·2 +4·5	−18	+10	+10	−15·5	−1·3	−5·2	−3·5 −3·9
−23	+7·5	+17·5	−13·5	−4·8	−1·0	+4·7 +3·8	−19	+13	+14	−17·5	+ ·1	−4·8	−5·3 −4·9
−21	+6	+19	−13·5	−4·9	− ·4	+3·9 +4·5	−17·5	+ 9	+10·5	−20	−1·7	−7·7	−3·1 −6·0
−24·5	+7	+15	−14	−5·7	−2·3	+5·3 +3·4	−17	+ 9	+11	−14	−1·5	−3·9	−6·2 −2·4
−20·5	+8·5	+16·5	−13	−3·2	−1·1	+ ·9 +2·1	−17	+ 9	+ 8·5	−19	−1·5	−7·9	−2·4 −6·4
− 0·5	+5	+14·5	−13·5	−5·3	−2·2	+4·2 +3·1	−17	+ 6·5	+ 8	−18	−3·0	−7·5	−4·9 −4·5
−23	+6·5	+13	−12·5	−5·4	−2·2	+3·2 +3·2	−18	+ 7	+ 8·5	−18	−3·1	−7·3	−4·4 −4·2
−20·5	+4·5	+14·5	−11·5	−5·6	−1·0	+3·4 +4·6	−15	+ 4·5	+ 8	−19	−3·4	−8·1	−3·9 −4·7
−20·5	+6	+15	−10·5	−4·7	− ·2	+3·7 +4·5	−14·5	+ 5	+ 9	−16·5	−2·9	−6·2	−5·2 −3·3
−18·5	+5·5	+15·5	−11·5	−4·2	− ·6	+4·0 +3·6	−18	+ 8·5	+12	−17·5	−2·2	−5·6	−4·0 −3·4
−19	+6	+14	−12·5	−4·1	−1·8	+3·5 +2·3							
					Mean	+3·58						Mean	−4·29

Calculated reading for equilibrium position 24″·43410.

TABLE I.—Set No. 1. (3rd April, 1890)—continued.

	t_1.	t_2.	Mean temperature.
Spherometer reading for mercury surface corresponding to I. (1) and I. (2) 36 + 327°·3 . . . . . at	15·46	15·5	15·48
Spherometer reading for mercury surface corresponding to I. (3) and I. (4) 29 + 341°·2 . . . . . at	15·55	15·5	15·525

Number of experiment.	c_5.	t_1.	t_2.	θ.	N.	Number of experiment.	c_6.	t_1.	t_2.	θ.	N.
I (3) *a*.	15·5 +485	15·61	15·6	15·5 15·55	13	I (3) *β*.	15·5 +505	15·61	15·6	15·55 15·65	13

E.		W.		E.	W.	W.–E.	E.		W.		E.	W.	W.–E.
−17	+14	+17	−6·5	+1·5	+3·0	+1·5	−13	+19	+18·5	− 8	+6·1	+2·7	−3·4
−18	+14	+23·5	−9·5	+1·1	+3·8	+1·9 +2·7	−13	+20	+24·5	− 8·5	+6·7	+4·8	−4·0 −1·9
−16·5	+15	+25	−9·5	+2·3	+4·4	+1·5 +2·1	−12·5	+20	+18·5	− 7	+6·9	+3·3	−2·1 −3·6
−13	+14·5	+24·5	−6	+3·4	+6·3	+1·0 +2·9	−11	+18·5	+24·5	−10·5	+6·6	+3 6	−3·3 −3·0
−15·5	+19	+26	−7·5	+5·1	+6·0	+1·2 + ·9	−14	+20	+21·5	− 9	+6·3	+3·3	−2·7 −3·0
−15·5	+18	+24·5	−6·5	+4·5	+6·0	+1·5 +1·5	−12·5	+19	+23	−11	+6·3	+2·7	−3·0 −3·6
−13·5	+15	+24	−8	+3·5	+4·9	+2·5 +1·4	−13	+19	+22	−11	+6·1	+2·3	−3·4 −3·8
−15	+15	+23	−5	+2·9	+6·3	+2·0 +3·4	−13	+16	+20·5	−10·5	+4·3	+2·0	−2·0 −2·3
−15	+16·5	+23·5	−8	+3·8	+4·7	+2·5 + ·9	−14	+20	+21	− 9	+6·3	+3·1	−4·3 −3·2
−18	+15·5	+23·5	−7·5	+2·0	+5·0	+2·7 +3·0	−12·5	+18·5	+25·5	−11	+6·0	+3·7	−2·9 −2·3
−15	+14·5	+24	−6	+2·6	+6·1	+2.4 +3·5	−12	+19	+19·5	− 7	+6·5	+3·7	−2·8 −2·8
−11·5	+15	+24	−7·5	+4·3	+5·2	+1·8 + ·9	−10·5	+17	+21·5	− 7	+5·9	+4·5	−2·2 −1·4
					Mean	+1·99						Mean	−2·91

Calculated reading for equilibrium position 15″·99312.

TABLE I.—Set No. 1. (3rd April, 1890)—continued.

Number of experiment.	c_7.	t_1.	t_2.	θ.	N.	Number of experiment.	c_8.	t_1.	t_2.	θ.	N.
I (4) α.	24·5 +340	15·61	15·6	16·1 16·15	21	I (4) β.	24·5 +360	15·61	15·6	16·15 16·2	21

E.		W.		E.	W.	W.–E.	E.		W.		E.	W.	W.–E.
−10	− ·5	+ 8·5	− 5	−4·3	+ ·4	+4·7	− 9·5	+3	+ 6	− 9	−2	−3	−1·0
−14	+3	+11	− 5·5	−3·9	+1·2	+4·3 +5·1	−11	+6·5	+ 8	− 9	− ·6	−2·1	−2·4 −1·5
−15·5	+7·5	+14·5	− 8	−1·8	+1·1	+3·0 +2·9	− 9	+4	+10	−11	−1·2	−2·5	− ·9 −1·3
−14·5	+3	+10·5	− 8·5	−4·1	− ·8	+5·2 +3·3	−10·5	+3·5	+ 4·5	− 5·5	−2·1	−1·5	− ·4 + ·6
−16·5	+5	+11	−10	−3·7	−1·5	+2·9 +2·2	− 8	+4·5	+ 5	− 7·5	− ·5	−2·5	−1·0 −2·0
−16·5	+5·5	+14	− 8·5	−3·4	+ ·6	+1·9 +4·0	− 9·5	+8·5	+ 8·5	− 8·5	+1·2	−1·6	−3·7 −2·8
−15·5	+2	+ 8·5	− 6	−5·1	− ·2	+5·7 +4·9	− 8	+5	+ 8	− 8·5	− ·2	−1·8	−1·4 −1·6
−11	+ ·5	+ 9	− 3	−4·1	+1·8	+3·9 +5·9	−11	+6·5	+ 5·5	−10	− ·6	−3·8	−1·2 −3·2
−15	+2	+10	− 8	−4·9	− ·7	+6·7 +4·2	− 8·5	+8·5	+10	− 5	+1·6	+1·0	−5·4 − ·6
−15	+4	+13·5	−12·5	−3·7	−2·0	+3·0 +1·7	− 8	+5·5	+ 8·5	− 8·5	+ ·1	−1·6	+ ·9 −1·7
−17	+3	+ 8·5	− 5·5	−5·1	+ ·1	+3·1 +5·2	− 9	+8	+10·5	− 9·5	+1·1	−1·4	−2·7 −2·5
−15·5	+3·5	+ 9	− 8	−4·2	−1·1	+4·3 +3·1	− 8·5	+4	+ 7	− 8	−1·0	−2·0	− ·4 −1·0
−13	+4·5	+15	−11·5	−2·6	− ·8	+1·5 +1·8							
					Mean	+3·78						Mean	−1·62

Calculated reading for equilibrium position 24″·85400.

Reduction of the Observations.

§ 31. Let L = the distance between the electrodes in any equilibrium position.
n = the corresponding rate of rotation (number of revolutions per second) of the disc.
t = the temperature of the mercury in the trough.
θ = the temperature of the tuning-fork.
P = the vibration frequency of the fork at 15°·5 C.
P_θ = the vibration frequency of the fork at temperature θ.
ρ = the specific resistance of mercury at 15°·5 C.
ρ_t = the specific resistance of mercury at temperature t.
A = the sectional area of the mercury column at 15°·5 C.
A_t = the sectional area of the mercury column at temperature t.
k = the temperature coefficient of the tuning-fork vibration frequency [= − ·00011].
α = the temperature coefficient of the specific resistance.
γ = the cubical coefficient of expansion of mercury.
N = the number of teeth in the row on the cylindrical drum viewed through the fork slit.

[The difficulty of introducing a correction for the changes in the trough consequent on the small variations of temperature during the observations has been explained in § 15. In all that follows the dimensions of the trough are assumed to be constant.]

We have

$$Mn = \frac{L\rho_t}{A_t}$$

$$n = \frac{2P_\theta}{N} = \frac{2P(1 + k\overline{\theta - 15{\cdot}5})}{N}$$

$$\rho_t = \rho_0(1 + \alpha\overline{t - 15{\cdot}5})$$

$$A_t = A(1 + \gamma\overline{t - 15{\cdot}5})$$

$$\therefore 2MP\frac{1 + k\overline{\theta - 15{\cdot}5}}{N} = \frac{L\rho}{A}(1 + \overline{\alpha - \gamma}\,\overline{t - 15{\cdot}5})$$

or

$$2MP\nu = \frac{L\rho}{A}$$

where

$$\nu = \frac{1 + k\overline{\theta - 15{\cdot}5} - \overline{\alpha - \gamma}\,\overline{t - 15{\cdot}5}}{N}.$$

Let ν_1, ν_2 be the values of ν for two equilibrium positions of the movable electrode separated from one another by a distance l, the quantity of mercury in the trough being the same for both positions.

Then

$$2\mathrm{MP}\,(\nu_1 - \nu_2) = \frac{l\rho}{\mathrm{A}}$$

or

$$2\mathrm{MP}\mu = \frac{l\rho}{\mathrm{A}}, \text{ where } \mu = \nu_1 - \nu_2$$

or

$$2\mathrm{MPA} = s\rho, \text{ where } s = \frac{l}{\mu}.$$

In each complete set of observations four equilibrium positions are taken, two with the mercury in the trough at one level, two at another.

Let

s_1, A_1 be the values of s and A with the mercury at the first level,

and

s_2, A_2 their values when new mercury has been added.

Then

$$2\mathrm{MP}\,(\mathrm{A}_2 - \mathrm{A}_1) = (s_2 - s_1)\,\rho.$$

But

$$\mathrm{A}_2 - \mathrm{A}_1 = b\,(h_2 - h_1),$$

where

b = the mean breadth of the trough over the length used,

and

$h_2 - h_1$ = change of level of the mercury surface, assuming that before and after the addition of fresh mercury the temperature is 15°·5 C. ;

therefore we have finally

$$\rho = \frac{2\mathrm{MP}b\,(h_2 - h_1)}{s_2 - s_1}.$$

The results of calculation are given in Table VI. (p. 40).

In the column marked c the actual readings on the long screw and wheel H (Plate 2, fig. 5) are recorded. The first of the two columns marked l contains the distances passed over by the movable electrode, reckoned by taking the differences of these readings. The second of the columns marked l contains these differences corrected by reference to the standard bars (*vide* § 11).

In the column marked $h_2 - h_1$ are placed the actual differences of height of the two mercury surfaces obtained by simple interpolation from the spherometer calibration table in § 29 ; in the column next to it these differences corrected for temperature as follows :—

Let σ_1 = the height of the mercury surface above the base of the trough, when the first spherometer reading is taken at temperature t_1,

and σ_2 = the height of the mercury surface above the base of the trough, when the second spherometer reading is taken at temperature t_2.

Then $\sigma_2 - \sigma_1$ = the actual difference of height recorded in the column marked $h_2 - h_1$,

$$\text{and } (h_2 - h_1)_{15\cdot5} = \sigma_2 - \sigma_1 - \gamma\,(\sigma_2\,\overline{t_2 - 15\cdot5} - \sigma_1\,\overline{t_1 - 15\cdot5}).$$

Approximately

$$\sigma_1 = \cdot75 \text{ in.}$$
$$\sigma_2 = 1\cdot25 \text{ in.}$$

and these approximate values may be used in calculating the correcting term.

It will be observed that the greatest value of the correction is about 4 in 50,000.

In making the temperature corrections, the following values of the temperature coefficients have been taken :—

$$k = -\cdot00011\ ^*$$
$$\lambda_{15\cdot5} = \cdot0001798\ \dagger$$
$$\alpha_{15\cdot5} = \cdot0009076 \text{ (\textit{vide infra})}.$$

Let R_t = the resistance of a column of mercury in a glass tube at $t°$,

R_0 = the resistance of the same at 0°.

Then taking the formula

$$R_t = R_0(1 + 0\cdot0008649\ t + 0\cdot00000112\ t^2)\ddagger \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

we have $R_{15\cdot5} = R_0 \times 1\cdot013675$.

Let l_t be the length of the glass tube at $t°$, and s_t be the sectional area of the internal channel at the same temperature.

$$\text{Then } R_t = \frac{l_t\rho_t}{A_t} = \frac{l_0\rho_0}{A_0}(1 + \overline{\alpha - \beta}\,t) = R_0(1 + \overline{\alpha - \beta}\,t),$$

α being the temperature coefficient of the specific resistance,

and β the linear coefficient of expansion of the glass.

Hence $(\alpha - \beta) \times 15\cdot5 = \cdot013675$,

and $\alpha \times 15\cdot5 = \cdot013799$, taking $\beta = \cdot000008$.

We have then $\rho_{15\cdot5} = \rho_0 \times 1\cdot013799$.

The factor 1·013799 was used in calculating the specific resistance at 0° C. from that at 15°·5 C.

The value of $\alpha_{15\cdot5}$ above given is obtained from (ϵ), correction being similarly made for the fact that the formula applies to mercury in a glass tube.

* Macleod and Clarke, 'Phil. Trans.,' 1880.

† Regnault.

‡ Mascart, Nerville and Benoit, 'Journal de Physique,' 1884, p. 230.

TABLE VI.—The Results of Calculation.

Set No.	N.	t.	θ.	ν.	c.	l. (uncorrected).	l.	μ.	S.	$s_2 - s_1$.	$h_2 - h_1$.	$h_2 - h_1$, corrected to 15°·5 C.	Log. $\frac{h_2 - h_1}{s_2 - s_1}$	Sp. Res. in $\frac{(\text{inch})^2}{\text{sec.}}$ at 15°·5 C.	Sp. Res. in $\frac{(\text{cent.})^2}{\text{sec.}}$ at 15°·5 C.	Sp. Res. in $\frac{(\text{cent.})^2}{\text{sec.}}$ at 0° C.
1	8	15·47	15·10	·1250078	15·1708	9·2632	9·2622	·0480834	192·629	109·670	·49672	·49671	$\bar{3}$·6560148	14787·8	95,402	94,103
	13	15·48	15·47	·0769244	24·4341											
	13	15·60	15·56	·0769167	15·9931	8·8609	8·8588	·0293047	302·299							
	21	15·60	16·15	·0476120	24·8540											
2	8	15·52	15·70	·1249957	15·2583	9·2282	9·2274	·0480785	191·924	111·274	·50385	·50382	$\bar{3}$·6558779	14783·2	95,372	94,074
	13	15·51	16·14	·0769171	24·4866											
	13	15·51	15·70	·0769210	15·9213	8·8876	8·8854	·0293058	303·198							
	21	15·50	16·20	·0476152	24·8090											
3	8	15·49	15·69	·1249979	15·1531	9·2694	9·2683	·0480775	192·778	110·314	·49961	·49957	$\bar{3}$·6559683	14786·3	95,392	94,093
	13	15·51	15·74	·0769204	24·4225											
	13	15·57	14·90	·0769244	15·9294	8·8849	8·8826	·0293065	303·092							
	21	15·56	15·30	·0476178	24·8143											
4	8	15·64	15·87	·1249823	15·1278	9·2764	9·2754	·0480732	192·943	109·902	·49748	·49745	$\bar{3}$·6557453	14778·7	95,343	94,045
	13	15·64	16·19	·0769091	24·4041											
	13	15·36	15·54	·0769303	15·9434	8·8780	8·8757	·0293077	302·845							
	21	15·39	15·54	·0476226	24·8214											
5	8	15·19	15·85	·1250232	15·2458	9·2279	9·2270	·0480836	191·894	110·749	·50117	·50116	$\bar{3}$·6556345	14774·9	95,318	94,021
	13	15·17	15·70	·0769396	24·4737											
	13	15·56	16·44	·0769115	15·9553	8·8708	8·8678	·0293013	302·643							
	21	15·57	16·70	·0476102	24·8261											
													Mean	14782·2	95,365	94,067

§ 32. The final result arrived at is that the specific resistance of mercury at 15°·5 C. is equal to

$$14782{\cdot}2\,\frac{(\text{inch})^2}{(\text{sec.})} = 95365\,\frac{(\text{centimetre})^2}{(\text{sec.})},$$

and hence that the specific resistance of mercury at 0° C. is equal to

$$94067\,\frac{(\text{centimetre})^2}{(\text{sec.})}.$$

The probable error calculated from the five values given by the individual sets of observations is $\pm$ 10.

It follows that an ohm is equal to the resistance of a column of mercury of 1 square mm. sectional area, and 106·307 centimetres long.

§ 33. In conclusion it may be useful to consider the improvements that might be introduced in designing new apparatus for a further and more accurate determination of the specific resistance of mercury by this method.

Attention may, in the first place, be directed to the brushes in order to obtain steadiness of the galvanometer needle when the galvanometer circuit is complete, the battery circuit broken, and the disc in rotation. The perforated brush fed by a constant stream of mercury from a cistern of adjustable height goes far to secure this when the disc has been carefully ground true in place; and if, instead of one such brush, three or four were placed at different points of the circumference of the disc, as suggested by Rowland, the result would probably be sufficiently good.

In determining the equilibrium positions the variation in W$-$E (*vide* § 22) corresponding to a given displacement of the movable electrode is proportional to the current through the standard coil and the mercury trough; and the accuracy with which the equilibrium position can be determined may therefore be increased by increasing the current used. But there are two considerations which limit the magnitude of the current, viz., first and mainly, the unsteadiness of the galvanometer needle due to small variations in the rate of rotation of the disc, an unsteadiness which increases proportionately when the current is increased; and secondly, the heating of the wire of the standard coil. The heating of the wire of the coil may be diminished by increasing its sectional area; and there is no objection to this provided the coil is large enough to allow a sufficient number of turns in a single layer to produce an induction coefficient of sufficient magnitude. The main point needing attention, if a large current is to be advantageously used, is the rate of rotation of the disc. The more constant this can be made, the greater will be the current that can be employed, and therefore the more accurate will be the determination of the equilibrium position. In designing new apparatus no effort should be spared to compass uniformity in this respect. The variations in speed depend on variations in the lubrication of the bearings, and in the friction between the brushes of the electromotor and the com-

mutator, and in less degree perhaps in the friction between the brush or brushes at the disc and the disc circumference. A forced system of lubrication at all the bearings would probably effect a great improvement; the brushes should press as lightly as possible on the commutator of the electromotor, which ought to be newly and very carefully turned; and a heavy fly-wheel should also be used.

In regard to the standard coil, the silk covering of the wire introduces difficulty into the measurement of the mean radius, owing to the compressibility of the covering and its varying thickness. It would be preferable to have a coil of naked wire wound in a screw thread cut on a coil cylinder made of insulating material. But it does not seem easy to find an insulating material that will retain its shape when made into a cylinder of such large dimensions. Wood naturally suggests itself, but it would be difficult to prevent it from altering from time to time with varying atmospheric conditions. Possibly a coil cylinder of wood, built up of a great many pieces, and supported on both sides by heavy phosphor bronze plates, might give satisfactory results. If marble, after a prolonged immersion in molten paraffin wax, is a sufficiently good insulator, it might be tried. There is, however, always the alternative of using covered wire on a metal cylinder of such dimensions as to make the errors of measurement due to the silk covering of the wire insignificant to the degree of approximation required.

The trough, as has been already indicated (§ 6), may probably be more successfully made of plate glass or marble. There seems no reason why its dimensions should not be much more considerable than those of the trough used in the experiments described, in which case their accurate measurement would be facilitated. The sectional area of the mercury column may be as great as seems good, provided the length of the trough is sufficient to make the equipotential surfaces plane in the part of the trough which the movable electrode traverses. Further, this part of the trough is the only part that need be prepared with the utmost accuracy. It might, therefore, be well to make the trough in three parts, the middle portion being prepared with its base and sides carefully made plane, while the end portions need not receive the same attentive workmanship. With a large trough three or four feet might be traversed by the movable electrode, instead of some ten inches as in the present experiments.

In any final determination of the specific resistance to one part in ten thousand, an endeavour will doubtless be made to estimate it still more closely, and to obtain a decimal place further as a mean of a number of determinations. In any such attempt great care will have to be taken to maintain all parts of the apparatus (except the motor, fly-wheel, and connecting shafting) at definite and ascertainable temperatures, in a room specially designed for that purpose.

The author desires to acknowledge the great help he has received at all stages of these experiments from his assistant, Mr. S. T. Harrison; and to thank Mr. J. Gavey, of the South Wales Postal Telegraphic Department, for kindly placing at his disposal the telegraphic apparatus required.

II. *On the Determination of some Boiling and Freezing Points by means of the Platinum Thermometer.*

By E. H. Griffiths, *M.A., Sidney Sussex College, Cambridge.*

Communicated by R. T. Glazebrook, *F.R.S.*

Received May 27,—Read June 19, 1890.

[Plates 4, 5.]

1. Callendar, in his exhaustive comparison of the platinum and air thermometers ('Phil. Trans.,' A, 1887), suggested the introduction of the term "platinum temperature" (denoted by *pt*).

Pt is obtained by assuming the temperature to vary as the resistance. Thus

$$pt = \frac{R - R_0}{R_{100} - R_0} \times 100.$$

A curve is then constructed, having $t - pt$ (t being temperature as indicated by the air thermometer) as the ordinate and pt as abscissa. I have adopted this method of expressing my results throughout this paper.

If Callendar's conclusion is correct, viz., that whatever be the coefficient of increase in resistance from 0° to 100° of the platinum wire used, then for the same t all wires will give the same pt, the difficulty of determining high temperatures with accuracy has been overcome.

Unfortunately my experiments have driven me to the conclusion that the pt given by platinum thermometers made from different samples of wire differ considerably for the same value of t. However, in my thermometers the wire throughout is in contact with glass or asbestos, or both, and its behaviour is probably much influenced thereby. In Callendar's thermometers the coil was in contact with the glass only at its extremities.*

Later on I give particulars of the thermometers used in these experiments.

I was anxious to adopt a form of thermometer which would (1) quickly assume the surrounding temperature, (2) occupy little space, and (3) bear a reasonable amount of

* See note in Appendix (p. 70).

rough usage. I believe that the thermometers referred to hereafter as E, F, and G fulfil these requirements.

2. I proposed to graduate the thermometers by exposing them to the vapour of certain substances under known pressures, the substances selected being as follows :—

Table I.

	Temperature.	Authority.	References.
(1) *Melting ice*	0°		
(2) *Steam at* 760 *mm.*	100		
(3) Aniline at 760 mm.* . . .	184·41	Ramsay .	'Chem. Soc. Journ.,' vol. 47
(4) *Naphthalene at* 760 *mm.* . .	218·06	Crafts .	'Paris Soc. Chimique Bull.,' vol. 39
(5) Methyl salicylate at 760 mm.	222·88	Ramsay .	'Chem. Soc. Journ.,' vol. 47
(6) *Benzophenone at* 760 *mm.* .	306·08	Crafts .	'Paris Soc. Chimique Bull.,' vol. 39
(7) Mercury at 760 mm. . . .	357·25	Regnault	'Mémoires l'Académie des Sciences,' vol. 26
	358·2	Ramsay	'Chem. Soc. Journ.,' vol. 47
(8) *Sulphur at* 760 *mm.* . . .	448·38	Regnault	'Mémoires l'Académie des Sciences,' vol. 26

The change in temperature of the vapour of these substances due to changes in pressure appeared also to have been satisfactorily determined.

In the following pages I endeavour to show how the adoption of the numbers given for (1), (2), (4), (6), and (8) in the above table was forced upon me to the exclusion of the others.

The Apparatus Used.†

3. The resistance box (of the four dial pattern) was one constructed for me by Messrs. Elliott in 1886. I have had occasion to test this box severely during some investigations on which I have been engaged for the last two years, and I believe it to be a most reliable instrument. Messrs. Elliott were good enough to ascertain for me the coefficient of alteration per 1° C. for each set of dials. Although all the wire used in the box was from one casting, these coefficients differ slightly and are as follows :—

Table II.

Unit dial	= ·000413 per 1° C.	box right at 17°.
10 ohm ,,	= ·003733 ,,	
100 ,, ,,	= ·03773 ,,	
1000 ,, ,,	= ·38036 ,,	

* Most of these substances were suggested to me by Messrs. Heycock and Neville. I wish that I could adequately express my indebtedness to these gentlemen, who have not only assisted me in my observations, but have freely placed at my disposal the apparatus used by them in their investigations on the melting point of alloys.

† All the apparatus used in these experiments, with the exception of the galvanometer and resistance box, was constructed by Mr. F. Thomas, of Jesus Lane, Cambridge, and, like all other work that he has done for me, leaves nothing to be desired either in design or workmanship.

The accuracy of the coefficients will best be understood by reference to the Tables in the Appendix (pp. 65–69).

The chief difficulty has been that of ascertaining the real temperature of the coils, for the mere suspension of a thermometer within the box afforded but little information as to the temperature of the wires, unless the temperature of the box was steady.

The rapid changes of temperature in the air of the room in which the experiments were conducted made it necessary to place the resistance box, etc., in an adjoining room, which, unfortunately, had only an unprotected iron roof. Attempts were made to keep the temperature constant by means of a gas stove, but with little success.

A large copper bath with double walls was constructed, the bottom of the resistance box was cut out, and the top, with the exception of openings for the dials, was covered with a board, the under part of which was lined with swansdown. A gas jet beneath the bath was controlled by a mercury gas regulator of the usual form, but having a very large bulb about a half-inch broad and eight inches long, reaching from nearly the top to the bottom of the bath. This maintained the bath night and day at a temperature which was never observed to vary by ·1°.

However, the radiation from the top of the box could not be prevented, and during some of the frosty nights near the end of March its temperature fell as low 16°.

The thermometer in the box was coated with paraffin of about the same thickness as that on the coils. Later, Messrs. Elliott constructed for me a bobbin in all respects similar to those in the box, and the thermometer was placed with its bulb in the centre of this coil. It appeared probable, therefore, that its rate of change would be about equal to that of the bobbin.

The observations made after March 21st, from which date the regulator was constantly at work, showed a greater agreement than those taken before that date, and as the greatest divergence from the normal temperature (17°) was ·8°, the residual errors due to temperature could not in any case be very great.

4. The galvanometer was a high-resistance one, made for me by the Scientific Instrument Company. Its total resistance was between 8000 and 9000 ohms; however, its coils being connected in parallel arc, its resistance was about 2000 ohms. Although this was far too high a resistance, as compared with the other resistances of the bridge, I considered it advisable to use it, an important consideration being that any thermal effects were unlikely to affect it. The galvanometer was nearly dead beat, and was so arranged as to be extremely sensitive. The scale was about two yards from the galvanometer, and thus a very small deflection was easily noted.*

For some time commutators were used, all connections reversed, and the observations repeated to eliminate the effect of thermal currents. In no case, however, could any such effect be traced, so that in my later experiments I only reversed the connections occasionally.

A double contact key of the usual kind was used.

* A low-resistance galvanometer was tried and rejected.

It was found necessary not only to place the galvanometer on an insulating stand, but also the batteries and key. All connecting wires were run through india-rubber or glass tubes. Until this was done the galvanometer showed a permanent deflection after the wires were attached to the batteries. The whereabouts of this leakage I never discovered.

5. The apparatus was placed in rooms adjoining the Sidney Laboratory, and I was much troubled by the vibration caused by the traffic. In fact it was only possible to take some of the observations (such as those in which I wished to trace the process of surfusion during the freezing of metals) late at night. One great difficulty I met with at the outset, *i.e.*, the warming of the thermometer-coil by the current used when determining its resistance, and since it was necessary, during some of the observations on surfusion, to maintain the current for several seconds at a time, the point was one of considerable importance.

Callendar mentions that he used six Leclanché cells without any trouble of this kind. I found it impossible to use two unless I lowered the current by introducing extra battery resistance. The effect was clearly traceable if too strong a current was used. If the arms of the bridge were in equilibrium and the key held down, the spot would gradually slide away in the direction indicating a steady increase in the resistance of the coil. A few experiments showed that this rate of increase was nearly as the square of the current, thus indicating the cause. Again, the stouter the wire in the thermometer, the greater the E.M.F. that could be used without producing this effect. For instance, the thermometer A, in steam under 755 mm., gave a reading of 14·068 when used with one Leclanché cell, and of 14·080 when used with two Leclanché cells. The same difference was observable throughout all the readings of A, B, and C, at different temperatures. This effect was particularly marked in E* (whose coil was formed of exceedingly fine wire), particulars of which I have given later on.

I found it convenient in my later experiments to work with a large number of cells, introducing into the battery-circuit sufficient resistance to reduce the current to harmless dimensions.

On the determination of the "fixed points."

6. *Freezing-point of water.* The method described by Guillaume (see 'Traité Pratique de la Thermométrie de Précision') was adopted with this addition, that the vessel filled with finely powdered ice was surrounded by a larger one packed with large lumps. In some of the experiments the ice in the inner vessel was drained, in others, a mixture of ice and water was used. Snow was also tried (see Table XVIII.). In all cases the ice had been thoroughly washed with distilled water.

Boiling-points. A hypsometer of the usual form was used for water. The gauge connected with the instrument showed that the internal pressure rarely exceeded the external by more than 1 mm. of water.

The remaining boiling points were all determined in glass vessels. Round bottomed, long necked flasks, having a capacity of about $2\frac{1}{2}$ litres, were used for aniline, naphthalene, and sulphur; somewhat smaller ones for the remaining substances. At about $1\frac{1}{2}''$ from the top, a side tube was fitted into the neck, and through this tube the vapours passed into the condenser. The bulb of the thermometer just projected into the body of the flask (in no case touching the liquid), and the vapour on its way to the condenser had to pass the whole of the immersed portion of the thermometer. The top of the flask and the mouth of the condenser were closed with cotton wool or asbestos cloth, in which the vapours condensed, while the risk of increased internal pressure was diminished.

7. The difficulty with regard to super-heating was overcome as follows: An oven was built up with asbestos cards and in the roof a circular hole was cut in which the base of the flask was placed. The surface of the liquid in the flask was always maintained at a higher level than the rim of this hole, and it was, therefore, impossible for the flame to play on any portion of the glass above the liquid. To prevent radiation through the liquid a smoky flame was used until the exposed portion of the flask was covered with carbon.

Asbestos cloth was then wound round the flask, completely covering it from the roof of the oven to the exit tube, and on removing the cloth after distillation no signs of condensation were observable on the sides of the flask. From the time that these precautions were taken no indications of super-heating presented themselves, the temperature during distillation being independent of the size of the flame.

Every endeavour was made to secure pure specimens of the selected substances.

In most cases the observations were repeated from the commencement of distillation until about two-thirds of the sample had distilled over, the process was, therefore, practically, one of fractional distillation. Occasionally the distillate was separated after about half the sample had come over, and the residue and distillate experimented upon separately; and in one or two cases the process was repeated on the portions thus separated.

The only substances which showed slight signs of change when thus treated were methyl salicylate and triphenyl methane.

It is worthy of notice that the same specimens of naphthalene and benzophenone were boiled or distilled between twenty and thirty times, and yet their boiling-points showed no signs of change—a strong argument in favour of the purity of the samples, and of their adoption for the determination of fixed points.

As large a quantity as possible was used in each case.*

* Crafts, in the paper referred to in Table I., writes as follows:—" . . . La naphtaline du commerce ne varie plus de 0°·1 dans son point d'ébullition quand elle a été purifiée, par une méthode quelconque, jusqu'à ce que la substance fonde à 79°·9–79°·6. La même chose est vraie pour la benzophénone préparée par l'oxychlorure de carbone, quand son point de fusion est amené par une purification convenable à 48°–47°·7. On voit que des limites considérables peuvent être tolérées dans

The aniline, naphthalene, and sulphur (when liquid) measured nearly 1 litre; the mercury about 200 c.c. (2600 grms.); the quantities of benzophenone and methyl salicylate were rather smaller. In those cases in which different specimens were tried, the results were the same (see Table XIV.).

Comparison of readings taken under different pressures.

8. If R_t be the resistance at a temperature $t°$ C.,

then $\frac{R_{t_2} - R_{t_1}}{t_2 - t_1}$ will give the approximate increase per 1° C. at a temperature $\frac{1}{2}(t_1 + t_2)$.

The accuracy of the result depends of course on the nearness of t_1 and t_2.

Observing the pressure (during my first series of observations with each thermometer) at the time of the determination of the B.P. of water, naphthalene, benzophenone, and sulphur, I was able (from the data supplied by Crafts and Regnault) to determine the actual temperatures t_1, t_2, &c., corresponding to observed resistances R_1, R_2, &c. The resistance at 0° being also known, the approximate increase in resistance per 1° at temperatures $\frac{1}{2}(t_0 + t_1)$, $\frac{1}{2}(t_1 + t_2)$, &c., could be deduced.

After plotting the points thus obtained I was enabled to determine the approximate increase per 1° at intermediate points.

On examination of the change in temperature due to increased pressure as given by different observers (in the case of the selected substances) it was evident that however they might differ in regard to the actual temperature, they were in practical agreement as to the *change* over a range of about 30 mm. above and below 760 mm. For instance, both Regnault and Ramsay agree in giving ·0760° per 1 mm. as the change in mercury vapour through the above range, although they differ by nearly 1° in the actual temperature.*

les points de fusion sans que le point d'ébullition change, ou, en d'autres mots, des impuretés font varier plus vite le point de fusion que le point d'ébullition; et cette qualité, qui donne un moyen sûr de reconnaître la pureté, est indispensable pour l'usage en question."

On May 11th I determined the melting-points of the samples of naphthalene and benzophenone used during the experiments. The thermometer used was a small sensitive one of Geissler's.

The results were as follows:—

Benzophenone.

Melting-point of a portion which had not been boiled or distilled	(1)	47·8.
	(2)	47·9.
Melting-point of a portion which had been used throughout all the experiments		47·8.

Naphthalene.

Melting-point of a portion which had not been boiled or distilled	79·7.
Melting-point of a portion which had been used throughout all the experiments .	79·65.

* A considerable variation in this coefficient would have but a small effect on the fifth figure of the

Tables were thus constructed, giving in each case the change in resistance of the wire for a difference of 1 mm. pressure in the vapour of the substance.

I give the tables used for comparison of observations with E, F, and G; similar ones having been constructed for the other thermometers*:—

TABLE III.

Substance, vapour of.	Temperature of vapour at 760 mm.	Difference in temperature due to difference of 1 mm. in pressure (K).	Increase in resistance due to an increase of 1° in temperature.			± for each 1 mm. difference in pressure (K × 1).		
			I_E	I_F	I_G	E	F	G
Water	100°	·0375	·00913	·01481	·01558	·00034	·00056	·00058
Aniline	184·4	·053	·00884	·01432	·01508	·00047	·00076	·00080
†Naphthalene . .	218·06	·058	·00871	·01410	·01486	·00051	·00081	·00086
Methyl salicylate .	222·88	·057	·00870	·01406	·01482	·00050	·00080	·00084
†Benzophenone . .	306·08	·0636	·00834	·01343	·01420	·00053	·00085	·00090
Mercury . . .	357·25	·076	·00810	·01298	·01377	·00062	·00099	·00105
Sulphur . . .	448·34	·082	·00759	·01211	·01293	·00062	·00099	·00106

resistance. For example, if ·0780 be taken it would produce a change of ·0106 per 10 mm. in the resistance of G instead of ·0105, the number obtained by assuming it to be ·0760.

* Subsequent experiments have shown that the approximate corrections thus obtained are sufficiently accurate.

For example, the resistance of G at 200° C. is 8·0639,

and „ „ „ 0° C. is 4·9543,

$$\therefore \frac{8{\cdot}0639 - 4{\cdot}9543}{200} = \text{approximate increase per } 1° \text{ at } 100° = \mathbf{{\cdot}0155}.$$

Now (see Table XIX., experiments 6, 7, and 14 and 15) the true resistance

of G in steam under 746·8 mm. = 6·5321,

„ „ „ 772 „ = 6·5469,

∴ Difference for 25·2 mm. = ·0148.

Now the difference in temperature of steam at 746·8 and 772 is given by the accepted authorities as ·945°.

∴ Increase per ·945° = ·0148.

∴ „ „ 1° (at 100°) = **·0156**.

Here the result as obtained by experiment differs from the approximate result by only 1 in 156 over a range of 25·2 mm., which is the extreme variation in pressure met with during the experiments, and this difference would only cause a change of 1 in the fifth figure of R.

The value of $t_2 - t_1$ used in Table III. was in no case so large as 200 (greatest value 142·3), consequently the above is an extreme case.

The value given by Table III. for G at 100° = ·01558.

† CRAFTS gives the following formulæ for the boiling-points of these substances:—

$$\text{Naphthalene} \quad t = 218{\cdot}06 + \frac{H - 760}{17{\cdot}1},$$

$$\text{Benzophenone} \quad t = 306{\cdot}08 + \frac{H - 760}{15{\cdot}8}.$$

Freezing Points of Metals.

9. The apparatus used was similar to that described by Messrs. Heycock and Neville in their paper on the melting-point of alloys ('Chem. Soc. Trans.,' May, 1890). A large iron block, about 8 inches in height, had a circular hole about $1\frac{1}{8}$ inch in diameter, bored from the top to within 2 inches of the bottom.* The capacity for heat of this block was very great, and, therefore, its rate of alteration in temperature very slow. The block stood on a large Fletcher's burner and was surrounded by two circular burners, thus the heating effect was fairly uniform. A circular ring, having a slight spiral twist, and worked by a water-motor, moved rapidly up and down through the melted metal, stirring it thoroughly. Down the centre passed the thermometer, its lower end reaching nearly to the bottom of the hole.

The space above the melted metal was filled with paraffin, oxidation was thus prevented, and that portion of the rod of the stirrer which passed out of the metal during the ascent, kept at an even temperature.

It was found possible to use paraffin with all the metals employed, except zinc, although, in some cases, the "fuming" was excessive. Zinc was, in consequence, a difficult metal to work with.

I endeavoured to prevent oxidation and surface cooling by covering its surface with finely powdered charcoal. The result was not entirely satisfactory, and I found it impossible to prevent a certain amount of oxidation during the experiments with that metal. I do not know that this would affect the freezing-point (unless the oxide was soluble in the molten metal), but it prevented any surfusion. The temperature throughout the process of solidification was, however, very constant, and could be determined without difficulty.

From 400 to 600 grms. of metal were used in each case, and as far as time permitted, I tested samples from different makers. It appears to be fairly established that, with certain rare exceptions, the presence of any foreign metal will be shown by the lowering of the freezing-point; therefore, when different samples give different freezing-points, I have adopted the higher.† The phenomenon of surfusion was, in some cases, very marked. When taking observations, the key was held down and the resistance of the box lowered as the metal cooled, so as to maintain the spot in its zero position. When solidification commenced, the process had to be reversed, and the resistance rapidly increased; for example, in one experiment on Sn the resistance of E fell to 5·1330, and then rapidly rose to 5·1578. When making these observations I was in a different room from that in which the metal was cooling, and this rise was my only indication. It was, however, unmistakable, and, after the first few trials, easily measured. The stirrer was usually arrested by the solidification of the

* The surface of this hole and the stirrer were coated with Fe_3O_4 by raising the block to a high temperature and passing steam into the hole for several hours.

† See freezing-point of Pb, Table XV.

metal within 30 to 60 seconds after the rise. In those cases where surfusion was absent, the cessation of cooling was sufficiently marked to allow of accurate determination.

Thermometers A, B, C, and D.

10. The platinum wire used for A, B, and C had a diameter of ·005 inch. Glass tubing was drawn out until of just sufficient bore to admit this wire, and was then cut into lengths of about $1\frac{1}{4}$ inch. The wire was then threaded through these pieces, the ends of which were heated until they adhered to the wire. About two yards of the platinum wire were thus covered, and a stout silver wire was hard-soldered on to each end of it. One of these pieces of silver wire was passed through a fine glass tube about 18 inches long, placed within a second glass tube of about $\frac{1}{4}$ inch in diameter; and the second silver wire was passed up the annular space between these two tubes. The outer tube was widened at its lower end to $\frac{1}{2}$ inch external diameter, the glass-covered platinum wire was folded up at the junctions of the glass threads and thrust into this wider portion, which was then fused at the open end. The bulb thus formed was heated to a red heat, and allowed to cool slowly. The whole apparatus had much the size and appearance of a sensitive mercury thermometer, with a bulb about $1\frac{3}{8}$ inch long.

The upper end was fixed to a small board, and the ends of the projecting silver wires were soldered to two heavy brass connecting screws, insulated by ebonite.

The same sample of platinum wire was used in all three.

The silver wire in A was "standard silver," in B and C pure silver.

The total resistance of the silver wires was as follows:—

$$A = \cdot100\ ;\ B = \cdot087\ ;\ C = \cdot101 \text{ ohm at } 17^\circ \text{ C.}$$

From the connecting-screws well-insulated stout copper wires ran to the resistance box, the same wires being used in all experiments with these thermometers. The resistance of the wires and the connecting-screws was ·089 ohm. The external resistance (meaning thereby all resistance except that of the platinum wire in the bulb) was of no consequence provided that it remained constant, nor would its variation be of any consequence provided that it was regular, which would probably be the case if the same portion of the stem was always immersed.

I scarcely anticipated that my measurements would be sufficiently exact to render appreciable the alteration in resistance due to different immersions of the stem, the coefficient of increase of silver being less than that of platinum, and the external resistance only about $\frac{1}{150}$ of the coil. Herein I was mistaken, and I afterwards ascertained that the consequent error might amount, in some cases, to nearly ·2° C.*

* For example:—*Thermometer B; Vapour of H_2O; Barometer at* 770 *mm.*

Bulb only just within hypsometer . . . . . Resistance = 17·467

„ and 14 in. of stem within hypsometer . . . . „ = 17·477

This was an extreme case, and here the difference = ·01 ohm = ·2°, *nearly.*

However, in nearly all my experiments with these thermometers the same fraction of the stem was immersed, and the error due to this cause inappreciable. This difficulty could, of course, be met by using stouter silver wires.

A large number of determinations were performed with thermometers A, B, and C, and they have remained quite constant although roughly used. I have compared them with several large bulb mercury thermometers, and find that they assume the surrounding temperature more rapidly than some and less rapidly than others.

To illustrate the close correspondence of the experimental results, and also the accuracy of the corrections used for the variations in the temperature of the box and the height of the barometer, I extract the following page from my note-book. (The resistances in all cases include the exterior resistance.)

Table IV.—Resistance of A at Temperature 100° C.

Date.	Observed resistance.	Temperature of coils.	Resistance after correction for temperature of coils.	Barometer.	Correction for barometer.	Resistance at 100°.
				mm.		
Jan. 25th, 1890	18·585	16°·5	18·582	749·5	+ ·017	18·599
„ „ „	18·584	16·5	18·581	749·5	+ ·017	18·598
„ 27th „	18·622	14·5	18·605	762·4	— ·004	18·601
„ „ „	18·623	14·5	18·606	762·4	— ·004	18·602
„ 28th „	18·628	12·2	18·594	755·0	+ ·008	18·602
„ 29th „	18·649	12·4	18·617	771·0	— ·018	18·599

Although I afterwards discarded this form of thermometer, I have described it fully, because it is inexpensive, easily made, and may be used in all cases where an accuracy bounded by 1° C. is sufficient.

11. The results obtained from these thermometers are as follows (I consider it unnecessary to give the records of individual experiments) :—

Table V.

In all cases one Leclanché cell was used, and the arms of the bridge $\frac{1000}{10}$.	Resistances of A, B, C, after correction for temperature of coils and height of barometer.		
	A.	B.	C.
Melting ice	14·068	13·202	14·552
B.P. of H_2O	18·601	17·462	19·247
„ aniline	22·283	20·918	23·060
„ naphthalene	23·720	22·272	24·544
„ methyl salicylate	23·934	22·475	24·770
„ benzophenone	27·389	25·716	28·344
„ mercury	29·452	27·653	30·480
„ sulphur	32·915	30·914	34·067

Denoting the platinum temperatures by pt_a, pt_b, and pt_c, respectively, then

$$pt_a = \frac{R - 14{\cdot}068}{4{\cdot}533}; \qquad pt_b = \frac{R - 13{\cdot}202}{4{\cdot}260}; \qquad pt_c = \frac{R - 14{\cdot}552}{4{\cdot}695}.$$

Table V. gives us the following results :—

TABLE VI.*

	pt_a.	pt_b.	pt_c.	Assuming values of t given in Table I., then		
				$t - pt_a$	$t - pt_b$	$t - pt_c$
B.P. (at 760 mm.) aniline	181·19	181·12	181·18	3·22	3·29	3·23
,, ,, naphthalene . .	212·88	212·91	212·84	5·18	5·15	5·22
,, ,, methyl salicylate	217·60	217·67	217·64	5·28	5·21	5·24
,, ,, benzophenone .	293·81	293·76	293·78	12·27	12·32	12·30
,, ,, mercury . . .	339·31	339·26	339·27	17·94	17·99	17·98
,, ,, sulphur . . .	415·68	415·76	415·66	32·70	32·62	32·72

12. On plotting these numbers it was evident that they were not as concordant as might have been expected, considering the close agreement in the experimental results. No difficulty was experienced in drawing a smooth curve through the points determined from melting ice, steam, naphthalene, benzophenone, and sulphur, but the curves thus drawn passed *above* mercury (REGNAULT) and methyl salicylate (RAMSAY), but *below* aniline (RAMSAY).

Again, RAMSAY'S boiling-point for mercury (358·2) gave a value of $t - pt$, which fell considerably *above* the curves. Further, it was found impossible to draw a consistent curve through the points obtained from melting ice, steam, aniline, methyl salicylate, and mercury, even if the remaining points were disregarded.

The experiments were repeated with great care, and all the precautions mentioned in Sections 3 to 7 were adopted (with the exception of the thermo-regulator to the resistance box). The results, however, were practically the same.

An equation of the form $y = ax + bx^2 + cx^3 + dx^4$ was taken (where y, the ordinate, $= t - pt$, and $x = pt$), the values of y and x given by the boiling-points of water, naphthalene, benzophenone, and sulphur were substituted, and some intermediate points then deduced.

Curves A, B, and C (Plate 4) were then drawn on a large scale, the vertical spaces (on the original plan) representing ·1° and the horizontal ·5°.†

* All these thermometers were constructed from the same sample of platinum wire and glass, and the resistances of the coils are nearly equal. The results are sufficiently close to lead to the conclusion that the pt is the same for A, B and C at any given temperature, the greatest variation from the mean pt being ·06°.

† By means of curve A the mercury thermometers used by Messrs. HEYCOCK and NEVILLE were

The values of the boiling-points of the remaining substances, as obtained from curves A, B, and C, are as follows:—

TABLE VII.

	Aniline.			Methyl salicylate.			Mercury.		
	A.	B.	C.	A.	B.	C.	A.	B.	C.
pt.	181·19	181·12	181·18	217·60	217·67	217·64	339·31	339·26	339·27
$t - pt$ . . .	3·13	3·15	3·11	5·48	5·45	5·52	18·30	18·33	18·38
t	184·32	184·27	184·29	223·08	223·12	223·16	357·61	357·59	357·65
Mean *t* . .	184·29			223·12			357·62		

On comparing these curves with one plotted from the numbers given by CALLENDAR ('Phil. Trans.,' A, 1887), it was evident that they differed considerably: the rise is steeper, and $t - pt$ is invariably a larger quantity. This difference may be in some measure due to the behaviour of the glass with which the wire is in contact throughout. If, however, this is the case, it will not affect the temperatures deduced from the curve, provided that the resistance remains constant for each temperature. I was anxious to see if a thermometer constructed from a different sample of wire would give the same platinum temperatures, and, if these differed, to ascertain if the resulting values of *t* differed.

13. *Thermometer D.*—The platinum wire used was ·007 inch diameter. No silver or solder was used within the instrument. A thick rod of platinum ran through the inner tube, and down the annular space between the two glass tubes was passed a tube of platinum foil, terminating about $1\frac{1}{2}$ inch from the lower end. One end of the fine wire was welded to the rod, and the wire was then wound as a spiral round the lower $1\frac{1}{4}$ inch of the inner tube. This spiral was then entirely covered by fine threads of glass (laid lengthwise), and over them the wire was wound downwards to the end. This process was repeated until five layers had been formed, and the end of the coil was then welded to the platinum tube. The glass bulb enclosing the coils was about the same size as in A, B, and C. This coil contained nearly 9 feet of wire.

Very few experiments were performed with this thermometer as, owing to the large mass of glass in the bulb, it was very slow in acquiring the temperature of the surrounding vapour. I, however, obtained one reading at all the previous points—with the exception of methyl salicylate.

graduated. I believe that they have given an account of the method adopted in their paper ('Chem. Soc. Trans.,' May, 1890).

TABLE VIII.—Thermometer D.

	Resistance after correction for temperature of coils and height of barometer.	pt_d.	$t - pt_d$.
Melting ice	14·953	0	0
B.P. of H_2O	19·501	100	0
,, aniline	23·155	180·34	..
,, naphthalene . . .	24·588	211·85	6·21
,, benzophenone . .	28·220	291·71	14·37
,, mercury	30·248	336·30	..
,, sulphur	33·653	411·17	37·21

Plotting the same 5 points as before, and deducing from the curve (curve D) the remaining temperatures, we get

Aniline . . . . . $180·34 + 3·87 = 184°·21$
Mercury . . . . . $336·30 + 21·24 = 357°·54$.

Curve D (Plate 4) follows quite a different path from Curves A, B, and C (the platinum temperature of the vapour of S differing by as much as $4°·5$), yet the values of t, deduced from the curves, differ by less than $·1°$. It must be remembered that the above represent solitary determinations, and, therefore, the resulting values are of little importance in themselves. Nevertheless, they point to the conclusions (see Curves A, B, C, and D) :—

(1.) That the platinum temperatures obtained by using different samples of wire, differently arranged, are not equal.

(2.) That the "fixed points" (viz., those obtained by assuming previous determinations of melting ice, steam, naphthalene, benzophenone, and sulphur) give, in each case, a consistent curve.

(3.) That the temperatures of the vapour of aniline, methyl salicylate, and mercury, deduced from these curves, are practically identical, but differ slightly from previous determinations.

These conclusions were borne out by subsequent experiments.

14. A different form of thermometer was now adopted.

I obtained from Messrs. JOHNSON and MATTHEY a thick platinum tube, $\frac{1}{16}$th inch internal diameter, and $16\frac{1}{2}$ inches long, having a resistance of about ·04 ohm. Through this a hard glass tube was passed, projecting $1\frac{3}{4}$ inch at one end. Round this projecting portion was wound a slip of asbestos paper* (obtained from BELL

* The non-conductivity of this asbestos paper was exposed to severe tests. As compared with glass, it proved much the better insulator. A tube formed of it was placed within a hard glass tube, and two tightly-fitting wires thrust within until the ends were as close as possible without contact. This was placed in the circuit between four Leclanché cells and the galvanometer. It was then slowly heated to a bright red heat, but no deflection was observed. If, however, the wires were allowed to touch the glass, the deflection was considerable.

and Co.) until the thickness of the roll of paper was equal to that of the platinum tube.

On this roll of asbestos the wire was wound, one end being welded to a projecting slip on the lower end of the platinum tube, the other to a bundle of thick platinum wires passing down the centre glass tube.

The coil consisted of about thirty turns. The coil and the asbestos were then heated to a red heat and allowed to cool gradually. A thin hard glass tube, closed at the lower end, was then slipped over the whole. The top portion of the instrument was filled with paraffin, and the upper $8\frac{1}{2}$ inches surrounded by an outer tube covered with asbestos yarn to prevent surface condensation. Water was kept running through this outer tube during each experiment, the change in resistance of the connections caused by the changes of temperature in the tap water being less than could be measured. One advantage of this form of thermometer is, that, owing to the diameter being the same throughout, it can be thrust through perforated corks, etc. The asbestos roll being of an elastic nature, presses the coil against the exterior tube, and thus the instrument is extremely sensitive. If the hand be placed within an inch or two of the coil, when at the ordinary temperature of the room, the spot of light will be deflected off the scale.

In order still further to reduce the external resistance, silver wires, running from the upper end of the platinum tube to within 1 inch of its lower end, were hard-soldered to the platinum connections. Similar wires were added in the same manner to the platinum wires in the inner tube. Since the resistance of the platinum is so small, any current caused by differences of temperature within the thermometer will be short circuited, and thus will not visibly affect the galvanometer. As the silver wires terminated about $1\frac{1}{2}$ inch above the coil, the flow of heat along them was not likely to influence its temperature.

Thermometers thus constructed have proved satisfactory in every respect, and can be safely exposed to any temperature up to a dull red heat.

Thermometers E, E, F, and G.*

15. I obtained from Messrs. Johnson and Matthey some exceedingly fine platinum wire—diameter ·001 inch, and a resistance of about 48 ohms per foot. Great difficulty was experienced both in welding this wire to its connections and in winding the coil. A piece of this wire, about 20 inches in length, was wound into a coil $1\frac{3}{8}$ inch in length. The external diameter of the thermometer was only $\frac{3}{16}$ inch, and was the same throughout its length. Eight inches projected below the condenser tube. The total resistance from the coil to the connecting screws at the top of the instrument was about ·021 ohm. About two-thirds of the thermometer was kept at a constant temperature by the tap water. Thus the part exposed to change of temperature was about ·007 ohm, and as the greater portion of it was always immersed, any variation caused by difference of immersion could be safely disregarded.

The same remarks apply to E, F, and G, which were constructed in the same manner, except as regards the wire used for the coil.

Before using these thermometers the wires running to the resistance box in the next room were replaced by thick copper rods, carefully insulated, the resistance of which, and all connecting screws, was ·014 ohm.

These rods were only subjected to the same changes of temperature as the air of the room, and any difference thus produced was not measurable. E*, however, had one fatal defect. The capacity for heat of the coil was so small that the difficulty previously referred to (the raising of its temperature by the current used when determining the resistance) compelled the reduction of the E.M.F. used to such an extent, that the deflection of the galvanometer was too slight to be accurately measured.* I determined, however, before destroying the thermometer, to ascertain the resistance at the "fixed points," using seven cells with a battery resistance of 1000 ohms, and using $\frac{10000}{100}$ for the arms of the bridge.

The results were as follows:—

TABLE IX.

	Corrected resistance.	*pt.*	$t - pt.$
Melting ice	43·795	0	0
Steam (760 mm.)	56·733	100	0
Aniline (760 mm.)	67·353	182·08	
Naphthalene (760 mm.)	71·524	214·32	3·74
Benzophenone (760 mm.)	82·098	296·06	10·02
Sulphur (760 mm)†	98·010	419·04	29·34

* For example

Arms of bridge.	E.M.F.	Observed R.	
$\frac{10000}{100}$	2 Leclanché cells	56·927	(1)
$\frac{10000}{100}$	1 Leclanché cell and 100 ohms placed in battery connection	56·887	(2)
Again, $\frac{10000}{100}$	7 Leclanché cells and 1000 ohms in battery connection	56·875	(3)
$\frac{1000}{10}$	7 Leclanché cells and 1000 ohms in battery connection	56·925	(4)
Again, $\frac{10000}{100}$	2 Leclanché cells	43·862	(5)
$\frac{1000}{100}$	1 Leclanché cell and 100 ohms in battery connection	43 762	(6)

(1) and (2) the thermometer in steam under same pressure (781).
(3) and (4) „ „ „ „ (772·5).
(5) and (6) „ in melting ice.

The fall in the resistance of the coil due to a diminution in the current passing through it, is here very marked.

† I omitted to determine mercury B.P. by means of E*.

This curve (Curve E*, Plate 4,) differs considerably from the preceding curves; it, however, approaches more nearly to the curve plotted from CALLENDAR'S numbers. Ascertaining from it the temperature of aniline B.P. at 760 we get 182·08 + 2·16 = 184·24, thus bearing out the conclusions arrived at in section 13.

The thermometer was reconstructed and a wire of diameter ·007 inch (the same sample of wire as that used for D), and length about 20 inches, was substituted for the previous coil. This thermometer was labelled E.

Diameter of wire in F = ·005 inch (the same sample of wire as that used for A, B, and C).

F was destroyed during some experiments with melted Zn and was replaced by G, whose coil was constructed of a rather greater length of the same wire. Both F and G had an external diameter of a quarter of an inch. As regards the external resistance, length, etc., they practically coincide with E*.

16. A series of observations led me to the conclusion that the current produced by the following arrangement was the greatest that I could use if I wished to prevent any observable rise in temperature due to the current.

F. 7 Leclanché cells and 500 ohms introduced into battery, arms of bridge $\frac{10000}{10}$.
E. ,, ,, 200 ,, ,, ,, ,, ,,
G. ,, ,, 500 ,, ,, ,, ,, ,,

These conditions were maintained throughout all the remaining experiments.

TABLE X.—Resistances of E, F, and G after correction for temperature of coils and height of barometer.*

	E.	F.	G.
Melting ice	3·0569	4·7662	4·9543
Steam	3·9848	6·2715	6·5399
B.P. of aniline	4·7384	7·4922	7·8288
,, naphthalene	5·0355	7·9709	8·3351
,, benzophenone	5·7896	9·1833	9·6175
,, mercury	6·2149	9·8668	10·3410
,, sulphur	6·9276	11·002	11·552

From Table X. we get the following results:—

* Tables of the individual experiments from which these numbers are obtained are given in the Appendix (pp. 65–69), except those of aniline and mercury which are given in Table XIV.

TABLE XI.

	pt. E.	*pt.* F.	*pt.* G.	$t - pt.$ E.	$t - pt.$ F.	$t - pt.$ G.
B.P. of aniline	181·24	181·08	181·28	..	..	..
„ naphthalene . . .	213·27	212·89	213·22	4·79	5·17	4·84
„ benzophenone . .	294·57	293·44	294·10	11·51	12·64	11·98
„ mercury	340·40	338·83	339·72	..	..	..
„ sulphur	417·22	414·25	416·10	31·16	34·13	32·28

These numbers (with the exception of those deduced by assuming the standard numbers for aniline and mercury) give, as in previous cases, a smooth curve, although $t - pt$ differs considerably in the last three columns.

Assuming, as before, $y = ax + bx^2 + cx^3 + dx^4$, and substituting the values of y and x given by the above table, we get

TABLE XII.

	E.	F.	G.
a	− ·026724	− ·025479	− ·0229155
b	·00032537	·00029505	·000260768
c	− ·00000070247	− ·00000050	− ·0000004013
d	·00000000121	·0000000010	·00000000085

From Table XII. we get the following values of y:—

Table XIII.—Values of $y = (t - pt)$.

	E.	F.	G.
When $x = 50$	− ·60	− ·59	− ·54
$x = 150$	1·56	1·63	1·49
$x = 250$	7·40	8·16	7·61
$x = 350$	18·60	20·74	19·47
$x = 450$	39·47	43·74	41·22

Ten points were thus obtained on each curve, five by direct experiment (Table X.) and five from the equation.

The curves (Curves E, F, and G, Plates 4 and 5) run without the slightest difficulty

through these points. They are plotted on a scale in which a vertical distance of $\frac{1}{2}$ inch corresponds to a difference of 1° in y, and its value can be read with certainty to ·02°.

The marked difference in the coefficients of the equations may be due (1) to differences in the quality and thickness of the platinum wire used; (2) to differences in the total external resistance; (3) to differences in the asbestos and glass, different samples being used in each.

That, under the same circumstances, the resistance of each thermometer remains constant is, I believe, clearly shown by the Tables given in the Appendix.

The temperatures of the boiling-points of aniline and mercury (at 760 mm.), deduced from curves E, F, and G (using the numbers given in Tables X. and XI.), are as follows:—

		E.	F.	G.	Mean t.
Aniline.	pt =	181·24	181·08	181·28	
	y =	2·96	3·15	2·95	
	t =	184·20	184·23	184·23	184·22*
Mercury.	pt =	340·40	338·83	339·72	
	y =	17·24	18·85	17·90	
	t =	357·64	357·68	357·62	357·65*

The greatest variation from the mean value being 0°·03.

My attention was, henceforward, confined to determining various temperatures by means of these curves.

After having used thermometer F for the determination of the boiling-point of methyl salicylate and the melting-points of Sn and Pb, it met with an accident, and thereafter I was dependent upon E and G.

At regular intervals I tested these thermometers by means of a hypsometer, but was not able to detect any alteration in them.

I give particulars of the remaining experiments in full, without omission, as I wish to show how comparatively insignificant the variations are in each case.

I hope that the replacement of glass by porcelain tubes will hereafter enable me to determine the freezing-points of metals melting at higher temperatures.

* Mean of previous results (curves A, B, C, D, and E*) :—

Aniline	184·25
Mercury	357·60

TABLE XIV.—DETERMINATION of the B.P. of Aniline, Methyl Salicylate, Triphenyl Methane, and Mercury.

Nature of experiment.	Date.	Therm.	R.	Temp. of coils.	R. after correction for temp. of coils (right at 17°) (see Table II.).	Ht. of bar.	Correction for bar. to 760 (see Table III.).	R. at 760.	Mean resulting *pt*.	Diff. taken from curve = *t* − *pt*.	*t*.	Mean *t*.	Previous determination.	Remarks.
	1890.			°		mm.						°		
B.P. of aniline	March 1	F	7·4959	17	7·4959	765	−·0038	7·4921					RAMSAY 184·41	Two different samples used—(A) supplied by HARRINGTON BROS., (B) supplied by BURGOYNE, BURBIDGES, and Co.
	,, 1	,,	7·4954	17·15	7·4958	765	−·0038	7·4920					THORPE 183·7	
	,, 2	,,	7·5066	14·6	7·5498	769·5	−·0073	7·4925	181·08	3·15	184·23			
	,, 22	,,	7·4893	16·9	7·4890	756·2	+·0029	7·4919						
A	,, 22	,,	7·4896	16·9	7·4893	756·2	+·0029	7·4922						
	,, 22	,,	7·4895	16·85	7·4892	756·3	+·0028	7·4920						
	,, 22	E	4·7368	16·85	4·7366	756·8	+·0015	4·7381						Curves (A, B, and C) gave 184·29
	,, 22	,,	4·7369	16·85	4·7367	756·8	+·0015	4·7382						Curve (D) gave 184·21
	,, 22	,,	4·7366	16·85	4·7364	756·8	+·0015	4·7379				184·22		Curve (E*) gave 184·24
	,, 26	,,	4·7369	17·1	4·7371	757·6	+·0011	4·7382	181·24	2·96	184·20			
	,, 26	,,	4·7371	17·1	4·7373	757·6	+·0011	4·7383						
	,, 26	,,	4·7380	16·65	4·7374	757·2	+·0013	4·7387						
B	,, 26	,,	4·7378	16·7	4·7373	757·2	+·0013	4·7386						
	,, 26	G	7·8268	16·9	7·8266	757·5	+·0020	7·8286	181·28	2·95	184·23			
	,, 26	,,	7·8269	17	7·8269	757·5	+·0020	7·8289						
B.P. of methyl salicylate	March 22	F	8·0403	17·05	8·0404	756·3	+·0029	8·0433	217·70	+ 5·51	223·21	223·19	RAMSAY 222·88	Not satisfied as to purity of sample; some time before temp. was steady.
	*March 22	E	5·0778	17	5·0778	756	+·0020	5·0798					CAHOURS 222	Curves (A, B, and C) gave 223·12
	,, 22	,,	5·0782	17	5·0782	756	+·0020	5·0802	218·08	+ 5·10	223·18			
	,, 22	,,	5·0784	17	5·0784	756	+·0020	5·0804						
B.P. of triphenyl methane	April 1	E	6·2138	16·6	6·2129	770·8	correction not known	..	340·18	17·20	357·38	357·35 (bar. 770·8)	KEKULÉ 355 (760)	Quantity (2 ozs.) too small for satisfactory determination
	,, 1	G	10·339	16·6	10·337	770·8	,,	..	339·47	17·86	357·33		CRAFTS 358 (754)	No data for correction of pressure
													SCHWARZ 330 (760)	
B.P. of mercury	March 7	F	9·8623	16·6	9·8608	753·7	+·0062	9·8670	338·83	18·85	357·68		REGNAULT 357·25	About 2600 grms. of mercury in flask
	,, 22	,,	9·8648	16·6	9·8633	756·5	+·0035	9·8668					RAMSAY 358·2	Curves (A, B, C) gave 357·62
	,, 22	E	6·2135	16·8	6·2130	757	+·0019	6·2149						Curve (D) gave 357·54
	,, 22	,,	6·2133	16·8	6·2128	757	+·0019	6·2147	340·40	17·24	357·64	357·65		
	,, 26	,,	6·2134	17·3	6·2141	758·8	+·0007	6·2148						
	,, 26	,,	6·2136	17·3	6·2143	758·8	+·0007	6·2150						
	,, 26	G	10·339	17·2	10·340	759	+·0010	10·341	339·72	17·90	357·62			
	,, 26	,,	10·339	17·2	10·340	759	+·0010	10·341						

* In every case where the date is repeated, the experiment was a distinct one, the substance being boiled again after cooling.

TABLE XV.—Freezing-points of Metals.

Nature of experiment.	Date.	Therm.	Observed R.	Temp. of coils.	R. after correction for temp. of coils.	Resulting *pt.*	Diff. taken from curve = ($t-pt.$)	*t.*	Mean *t.*	Previous Observers.	Remarks.
Freezing-point of Sn	Mar. 9*	F.	8·1660	16·9	8·1657					REIMSDYK 228·5	Weight of Sn, Mar. 9, 500 grms. Weight of Sn, Mar. 28, 600 grms. Surfusion was in many of the experiments *very* marked. Spec. of Sn supplied by Messrs. JOHNSON and MATTHEY.
	,, 9	,,	8·1662	16·9	8·1659	225·87	6·13	232·00		KUPFFER 230·	
	,, 9	,,	8·1659	16·9	8·1656					PERSON 232·7	
	Mar. 28	G.	8·5430	16·88	8·5427					CRICHTON 238·	
	,, 28	,,	8·5426	16·9	8·5423						
	,, 28	,,	8·5434	16·9	8·5431	226·30	5·72	232·02	232·03		
	,, 28	,,	8·5425	17·	8·5425						
	,, 28	E.	5·1578	17·05	5·1579						
	,, 28	,,	5·1576	17·1	5·1578	226·45	5·63	232·08			
Freezing-point of Bi A	Mar. 28	E.	5·4825	16·75	5·4820					PERSON 273·5	Weight of Bi used, Spec. A. 600 grms., Spec. B. 550 grms. Both these specimens were supplied by Messrs. JOHNSON and MATTHEY, but with an interval of nearly twelve months. Surfusion very slight, but temp. very steady from commencement of solidification.
	,, 28	,,	5·4822	16·72	5·4817	261·38	8·30	269·68		REIMSDYK 268·3	
	,, 28	,,	5·4823	16·7	5·4817						
B	April 2	,,	5·4820	16·8	5·4816						
	,, 2	,,	5·4826	16·8	5·4822	261·39	8·30	269·69	Spec. A.		
	,, 2	,,	5·4825	16·8	5·4821				and B.		
A	Mar. 28	G.	9·0960	16·8	9·0953				269·68		
	,, 28	,,	9·0962	16·8	9·0955	261·16	8·53	269·69			
	,, 28	,,	9·0958	16·8	9·0951						
B	April 2	,,	9·0955	16·8	9·0948						
	,, 2	,,	9·0960	16·8	9·0953	261·15	8·53	269·68			
	,, 2	,,	9·0959	16·8	9·0952						

* In every case where the date is repeated, the experiment was a distinct one, the substance being allowed to solidify, then re-melted and again cooled.

TABLE XV.—Freezing-points of Metals—(continued.)

Nature of experiment.	Date.	Therm.	Observed R.	Temp. of coils.	R. after correction for temp. of coils.	Resulting *pt.*	Diff. taken from curve = (*t*—*pt.*)	*t.*	Mean *t.*	Previous Observers.	Remarks.
Freezing-point of Cd A	Mar. 27	E.	5·9175	17·3	5·9182					PERSON 320·7	Spec. A. supplied by HARRINGTON BROS., weight used 500 grms. Spec. B. supplied by Messrs. JOHNSON and MATTHEY, weight used 459 grms. Surfusion very marked in the experiments on sample B.
	,, 27	,,	5·9176	17·3	5·9183	308·43	13·08	321·51		REIMSDYK 320	
	,, 27	,,	5·9174	17·35	5·9182				Spec. A. 321·50	VAN DER WYDE 325	
	,, 27	G.	9·8346	17·4	9·8361						
	,, 27	,,	9·8350	17·45	9·8366	307·92	13·57	321·49			
B	April 1	E.	5·9208	16·1	5·9199						
	,, 1	,,	5·9206	16·65	5·9195	308·58	13·12	321·70	Spec. B. 321·67		
	,, 1	G.	9·8400	16·65	9·8387						
	,, 1	,,	9·8405	16·7	9·8394	308·04	13·60	321·64			
	,, 1	,,	9·8400	16·75	9·8391						
Freezing-point of Pb A	Mar. 15	F.	9·4820	16·35	9·4797					PERSON 326·2	Spec. A. supplied by HARRINGTON BROS., granulated, 600 grms. used. Spec. B. supplied by Messrs. JOHNSON and MATTHEY, foil, 600 grms. used.
	,, 15	,,	9·4820	16·45	9·4800	313·14	15·13	328·27	Spec. A. 328·27	KUPFFER 334	
	,, 15	,,	9·4814	16·6	9·4800					QUINCKE 330	
B	Mar. 29	G.	9·9392	17·02	9·9393						
	,, 29	,,	9·9395	17·02	9·9396	314·40	14·39	328·79	Spec. B. 328·78		
	,, 29	,,	9·9392	17·05	9·9395						
	,, 29	E.	5·9789	16·65	5·9782						
	,, 29	,,	5·9788	16·75	5·9784	314·91	13·86	328·77			
	,, 29	,,	5·9787	16·9	5·9786						
Freezing-point of Zn	Mar. 31	G.	11·203	16·4	11·200					REIMSDYK 420	Spec. of Zn supplied by HARRINGTON BROS., 450 grms. used. No surfusion, but temp. during solidification very steady.
	,, 31	,,	11·200	16·6	11·198	393·90	27·31	421·21		WRIGHT and TUFF 420	
	,, 31	,,	11·202	16·6	11·200				421·23	PERSON 433·3. Other observers range from 342 (DANIELL) to 450 (BOUSSINGAULT)	
	April 1	E.	6·7204	16·8	6·7199						
	,, 1	,,	6·7202	16·8	6·7197	394·84	26·41	421·25			
	,, 1	,,	6·7205	16·8	6·7200						

Summary of Results.

17. (1.) Different thermometers, whose coils are formed of different specimens of platinum wire, encased in glass or wound on asbestos, do *not* give the same platinum temperature when at the same actual temperature.

(2.) The readings of each thermometer are constant, whenever its temperature is the same.

(3.) The platinum temperatures obtained by assuming CRAFTS' numbers for the B.P. of naphthalene and benzophenone, and REGNAULT'S for that of sulphur, are consistent with each other, and with 0° and 100°.*

(4.) Using the substances mentioned in (3) for the determination of the "fixed points," of platinum thermometers, the thermometers thus graduated agree in assigning the values given in Table XVI. as the air thermometer temperatures of the boiling and freezing points of the different substances. The difference between the numbers obtained by using different thermometers on the same specimen in no case affects the fourth figure of the results.

TABLE XVI.

		Temp. by air therm.
Boiling-points—		
(Bar. 760)	Aniline	**184·22**
„	Methyl salicylate	**223·19**
(Bar. 770·8)	Triphenyl methane	**257·35**
(Bar. 760)	Mercury	**357·65**
Freezing-points—		
	Tin	**232·03**
	Bismuth	**269·68**
	Cadmium	**321·67**
	Lead	**328·78**
	Zinc	**421·23**

APPENDIX.

I give, in full, tables showing the results of the individual experiments by which the "fixed points" on E, F, and G were determined.

* It must be remembered that these determinations were obtained by use of the air thermometer, whose readings agree with the mercury thermometer at 0° and 100°. It follows that all the temperatures ascertained by my thermometers are expressed in terms of the air thermometer. If subsequent investigations establish new values for the air-temperature at these points, the figures given in the Tables in Appendix would supply sufficient data for the correction of the temperatures deduced from them.

TABLE XVII.—Determination of "Fixed Points" on E. (Seven Leclanché Cells. 200 ohms in Battery Connection. Bridge $\frac{10000}{10}$.)

Nature of experiment.	No.	Date.	Observed R.	Temp. of coils.	R. after correction for temp. of coils (by Table III.).	Ht. of bar.	Correction for bar. 760 (by Table III.).	R. bar. at 760.	pt.	t.	$t-pt$.	Remarks.
				°C		mm.						
Melting ice . . .	* 1	March 22	3·0568	17	3·0568							In all cases the inner vessel filled with finely powdered ice and distilled water
	* 2	„ 22	3·0569	17	3·0569							
	* 3	„ 26	3·0570	17·1	3·0571							
	* 4	„ 26	3·0569	17·2	3·0571							
	* 5	April 2	3·0570	16·9	3·0569							
	* 6	„ 2	3·0571	16·9	3·0570							
				Mean	3·0570							
B.P. of water . . .	7	March 20	3·9826	15·65	3·9806	747	+·0044	3·9850				Thus $pt_e = \frac{R - 3·0570}{·9277}$
	8	„ 21	3·9829	16·1	3·9815	752·5	+·0027	3·9842				
	* 9	„ 23	3·9811	17·5	3·9819	751·5	+·0029	3·9848				18. E had before this experiment been immersed in melted Sn, Bi, Cd, and Pb
	*10	„ 23	3·9812	17·5	3·9820	751·5	+·0029	3·9849				
	*11	„ 23	3·9810	17·5	3·9818	751·5	+·0029	3·9847				
	*12	„ 24	3·9822	16·2	3·9810	749·6	+·0036	3·9846				
	*13	„ 25	3·9802	16·8	3·9799	746·5	+·0046	3·9845				
	*14	„ 25	3·9801	16·9	3·9800	746·5	+·0046	3·9846				
	*15	„ 25	3·9798	17·1	3·9800	746·5	+·0046	3·9846				19. This experiment was performed within half-hour after the immersion of E in melted Zn
	*16	„ 26	3·9843	17·25	3·9846	759·5	+·0002	3·9848				
	*17	„ 26	3·9844	17·25	3·9847	759·5	+·0002	3·9849				
	*18	„ 31	3·9894	16·45	3·9885	770·5	−·0036	3·9849				
	*19	April 1	3·9899	16·4	3·9889	772·2	−·0041	3·9848				
							Mean	3·9847				

TABLE XVII.—(continued.)

Nature of experiment.	No.	Date.	Observed R.	Temp. of coils.	R. after correction for temp. of coils (by Table III.).	Ht. of bar.	Correction for bar. 760 (by Table III.).	R. bar. at 760.	*pt.*	*t.*	*t*—*pt.*	Remarks.
				°		mm.						
B.P. of naphthalene .	20	March 21	5·0319	16·95	5·0318	753·5	+·0039	5·0357	213·29			
	21	„ 21	5·0317	17	5·0317	753·5	+·0039	5·0356	213·28			
	*22	„ 25	5·0306	17·1	5·0308	751	+·0046	5·0354	213·26	218·06	4·79	
	*23	„ 25	5·0307	17·1	5·0309	751	+·0046	5·0355	213·27	(CRAFTS)		
	*24	„ 25	5·0307	17·25	5·0312	752	+·0041	5·0353	213·25			
	*25	April 2	5·0382	17·1	5·0384	765·5	—·0028	5·0356	213·28			
	*26	„ 2	5·0380	17·1	5·0382	765·5	—·0028	5·0354	213·26			
								Mean	213·27			
B.P. of benzophenone	27	March 21	5·7875	16·6	5·7866	754·7	+·0028	5·7894	295·55			
	28	„ 21	5·7878	16·6	5·7869	754·7	+·0028	5·7897	295·58	306·08	11·51	
	29	„ 21	5·7880	16·6	5·7871	754·7	+·0028	5·7899	295·60	(CRAFTS)		
	*30	„ 25	5·7833	17·2	5·7837	749	+·0058	5·7895	295·56			
	*31	April 2	5·7922	17·1	5·7924	765	—·0027	5·7897	295·58			
								Mean	295·57			
B.P. of sulphur . .	32	March 21	6·9239	17	6·9239	754·5	+·0034	6·9273	417·17			
	33	„ 21	6·9237	17	6·9237	754·5	+·0034	6·9271	417·15	448·38	31·16	
	*34	„ 24	6·9200	17·25	6·9208	748	+·0074	6·9282	417·26	(REG-		
	*35	„ 24	6·9198	17·25	6·9206	748	+·0074	6·9280	417·24	NAULT)		
	*36	„ 25	6·9198	17·2	6·9203	748	+·0074	6·9277	417·21			
	*37	April 2	6·9300	17·2	6·9305	764·5	—·0028	6·9277	417·21			
								Mean	417·22			

* Numbers thus marked were obtained after the gas regulator and bath had been used for coils; the resulting numbers are therefore more trustworthy.

TABLE XVIII.—Determination of "Fixed Points" on F. (Seven Leclanché Cells. 500 ohms in Battery Connection. Bridge $\frac{10000}{10}$.)

Nature of experiment.	No.	Date.	Observed R.	Temp. of coils.	R. after correction for temp. of coils (by Table III.).	Ht. of bar.	Correction for bar. to 760 (by Table III.).	R. at 760.	Resulting *pt.*	*t.*	*t*−*pt.*	Remarks.
				°		mm.						
Melting ice	1	Feb. 26	4·7688	15·8	4·7666							No. 1. Ice in inner vessel drained
	2	,, 27	4·7745	12·6	4·7664							No. 2. Ice in inner vessel with water
	3	,, 28	4·7658	16·6	4·7652							No. 3. } Snow instead of
	4	March 1	4·7683	15·8	4·7661							No. 4. } ice
	*5	,, 22	4·7662	17	4·7662	..	..	..	..	0	0	No. 5. Same as 2 but ice very finely powdered
				Mean	4·7662							
B.P. of water . . .	6	Feb. 26	6·2839	15·4	6·2801	774·3	−·0081	6·2720				Thus $pt_f = \frac{R - 4·7662}{1·5053}$
	7	March 1	6·2760	15·5	6·2724	760·7	−·0004	6·2720				
	8	,, 20	6·2680	15·5	6·2644	747	+·0073	6·2717				
	9	,, 20	6·2679	15·6	6·2645	747	+·0073	6·2718				
	*10	,, 21	6·2694	16·15	6·2674	752·7	+·0041	6·2715				
	*11	,, 21	6·2693	16·2	6·2675	752·8	+·0040	6·2715				
							R. adopted	6·2715				
B.P. of naphthalene .	12	March 1	7·9758	16·6	7·9746	764·3	−·0035	7·9711	212·90			
	*13	,, 21	7·9660	16·8	7·9654	753	+·0056	7·9710	212·89	218.06	5·17	
	*14	,, 21	7·9658	16·8	7·9652	753	+·0056	7·9708	212·88	(CRAFTS)		
								Mean *pt.*	212·89			
B.P. of benzophenone	15	Feb. 27	9·2007	14·8	9·1931	771·5	−·0097	9·1834	293·45			
	16	,, 28	9·1955	16·5	9·1936	771·6	−·0099	9·1837	293·47	306·08	12·64	
	17	,, 28	9·1950	16·6	9·1935	771·6	−·0099	9·1836	293·46	(CRAFTS)		
	*18	March 21	9·1798	16·7	9·1788	754·7	+·0045	9·1833	293·44			
	*19	,, 21	9·1798	16·7	9·1788	754·7	+·0045	9·1833	293·44			
								pt. adopted	293·44			
B.P. of sulphur . .	20	March 6	11·004	15·5	10·998	757·2	+·003	11·001	414·19			
	21	,, 7	11·005	16	11·001	754·5	+·005	11·006	414·49	448·38	34·13	
	*22	,, 21	10·996	16·95	10·996	754	+·006	11·003	414·25	(REG-		
	*23	,, 21	10·996	16·95	10·996	754	+·006	11·002	414·25	NAULT)		
								pt. adopted	414·25			

The numbers marked * were taken after the gas regulator and bath had been used for coils; the resulting numbers are, therefore, more trustworthy.

TABLE XIX.—Determination of "Fixed Points" on G. (Seven Leclanché Cells. 500 ohms in Battery Connection. Bridge $\frac{10000}{10}$.)

(All observations with G were taken after the gas regulator, &c., had been used for coils.)

Nature of experiment.	No.	Date.	Observed R.	Temp. of coils.	R. after correction for temp. of coils (by Table III.).	Ht. of bar.	Correction for bar. 760 (by Table III.).	R. at 760.	*pt.*	*t.*	*t*—*pt.*	Remarks.
				°		mm.						
Melting ice . . .	1	March 26	4·9538	17·2	4·9542							In all cases inner vessel filled with finely-powdered ice and distilled water
	2	,, 26	4·9539	17·2	4·9543							G was left standing in the ice for two hours between observations 2 and 3
	3	,, 26	4·9536	17·35	4·9543							
	4	April 2	4·9542	17	4·9542							
	5	,, 2	4·9544	17	4·9544							
				Mean	4·9543							
B.P. of water . . .	6	March 25	6·5318	17·15	6·5322	746·8	+·0077	6·5399				Thus $pt_g = \frac{R - 4·9543}{1·5856}$
	7	,, 25	6·5316	17·15	6·5320	746·8	+·0077	6·5397				
	8	,, 26	6·5391	17·25	6·5397	759·5	+·0003	6·5400				
	9	,, 26	6·5389	17·25	6·5395	759·5	+·0003	6·5398				
	10	,, 26	6·5389	17·25	6·5395	759·5	+·0003	6·5398				(13) G had before this experiment been immersed in melted Bi, Cd, Sn, and Pb
	11	,, 27	6·5410	17·5	6·5422	764	—·0023	6·5399				
	12	,, 27	6·5409	17·55	6·5422	764·2	—·0022	6·5400				
	13	,, 31	6·5470	16·6	6·5460	770·5	—·0061	6·6399				(14) and (15) These experiments were performed after G had been immersed in melted Zn
	14	April 1	6·5484	16·35	6·5468	772	—·0070	6·5398				
	15	,, 1	6·5486	16·35	6·5470	772	—·0070	6·5400				
							Mean	6·5399				

TABLE XIX.—(continued.)

Nature of experiment.	No.	Date.	Observed R.	Temp. of coils.	R. after correction for temp. of coils (by Table III.).	Ht. of bar.	Correction for bar. 760 (by Table III.).	R. at 760.	*pt.*	*t.*	*t*—*pt.*	Remarks.
				°		mm.						
B.P. of naphthalene	16	March 25	8·3275	17·2	8·3281	752	+·0069	8·3350	213·21	218·06	4·84	
	17	„ 25	8·3276	17·2	8·3282	752	+·0069	8·3351	213·22	(CRAFTS)		
	18	April 2	8·3394	17·1	8·3396	765	−·0043	8·3353	213·23			
	19	„ 2	8·3390	17·1	8·3392	765	−·0043	8·3349	213·21			
								Mean	213·22			
B.P. of benzophenone	20	March 25	9·6076	17·1	9·6080	749·5	+·0095	9·6175	294·10	306·08	11·98	
	21	„ 25	9·6076	17·1	9·6080	749·5	+·0095	9·6175	294·10	(CRAFTS)		
	22	April 2	9·6226	17	9·6226	765·5	−·0050	9·6176	294·11			
	23	„ 2	9·6222	17	9·6222	765·5	−·0050	9·6172	294·09			
								Mean	294·10			
B.P. of sulphur . .	24	March 25	11·537	17·25	11·538	748	+·013	11·551	416·05	448·38	32·28	(26) and (27) Distilled for twenty minutes between these experiments; about ½ litre distilled over
	25	„ 25	11·539	17·2	11·540	748	+·013	11·553	416·15	(REG-NAULT)		
	26	April 2	11·557	17·15	11·558	765	−·005	11·553	416·15			
	27	„ 2	11·556	17·15	11·557	765	−·005	11·552	416·10			
								Mean	416·10			

Note on the Possible Causes of the Differences of pt as shown by Different Thermometers, when t is the same.

If the external resistance is constant, it cannot affect the value of $\frac{R_t - R_0}{R_{100} - R_0} \times 100$, whatever its magnitude.

It has been suggested that the discrepancy between my $t - pt$ curves and CALLENDAR'S, may be accounted for by the difference in the alteration of the "stem" resistance due to changes in its mean temperature. If such were the explanation, it is probable that the discrepancies would be more marked in A, B, C, and D (where the whole of the stem would be liable to alteration in temperature) than in E*, E, F, and G (where nearly the whole of the immersed portion was kept at the temperature of the tap water. No such contrast is, however, observable.

On comparison of the pt given by thermometer G with that given by CALLENDAR at the same temperature, I get the following numbers :—

Thermometer G	When $t = 440{\cdot}9$ then	$pt = 410{\cdot}4$
CALLENDAR (Table A., 5. Exp. 30).*	„ $t = 440{\cdot}9$ „	$pt = 417{\cdot}1$
	Difference	$6{\cdot}7$

Now the mean temperature of that small portion of the stem which was immersed and exterior to the condenser must, in all cases, have been somewhere between the temperature of the tap water and that of the coil.

Assuming that S_t, S_{100}, and S_0 would be the resistances if the temperature of the stem remained constant at the temperature of the tap water, and R_t, R_{100}, and R_0 the observed resistances, then $R_t > S_t$ and $R_0 < S_0$ $\therefore$ *a fortiori*, $R_t - R_0 > S_t - S_0$.

Assuming the mean resistance of the stem to be a function of t, then

$$\frac{R_t - R_0}{R_{100} - R_0} = \frac{S_t - S_0 + f(t)}{S_{100} - S_0 + f(100)}.$$

Denoting the pt as given by G by pt_g, and, as given by CALLENDAR by pt, then

$$pt_g = \left(pt + \frac{f(t)}{S_{100} - S_0}\right)\left(1 - \frac{f(100)}{S_{100} - S_0}, \text{ \&c.}\right)$$

$$= pt - pt\,\frac{f(100)}{S_{100} - S_0} + \frac{f(t)}{S_{100} - S_0}, \text{ \&c.}$$

$$\therefore pt - pt_g = pt\,\frac{f(100)}{S_{100} - S_0} - \frac{f(t)}{S_{100} - S_0}, \text{ \&c.}$$

Now the total stem resistance of G was about ·021 ohm at 0°.

* 'Phil. Trans.,' A., 1887.

In no case was the portion of the stem immersed and exterior to the condenser more than one-ninth of the whole stem. Assuming, however, that one-fourth of the whole stem is subject to variation, we get ·005 ohm as its resistance. Of this portion one end was at the temperature of the coil, the other at that of the tap water; it is probable, therefore, that its mean temperature was about $t/2$. Taking the circumference of increase of platinum as ·003, we can deduce a rough approximation (taking $S_{100} - S_0 = 1{\cdot}59$; see Table XIX.) for $pt - pt_g$ when $t = 440$.

We get

$$pt\,\frac{f(100)}{S_{100} - S_0} - \frac{f(t)}{S_{100} - S_0} = {\cdot}2, \text{ etc.}$$

The difference of 6·7 cannot, therefore, be due solely to the difference in the mean stem temperature.

A more probable explanation is to be found in the decrease in the resistance of glass as its temperature rises. The platinum tube and rod in the stem were separated by a thin glass tube only, of which the lower six inches were exposed to the extreme changes of temperature. Again, the coil was in contact with glass throughout its whole length. A fall in the resistance of this large section of glass sufficient to reduce the total resistance at 440°·9 from 11·5678 to 11·4616 ohms would account for the difference between pt_g and pt.

From Everett's 'Physical Constants,' I extract the following numbers :—

Specific resistance of glass at	200°C.	=	2230000×10^{10}
,, ,, ,,	300°	=	14500×10^{10}
,, ,, ,,	400°	=	7210×10^{10}

Assuming these numbers, the above fall in the resistance would be accounted for on the supposition that the total resistance of the glass at 200° did not exceed about 6×10^5 ohms, which, bearing in mind the large section, is not an impossible number. This explanation is borne out by the fact that Curve D is the steepest, showing the greatest falling off in the increase of resistance. In D the section of the glass conductor was much greater than in any of the others, the wire being in close contact with glass on *both* sides throughout the whole length of nine feet. Again, slight differences in the quality of the glass, or in the nature of the contact between it and the platinum would be sufficient (on this hypothesis) to account for the differences between the curves.

It is, however, conclusively shown by the experimental results that, whatever is the cause of the discrepancy, it does not affect the determinations of temperature made by these thermometers.

[January, 1891.—In a communication which Mr. Callendar and I had the honour of presenting to the Royal Society on December 18th, 1890, we gave the

results of an investigation into the boiling-point of sulphur. We found that the temperature by normal air thermometer of the saturated vapour of sulphur under a pressure of 760 mm. of mercury at 0° C. and $g = 980{\cdot}61$ C.G.S. (sea-level in lat. 45°), is $t = 444°{\cdot}53$ C.

It follows that the numbers given on page 64 require revision in the sense indicated by the note on that page.

A constant error of ·5 mm. in the barometer scale used by me has subsequently been discovered.

The numbers obtained by the revision thus rendered necessary are given by us in the communication above referred to.]

III. *On the Specific Heats of Gases at Constant Volume.*—Part I. *Air, Carbon Dioxide, and Hydrogen.*

By J. Joly, *M.A., B.E., Assistant to the Professor of Civil Engineering, Trinity College, Dublin.*

Communicated by Professor Fitzgerald, *M.A., F.R.S., F.T.C.D.*

Received September 2,—Read November 20, 1890.

[Plate 6.]

Introduction.

This paper is occupied with a consideration, from a purely experimental point of view, of the specific heats of three gases: (1) of air, chosen as being typical of the sensibly perfect gas, and one the properties of which have entered more into thermodynamical considerations than those of any other fluid; (2) of carbon dioxide, as typical of the imperfect gas, in the case of which there is also the advantage of an already extensive knowledge of its other properties; (3) of hydrogen, as a gas whose behaviour, as regards Boyle's law, suggests that it occupies a peculiar position among perfect gases.

The specific heats of these gases are, in this first notice, treated over pressures ranging approximately from 7 to 25 atmospheres. The range of temperature is not sensibly varied, the lower limit being that of the prevailing atmospheric temperature, the upper that of steam at ordinary pressures. The question of the relation between the specific heats of gases and their temperatures I hope to make the subject of a future notice.

For advice and assistance in all matters of difficulty, more especially in the consideration of the nature of the corrections to be applied to the experiments, I have to thank Professor Fitzgerald. Where my work touched on chemical science I had the friendly assistance of Professor Emerson Reynolds, and must thank him for his readiness to help me. In this connection, my thanks are also due to Mr. E. Werner. The labour of reducing the results of the experiments was much lightened by the kind help of Mr. H. H. Dixon, of Trinity College.

The Board of Trinity College, in providing a suitable room for my experiments, have rendered it possible for me to attain a far higher degree of consistency in my

results than was previously possible. Some only of the experiments recorded in this paper shared this advantage; their results lead me to hope that future work will possess a higher degree of accuracy.

The Government Grant Committee have enabled me to meet the expenses attending the construction of apparatus.

To the personal care which Messrs. Yeates and Son have devoted to the construction of the apparatus the practicability of the experiments may be said to be due. It is sufficient, as testimony of this, to refer to the qualities displayed by the vessels they constructed for holding the gases.

On the Possibility of Accurate Determinations.

In two previous communications made to the Royal Society,* I have inquired into the accuracy of the method of condensation as a means of investigating the specific heats of substances. I claimed for it a high degree both of truth and sensibility, comparing favourably with, if not transcending, that of all other methods. I might also have claimed for it the quality of elasticity in an important degree. With a delicate balance, the smallest mass may be dealt with. Increasing the power of the balance and the dimensions of the calorimeter, it is easy to deal with large masses or with bulky vessels impossible of treatment by other methods.

This is the method employed in the following investigations on the specific heats of gases. For its principles and for its mode of employment specially with reference to gases, I must refer to my paper on the Steam Calorimeter (*loc. cit.*), in which a plate will be found illustrating the large differential calorimeter used in my experiments. The spherical copper vessels, one of which in an experiment will contain the gas under a greater or less degree of pressure, are shown suspended in the calorimeter. The essential detail of construction, whereby the freedom of the wires suspending the vessel from the balance is secured, is figured in the adjoining plate given in that paper. There will be no need of further description of the apparatus here.

I assume here, on my former proofs, the reliability of the method, and pass on to the question of the degree of accuracy possible of attainment in this application of it. The possibility of such experiments has always been a question of degree. For with the amount of stress to which any suitable material for enclosing the gas may be safely subjected, the weight of the vessel must many times transcend that of the gas, so that, in spite of the rather high specific heat to be ascribed to the latter, the calorific capacity of the gas must remain only a fraction of that of the containing vessel. Nor by increasing the size of the vessel is the disproportion diminished. For, supposing the vessel spherical, the rate of stress on a great circle for a given pressure will increase as the diameter; the weight of the vessel as the square of the diameter,

* "On the Method of Condensation in Calorimetry" ('Roy. Soc. Proc.,' vol. 41, p. 352, *et seq.*); and "On the Steam Calorimeter" ('Roy. Soc. Proc.,' vol. 47, p. 218, *et seq.*).

if the thickness be maintained constant. But, assuming in each case the spherical vessel to be charged to the limiting safe stress, an increase in the diameter necessitates an increase in the thickness of the material proportional to the increase of stress, that is, proportional to the increase of diameter. Thus, finally, the weight of a vessel which is always at its limiting safe stress increases as the cube of the diameter; and this is the rate of increase of its volume or of the weight of the contained gas.

An inferior limit to the size of the vessel is, however, fixed in the case of the present experiments, by the sensibility of the ordinary chemical balance in estimating the small weight of steam which is condensed due to the thermal capacity of the contained gas; a superior limit by the increased risk on a chance explosion and the inconvenience of providing large quantities of steam for filling the calorimeter rapidly at the commencement of an experiment.

After various trials and experiments on larger vessels, two spheres of copper having an external diameter of about 6·7 cms. were designed. One, "A," to stand a test pressure of 500 lbs. on the square inch and a working pressure up to 250 lbs. The other, "B," to stand a test of 1000 lbs. and a working pressure of 500 lbs. A smaller margin of safety than this was not considered advisable as, at the temperature of the steam, a decreased resistance of the vessel was to be expected. The thickness of the walls of these vessels would be, by calculation, closely one-half and one millimetre respectively. And to these dimensions the spheres were as accurately as possible constructed.

Now if we take the case of sphere B and supposing it filled with air to a pressure of some 22 atmospheres at t°, rising to 30 atmospheres at t_2° or 450 lbs. per square inch, the following proportion obtains between the weight of steam thrown down by the vessel and the weight of steam precipitated by the contained air :—

B and its catchwater weigh nearly 100 grms. Its specific heat may be taken as 0·094. Over the range of 85° the condensation of B is therefore just 1·5 grms. The air in B weighs 4·2854 grms. Its condensation is 0·116 grm. Thus one-thirteenth of the precipitation is due to the air. But as we must be content to use the sphere B at pressures lower than this but still too high for sphere A, it is necessary to regard the ratio as descending as low as one to eighteen.

Forestalling speculations on the possibility of dealing with this one-thirteenth, say, to one per cent. of its value—or to $\frac{1}{1300}$th of the whole condensation, I give the results of an entire series of six experiments in which this ratio holds, in fact the series from which—taken at random—I have borrowed the foregoing figures. The mass of air, 4·2854 grms., affords over the mean range of 84°·52 a precipitation of 0·11629 grm. There is a want of balance between the thermal capacities of the vessels, making the observed precipitation 0·15217 grm. These are the mean results of the six experiments. To compare the experiments among themselves I divide the weights of precipitation observed in each case by the range of temperature for that experiment. The six numbers thus obtained represent the weight of steam thrown

down in each experiment per degree of range. The consistency of the experiments is thus truly represented by these numbers. They are :—

0·017941
0·018009
0.018070
0·018080
0·017906
0·018016

Mean precipitation divided by mean range

0·018004.

(See among the experiments, Air VIII.)

The departure from the mean in one case only amounts to so much as half per cent. From these figures it is evident that close measurements are practicable.

Regarding the mode in which these measurements are effected, I may in the first place refer to a former statement of my own that by this method of calorimetry observations of the thermal capacity may be carried to the one-thousandth part. This was based on experiments on minerals where surface conditions were varied by breaking up the specimens between successive observations.* Now, although it is probable that this degree of accuracy would not obtain in the case of successive experiments on large objects of low density, effected in the ordinary way, on the other hand the effect of the simultaneous differential arrangement is to limit or banish several sources of error. Any source of error common to the two spheres and to the gas is not accumulated on the gas. The gas simply bears its own share. Thus a mis-reading of temperature affects the result for the gas simply by the amount of the error, and will not, as in the case of successive differential experiments, by falsifying the specific heat of a mass thirteen times as effective, thermally, as the gas, throw thirteen times the error upon the gas when the allowance for the thermal effect of the sphere subsequently comes to be made. With this class of error are eliminated to a great extent effects of radiation and effects due to buoyancy. Theoretically, indeed, in the use of the simultaneous differential method, there is reason to consider the entire sensibility of the method directed to the evaluation of the effects of the gas as if isolated from its containing vessel. Practically this will not be fully attained. But it appears evident from all considerations that this application of the method to the calorimetry of gases is not straining the method; and the expectation is, I think, justified that in the use of this method the old difficulties will be found to be overcome. In the tables of experiments, given further on, discrepancies in many cases occur. But these cannot all be laid to the charge of the calorimetry. In the case of hydrogen

* "On the Method of Condensation," *loc. cit.*, p. 362.

nearly all these discrepancies are very certainly due to varying amounts of impurities in the samples of gas. Again the earlier experiments on air were made under difficulties which it would be needless to enumerate. The principal one, that of insufficient supply of steam, arising from an inadequate gas main, was not overcome till after the experiments on carbon dioxide were concluded. I did not think it necessary to repeat these, for their agreement *inter se* was fairly satisfactory, but in the case of the far more difficult problem, the variation of the specific heat of air, I did repeat, under the improved conditions. The results, from the present point of view, go to show the degree of precision attainable in the careful application of the method. I allude to experiments VIII.–XI., which are seen plotted on the diagram of experiments on air, Plate 6.

The method is delicate in its results. It is, I think, capable of dealing with some subtle questions in calorimetry, but, as in other methods of measurement, when the higher resources of the method are called upon, care and judgment are needed to guard against the entry of error.

The Vessel for Holding the Gas.

The earlier attempts at effecting the measurements of the specific heat were confined chiefly to efforts at overcoming two principal difficulties :—the construction of a vessel sufficiently light, strong, and staunch to contain the gas, and when this was accomplished, to secure the filling of it with *dry* gas at the requisite degree of pressure.

As regards the first difficulty it was soon found that vessels constructed of more than one piece were only reliable when joined with very carefully *brazed* joints. Several explosions, which but for precautions might have been disastrous, occurred owing to various forms of soft and hard soldered, overlap joints tearing open in the steam.

The ordinary form of stop-cock for admitting the gas proved useless. At high pressures it was impossible to avoid leakage. A screw valve was accordingly designed. This has proved very satisfactory, and may be so contrived as to add hardly more than one gramme to the mass of the sphere. Finally, copper was found to be so satisfactory in all respects that its use was decided upon. It is surprisingly tough, not scattering on explosion, but tearing open, and admits of being spun into very thin hemispheres. So far as thermal capacity is concerned, there is not much choice, the densities of metals being about inversely as their specific heats.

Spheres A and B are built up of hemispheres meeting within a narrow belt of their own thickness. This belt is widened at one point, and here the screw valve is soldered in its place. For this soft soldering is permissible and convenient. The belt is brazed in with the utmost care to secure a deep and staunch joint. The screw valve is shown in section in the cut (fig. 1). The valve is opened by screwing back the central pin, which is of hard tempered steel, terminating at one end in a burnished conical point,

which crushes into the soft brass when the valve is closed. The other end of the pin is fitted with a little eye for suspending the whole from the wire in the calorimeter. The sphere can be attached to the compression pump by the side tubulure. While being filled there will be leakage along the thread of the valve pin, if the thread be not choked by strong wax or shellac run in while the valve is warmed with a spirit flame. To avoid this trouble, and to still further lighten the valve, I devised the second form (fig. 2). In this the valve, as introduced into the calorimeter, terminates with the cross-hatched part of the figure. The remainder is a fitting for attaching it to the pump. The central spindle is surrounded by a stuffing box, and consists of a forked key for gripping and turning the valve pin. A longitudinal slot in the latter, extending just to the cone, allows of the free entry or exit of gas when the cone is lifted from its bed.

Fig. 1.

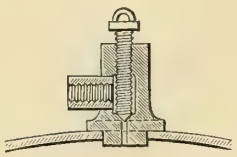
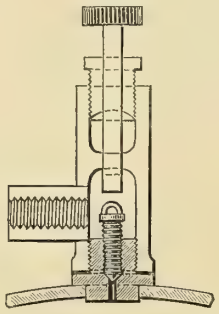


Fig. 2.



The sphere before use is attached to a small hydraulic hand-pump carrying a metallic gauge, and tested to the requisite pressure by pumping water into it. In this process the form of the vessel visibly becomes more spherical; the internal volume may increase nearly three per cent. Leakages are now detected and marked. Subsequently these are staunched with soft solder, applied while a vacuum is maintained in the sphere. The sphere will now probably be quite gas-tight, but it must be further tested when filled with gas by immersing it in water, and observing for some time if any bubbles grow upon its surface.

The catchwater of platinum foil with which each sphere is provided, is arranged to be easily removed. It is, in fact, hooked on to two little wire loops soldered to the sides of the sphere.

The idle sphere, or counterpoise, is not, of course, tested. It is only necessary to make sure that it is perfectly air-tight. It is closed when full of air at normal pressure, and is as similar to the active sphere as regards material, weight, and volume, as possible. The equality of volume should not be departed from to an extent exceeding three or four cubic centimetres. Their thermal values are adjusted by inserting calculated weights of copper wire into the idle sphere. It is easy to secure that this in the first instance shall be the lighter of the vessels. More than an approximate equality in the thermal capacities of the vessels is hardly requisite.

Differential experiments on the thermal behaviour of the spheres are in any case requisite at intervals during a research.

The Storage of the Gas.

The gas—the earlier experiments were on air—was at first pumped into the sphere by a single acting vertical-cylinder pump, of the kind formerly much in use for liquefying carbonic acid gas. It was found that the gas, although supplied dry to the pump, not only took up moisture from the oil lubricating the piston, but took up the oil itself. Two series of experiments, effected in the single calorimeter by the successive differential method, were carried out at this time on air. A filtering tube had been inserted between pump and sphere, but the odour of oil from the air when being released proved that this had failed to clean the air. The experience with the single calorimeter, too, showed that some improvement in dealing with the bulky vessels was desirable. The calculations for the error of buoyancy involving the density of the steam, of the air, and of the hygrometric state of the latter, as well as the thermal expansion of the sphere, were risky and laborious. And so the differential calorimeter was devised.

A larger and more effective filtering tube was next attached above the pump, and the experiments appearing in a preliminary note* were carried out. It was subsequently found that the dryness of air so stored was open to doubt, and it was resolved to seek some different means of storing the gas. The apparatus at present in use was accordingly designed.

This apparatus consists of a receiving apparatus and an oil pump. The compression pump formerly in use was converted into an oil pump by adding above the cylinder suitable valves and an expansion chamber, after the manner of the ordinary force-pump. The pump is driven by a flywheel turned by hand, and can pump oil up to pressures of 35 or 40 atmospheres. This pump is attached to the receiving apparatus (fig. 3) by the two tubes leading to the right of the figure.

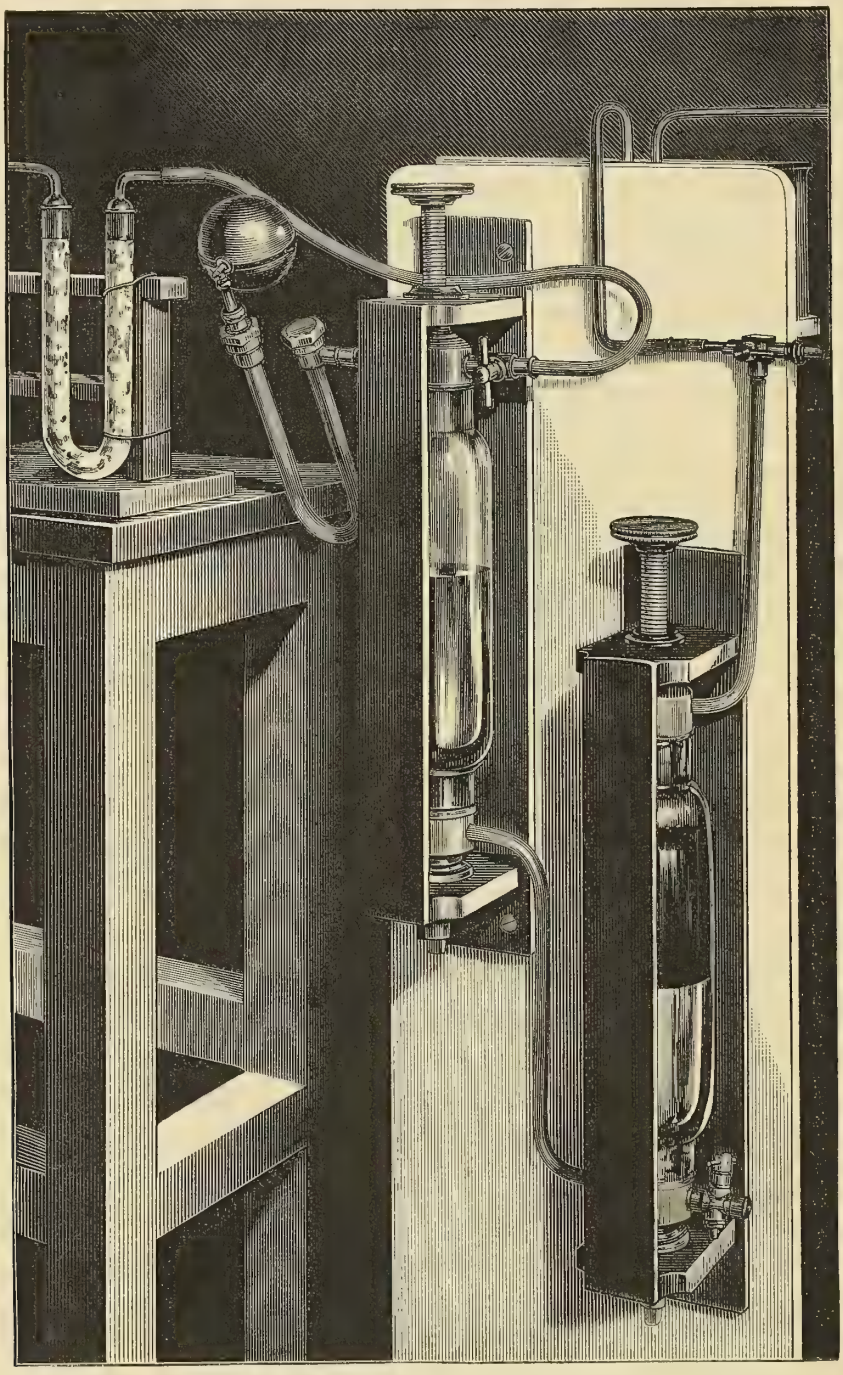
The receiving apparatus consists of two strong glass vessels—one placed at a level higher than the other. These receivers are 11 mm. thick in the walls. The internal volume of the upper one is 335 c.c., of the lower one 344 c.c. It is desirable that there should be this disparity of volume between them. These vessels, which are about 5 cms. in internal diameter, have, at the lowest estimate, assuming the glass to be well annealed, a bursting strength of some 180 atmospheres.

The glass receivers are carried in cast iron frames, furnished each with two slow-threaded screws, placed with their axes truly in line. The extremities of these screws are rounded, and bear against cast iron stoppers closing the ends of the receivers. In this way the uniform bearing of the stoppers against the ends of the receivers is secured. Leather washers, soaked in wax, are interposed next the glass. With this

* ‘ Roy. Soc. Proc.,’ vol. 45, p. 33.

arrangement, by screwing down the upper milled-headed screws, and without resort to the more powerful screws below, perfect immunity from leakage at the highest pressures is attained.

Fig. 3.



All the connections are effected through the stoppers, as is seen in the figure. The receivers communicate one with another by a bent wrought-iron tube, connecting the two undermost stoppers. The upper stopper of the lower receiver has a single con-

nection with an upright wrought-iron tube, terminating in a three-way stop-cock. Through this the lower receiver is put in connection with the high-pressure tube (of lead) leading to the oil pump, or through a glass tube with the little oil tank, placed above, and only partially seen in the figure protruding from behind the apparatus. Or the high-pressure tube from the pump may by this stop-cock be placed in connection with the oil-tank directly, the receiver being then closed. The lower stopper of this receiver is fitted with a wrought-iron stop-cock, for commanding the filling or emptying of the receiver with mercury.

The second and uppermost receiver has two stop-cock connections above; one, to the right, with the train of drying and purifying tubes through which the gas to be pumped is drawn before it enters this receiver; the other, to the left, leads to a high-pressure wrought-iron drying tube, of narrow bore (1 cm. *q.p.*), at the further extremity of which is the screw fitting for taking the tubulure on the valve of the copper sphere. Care is taken to keep the passages which are drilled in this uppermost stopper of the higher receiver, fine in bore, about 1 mm.

The method of applying the apparatus is as follows:—Clean dry mercury is let into the lower receiver. When this is just filled the mercury rises in the lower neck of the upper receiver. The tank must now be supposed to be filled with clean Rangoon oil, and the three-way cock connecting the pump and the receiver. On now working the pump oil is drawn from the tank through a glass tube (this is the higher right hand tube in the figure), and returned by the pump into the receiver. The mercury in this is displaced into the upper vessel, expelling the gas in it into the drying-tube and sphere. When the higher receiver is quite filled with mercury, owing to the greater content of the lower vessel, some three or four centimetres' depth of mercury remains in the latter. Closing now the left hand stop-cock of the upper receiver, setting the three-way cock to permit the oil in the lower one to flow into the tank, and finally opening the stop-cock connecting the upper receiver with the train of drying-tubes, the mercury flows back quietly by its own weight into the lower receiver, and gas is as quietly drawn into the upper one, to be again expelled, by the same operations, into the sphere.

The volume of the sphere being some 160 c.c. and of the drying-tube 31 c.c., one operation of the pump displaces sufficient gas to raise the pressure in the sphere by 1·6 atmospheres. In the knowledge of this I have dispensed with a pressure gauge in the use of this pump. But a gauge of small capacity attached between the pump and the high-pressure drying-tube is undoubtedly desirable, in case of leakage or of a mistake in reckoning the number of operations.

In the use of this pump the air only comes in contact with the mercury and glass of the upper receiver. After the pump has been in use some time, however, the mercury, judging from a certain dimming of the glass at the lower part of the receiver, appears to become to some extent greasy. This is much retarded by screwing a short tubulure of soft wood into the central perforation of the under stopper of the lower

vessel, extending some 2 cms. up into the mercury, which has all to flow out through this tubulure. This greasiness only does harm in so far as it may communicate moisture to the gas. No volatile hydrocarbon is taken up. To investigate this question a careful experiment was kindly carried out by Mr. WERNER in the chemical laboratory. A quantity of $2\frac{1}{2}$ litres of air compressed into the sphere, and which before entering the pump had been passed through a potash bulb to free it to a great extent from the normal CO_2 of the atmosphere, was very slowly allowed to escape through a heated combustion tube 40 cms. in length, filled with platinised asbestos, in order to burn any hydrocarbons into carbon dioxide. The issuing gas was bubbled slowly through baryta water. At the end of the operation a hardly appreciable opalescence (soluble in hydrochloric acid) was visible. As a small proportion of the normal CO_2 in air, escaping the potash bulb, would have caused the slight precipitation observed, there are no grounds for suspecting any appreciable impurity contracted in the receiver. The air compressed in this apparatus is, also, I may observe, quite free from odour.

The final drying of the compressed air is, however, necessary. Before the addition of the drying-tube (which is filled with dried asbestos and phosphoric anhydride), an experiment was made by passing 4·340 grms. of air (which had been dried through three large U-tubes of phosphoric anhydride, and compressed directly from the receiver into the sphere) through a carefully counterpoised drying-tube, which throughout the operation was maintained suspended upon the balance. An increase of 4·2 milligrammes in the weight of the tube was obtained. As some 2 milligrammes would, on received principles, saturate the volume occupied by the air in the sphere, this error could not be suffered to remain. The tube was therefore added. The experiment was subsequently repeated, 4·3527 grms. of air being passed through the carefully equilibrated drying-tube. The result was absolutely *nil*. Later, the experiment was again repeated on 2·331 grms. of air. The result in this case was unexpected. There was a slight, but appreciable, *loss* of weight. This can only be explained in the more complete drying of a minute plug of cotton wool placed in the tubulure of the drying-tube to hinder the free entry of air while the weighing was being effected.

It may be assumed from these observations, I think, that the various gases compressed in this pump, if delivered pure into the receiver, derived no impurity in being transferred to the copper sphere; and it will only be necessary to dwell in each case on the means taken for generating and purifying the gases.

Procedure in Making the Experiments.

The sphere having been repeatedly "washed out" with the gas to be dealt with, in a dry, pure state, is hung from the wire in the calorimeter against its counterpoise, the idle sphere, and the difference in weight observed. In the process of "washing out" the sphere is both heated (by a spirit flame) and exhausted between each

admission of the pure gas. During the first equilibration it is filled with the pure gas at normal pressure. It is now screwed up to the pump and the requisite quantity of gas pumped into it. It is then returned to the calorimeter, and when all is cold the added weight of gas—subject to a small correction—found by difference.

An experiment cannot be performed till the thermometer is stationary—meaning thereby a state in which the motion of the thermometer cannot be perceived in observations made at intervals of some ten minutes. All openings in the calorimeter are now closed and the balance set in active vibration to keep the air in the calorimeter stirred, and thus secure greater uniformity of temperature. I have lately in order to attain the same end used a small stirring apparatus consisting of a little fan driven round by clockwork. This is mounted at the extremity of a stopper which enters the calorimeter by the wide opening at the back, through which the steam is admitted. It is withdrawn at the moment of turning the current of steam into the calorimeter. The motion of the fan keeps the air in the calorimeter in active motion, while all openings are closed against the ingress of external air.

While steam is being got up in the boiler the thermometer is observed at intervals. It will generally be found to rise a little. The final reading before admitting steam is taken as that of the spheres. The variation will be small, seldom more than $\frac{1}{20}$th of a degree.

The observation of the behaviour of the spheres in the steam takes about five minutes. If loose weights have been inserted in the idle sphere it will be found that its final assumption of the steam temperature will probably be delayed beyond that of the filled sphere. The inequality in the rate of heating reveals itself in the gradual acquisition of weight of the idle sphere. This effect was remarkably conspicuous in the experiments upon hydrogen—a result depending, doubtless, on the high conductivity of that gas.* In the course of $3\frac{1}{2}$ minutes the rate of increase of the idle sphere is diminished so as to be no longer perceptible, but the balance is observed till the fifth minute closes. This supposes no "increment." The increment generally indicates a weak burner, or leaky joints, so that the steam is impure—not free from mist—or a preponderance of the radiation of one sphere over the other. The first evil is met by the use of a powerful burner, close-fitting joints and long preliminary clearing (7 to 10 minutes) of the boiler and steam ways with the live steam. The second evil is generally caused by one of the spheres becoming greasy from handling (generally the filled one, the other is seldom touched), and is met by freeing the surface from grease by washing well with dilute ammonia. Although occasionally troubled in my earlier experiments by this evil, I have now for many

* It might be suggested that this difference in the rate of heating of the spheres would give rise to an error in estimating the initial temperature, except in the case where this was actually stationary. I may observe, however, that a comparison of a number of the results of observations in the cases where t_1 was rising with the cases where t_1 was falling revealed no settled difference in one direction. The rate of change of t_1 is very slow.

months, using the calorimeter from twice to four times in the day, without intermission of a day, only observed it on two or three occasions. It can, indeed, be eliminated—very closely—by calculations based on continued observation of its value ("On the Steam Calorimeter," *loc. cit.*), but this necessitates the retention of the vessels in the steam an excessive time, which is to be avoided as the stress on the active sphere is then often very great.

The upper limit of temperature is most conveniently observed by a thermometer which is inserted in the calorimeter, in place of the low-range one, just before steam is let in.

When the experiment is concluded the spheres are simply dried in a soft clean cloth and laid aside to cool. The heat remaining in them completes the drying of their surfaces very thoroughly without further trouble.

When all is again cold, the spheres are restored to the calorimeter and an observation made on their equilibration. This is an assurance against leakage having occurred during the experiment. Absolute return to the former equilibration is, perhaps, exceptional, but the departure from it is generally small, some two or three-tenths of a milligramme, and will as often appear to indicate loss from the one sphere as from the other. When the number of experiments deemed requisite is completed a careful equilibration of the spheres is made and some or all of the gas let out. If it is desired to make a further investigation upon this sample of gas at a lower pressure, the desired amount is retained by suffering the escape of gas to occur slowly while the sphere is hung from the balance, the weights being so adjusted that there will be equilibrium when the proper amount of gas remains in the sphere.

Finally, after a series of investigations, by the addition of all the weighed quantities which have been suffered to escape, the weight of gas put in in the first instance is found. This value, as might be expected, is often found to differ by a small quantity from the first determined value of the weight. The difference is generally negligible, except in the case of hydrogen, where the total weight is small. In this case, as the second determination of W is more reliable than the first, it is accepted. In the first determination there is evidently some chance of the vessel contracting some impurity upon its surface in the process of filling.

The thermal comparison of the spheres at normal pressures is now effected. I have generally thought it inadvisable to increase the weight of gas dealt with by exhausting the active sphere. In the process of exhaustion there was risk of spoiling the weighings by contamination of the surface of the sphere. There was also the risk of a back-leakage of air into the sphere during the process; injurious, either by the fact that it was undried or as an impurity if other gas was being dealt with.

The number of observations made on the thermal equilibrium of the "empty" spheres is some five or six generally. In the earlier experiments these observations were more frequent, as trifling alterations on the valve or for leakage, &c., were then often necessitated.

Corrections to be Applied to the Experimental Results.

The direct results of experiment are subject to some corrections before they can be applied to the calculation of the specific heat of the gas. The conditions of experiment involve that the two spherical vessels have their specific heats compared when each is filled with gas at the pressure of one atmosphere, and again when a further weight (W grammes) of gas is compressed into one of them. It is necessary to inquire whether the presence of the added quantity of gas does not give rise to any thermal effect other than that due simply to its capacity for heat at constant volume between the limits t_1 and t_2.

1. *Correction for Thermal Expansion of the Vessel.*—The volume of the sphere increases during the experiment by the thermal expansion of the copper. The gas in the sphere is not, in fact, maintained at absolutely constant volume. The thermal effect of this on the added quantity of gas, W, is evidently that which is to be considered.

The initial pressure due to W is affected negatively by the increased volume of the sphere, positively by the rise of temperature. Work, ϵ, is done by the expansion of the gas through the increased volume of the sphere, and the thermal equivalent of this must be deducted from the experimental result. Calling the initial volume V_1, and the initial pressure P_1, at any moment during the rise of temperature.

$$P = \frac{P_1 V_1}{V}(1 + \alpha t) \quad \text{and} \quad V = V_1(1 + \beta t),$$

where α is the coefficient of increase of pressure at constant volume, and β the coefficient of thermal expansion of copper. From this a complete formula for the correction is found—

$$\epsilon = P_1 V_1 \left[\alpha(t_2 - t_1) + \frac{\beta - \alpha}{\beta} \cdot \log \frac{1 + \beta t_2}{1 + \beta t_1}\right].$$

But for this an approximation may be substituted. A good approximation is found to be—

$$\epsilon = \{P_1 V_1 \beta (t_2 - t_1)\}\{1 + \alpha - \beta \tfrac{1}{2}(t_2 - t_1)\}.$$

The correction is subtractive. A numerical estimate of its value is given further on.

2. *Effect of the Elastic Distension of the Vessel.*—In considering the last correction it was assumed, for the sake of simplicity, that the copper sphere is so rigid as not to yield to the increased pressure exerted by the gas as it rises in temperature.

To obtain exact data in calculating this correction, experiments were made on the shrinkage of the spheres when a known weight of gas was liberated. These experiments consisted in carefully weighing the sphere in water with a sinker before and after releasing a certain weight of air. These are given later, with other data connected with the spheres.

If the increase of volume by distension is estimated for the change of pressure $P_2 - P_1$, the usual equation for work of expansion gives the amount of the correction when reduced to its heat measure as a weight of steam precipitated. The correction is subtractive.

3. *Thermal Effect of Stretching the Material of the Sphere.*—The increased pressure of the gas in the sphere has the effect of distending the copper sphere, as just seen; this, according to well known principles, results in a cooling of the copper. Experiments have been made by JOULE, EDLUND, and others, on the amount of this cooling in wires exposed to suddenly-applied traction. The cooling of the copper in the present case is made good by the condensation of steam. An estimation, however, of the amount of this correction in an extreme case showed that it was too small to merit consideration. It is not, therefore, further considered.

4. *Correction for Buoyancy due to Distension in the Steam.*—The distension in the calorimeter has also to be considered, as occasioning another correction. The increased displacement in the steam will tend to decrease the apparent weight of the sphere, and so diminish the amount observed as the weight of steam precipitated due to the gas. The amount of the correction is the increase of volume due to rise of pressure multiplied by the absolute density of steam at the temperature t_2. It is additive.

5. *Correction for Buoyancy due to Distension at* t_1.—In that series of experiments in which the one sphere is filled with gas at high pressure, the air-to-steam displacement effect of the filled sphere is greater than it is in the series of experiments in which both spheres are at the same pressure. The distension due to W grammes of gas at t_1 is multiplied into the relative density of air and steam. As the correction results in a greater apparent increased weight of the filled sphere on the admission of steam into the calorimeter, the correction is subtractive.

6. *Correction for Unequal Thermal Capacity of the Spheres.*—This correction has been already alluded to. It is effected on a series of experiments the mean results of which are taken, giving a correction of $\pm \omega_1$ grammes of steam over a range $t_2 - t_1$. In applying it the amount of ω is adjusted for the mean temperature range of the series of experiments under consideration. It must be effected before the next correction.

7. *Reduction of the Weight of Precipitation to Vacuo.*—By multiplying the observed weight of precipitation less ω_1 (6) into the absolute density of steam, this correction is effected. It is additive.

The analogous correction for the displacement in air of the weights added to the balance to equilibrate the precipitated steam is unnecessary, if the weights are, as usual, of platinum or gold.

The total result of these corrections is to reduce the direct measurement by about 0·7 per cent. In estimating the relative importance of these corrections a few figures will be of service. I take the case of the series Air II., in which the stronger sphere B was used.

The data are :—

Volume of sphere B at 15° and 760 mm. internal pressure = 159·826 c.c. (by experiment).

Distension due to 4·3525 grms. of air at 12° C. (22·01 atmospheres) = 0·1732 c.c. (by experiment).

In the series Air II. :—

Mean $t_1 = 11{\cdot}72$, Mean $t_2 = 99{\cdot}73$, Weight of air = 4·8064 = W.

V_1 has first to be found, by a subtractive correction for temperature (of 0·024 c.c. using the coefficient of expansion $\beta = 0{\cdot}00004998$, applied to the measured volume at 15°, and by an additive correction for distension effected on a first rough estimate of pressure at t_1; 24 atmospheres in this case. This last correction amounts to 0·191 c.c. Finally therefore the value V_1 is found to be 159·993 c.c.

In estimating P_1, the weight of a volume of air = V_1, at the temperature 11°·7 and at the pressure 760 mm. is divided into W. This gives the approximate pressure P_1, due to W, as 24·266 atmospheres.

The thermal expansion over the range $t_2 - t_1$, is calculated to be 0·702.

From these values the correction (1) is first calculated. This reduced to its thermal equivalent in grammes of steam precipitated is found to amount to a correction of − 0·00090.

To estimate the second correction, that for the effect of elastic distension of the sphere, the pressure at the temperature t_2 is found by the coefficient of increase of pressure, α (Regnault). It is found to be 32·54, and the distension due to the increase is, by calculaton, 0·065 c.c. Taking V_1' as 160·695 and V_2' as 160·760, the correction is calculated and found to be − 0·0000658 grm. of steam.

The buoyant effect of the distension in the steam (4) is found to be + 0·000039 grm.; and the correction for the air-to-steam displacement effect of the distension due to P_1 (5) is − 0·000140 grm.

In this series, Air II., the weight of precipitation due to the unequal thermal values of the spheres is, on four experiments, found to be 0·00310 over the range 87·92. As this excess is on the idle sphere it is a + correction. Calculating the proportional correction over the range $t_2 - t_1$ and adding this to the observed precipitation, the correction for displacement of the precipitation in the steam is with sufficient accuracy applied to the result. It is + 0·000083 grm. The final result is a correction of − 0·000920 grm., which is taken as 1 milligrm., as the balance reads reliably no closer than to the fourth place of decimals. The true weight of precipitation to be ascribed to the calorific capacity of the air enclosed at constant volume, can now be arrived at.

Having calculated thus the corrections for one series of experiments effected in a particular vessel, the labour of future calculations may be much lightened by basing the new corrections on the old, proportionately to pressure and range of temperature.

In the calculation of the corrections in the case of carbon dioxide, which departs very sensibly from BOYLE's law, the estimation of pressure is based on the results of the experiments of ANDREWS ('Roy. Soc. Proc.,' vol. 24, p. 458) on the departure of this gas from conformity with BOYLE's law, and on its coefficient of expansion at constant volume at various pressures and temperatures. The last data are not as full as would be desirable.

8. A correction on the apparent weight of gas compressed into the sphere is necessary in the case of hydrogen where the total weight of gas is small. The correction arises from the distension of the sphere due to the pressure P_1, and is equal to the amount of distension multiplied into the density of the air at the time the weighing is effected. Whether it is positive or negative will depend upon whether the weight has been determined upon the occasion of filling the sphere or emptying it. It is negligible in the case of a gas so dense as air. Thus in the numerical example I have given the distension 0·191 c.c. affords a correction of 0·00023 grm. This, in a weight of gas of 4·8064 grms., is outside the value to five figures of the specific heat. In the case of hydrogen the same distension would arise for a weight of but one-third of a gramme. The correction has therefore been applied in all cases to the weights of hydrogen.

Calculation of the Density and Pressure.

On Plate 6 will be found charts giving a graphic representation of the results of the experiments. In these the mean result of each series is taken as ordinate, the mean absolute density (mass of one cubic centimetre) of the gas as abscissa.

The mean absolute density is the ratio W/V, where W is the *total* weight of gas enclosed in the sphere, V the mean volume during experiment. The total weight of gas is obtained with sufficient accuracy by adding to the measured weight of gas the calculated weight of a volume of the gas equal to the volume of the sphere, and at the temperature and pressure (approximately) obtaining in the "empty" sphere at the time when the weight of gas was being determined. The volume is obtained from the standard observations on the sphere, corrected for thermal expansion to the temperature $\frac{1}{2}(t_2 - t_1)$ (*i.e.*, to the mean temperature of the experiment), and for the distension due to the *total* weight of enclosed gas at the mean temperature of experiment as before.

This is a troublesome estimation, as it has to be independently calculated for each series.

Simultaneously with calculating the last constant we arrive easily at an estimate of the *actual* or total pressure in the sphere at the temperature $\frac{1}{2}(t_2 - t_1)$, that is, the pressure due to the total weight of gas in the sphere, as above, at the actual mean volume, as above, and at the mean temperature of experiment. The numbers obtained for the pressures in the sphere can only be considered as approximate. This

more especially applies to the case of carbon dioxide, for which gas I used, however, as far as possible the determinations of ANDREWS, as in the calculations (*ante*) of pressure and density for the evaluation of the errors.

Particulars of the Spheres used to contain the Gas and other particulars.

Sphere A.—Weight 41·105 grms., 0·4 grm. lighter than its counterpoise; internal volume 161·560 c.c. at 15°. This sphere appears to have increased permanently in volume from the effects of the experiments. The above is from a recent observation, and, although no comparably accurate observation was made before the experiments, a reliable observation of the difference between its external volume and the external volume of its counterpoise, made before the experiments, afforded 1·7 c.c.; one made recently gave 3·545 c.c., an increase of 1·845 in external volume.

Its shrinkage was determined recently in two experiments. In one the release of 2·5284 grms. of air at 14°·3 afforded a shrinkage of 0·2636 c.c.; in the second experiment the air released weighed 1·6046, and the shrinkage was 0·1723. Temperature of the air 16°. These results agree closely, and a mean result is accepted.

Sphere B.—Weight without catchwater 92·4 grms.; internal volume, measured June, 1890, = 159·826 at 15°; measured May, 1889 (before the experiments recorded in this paper) = 159·829 at 14°. There has therefore been no permanent set in the distension of this sphere. Its volume differs from that of its counterpoise by 0·580 c.c.; its weight by 6 grms.

Its distension has been measured by two experiments; one, in October, 1889, affording 0·1732 c.c. for a pressure of 22·01 atmospheres, or a weight of air 4·3525 at 12°. A second experiment (June, 1890) showed a shrinkage of 0·1013 for the release of 2·5728 grms. of air at 15°·4. These agree very closely; the first is taken for calculating the corrections.

The Thermometer.—The thermometer used in obtaining the initial temperatures throughout, is one of NEGRETTI and ZAMBRA'S, divided openly to tenths of a degree, and standardised at Kew. At the close of the experiments it was compared with a recently standardised thermometer, when no appreciable error was detected.

The upper limit of temperature was in the earlier experiments obtained from observation of a fine barometer corrected for capacity by a movable scale. Subsequently a standardised thermometer of NEGRETTI and ZAMBRA'S, reading to tenths, was used. Comparing its indications with the readings derived from the barometer, no appreciable difference was found to obtain.

Value ascribed to the Latent Heat of Steam.—The latent heat of steam is assumed to have the values determined by REGNAULT, and is derived from his formula, for each experiment. This constant has been investigated by ANDREWS (535·9 at 100°), BERTHELOT (636·2 at 100°, total heat), FAVRE and SILBERMANN (537·77 at 99°·81). These agree closely, but all are a little lower than REGNAULT'S results (536·5 at 100°),

and would tend to lower the results of the experiments by a very slight (inappreciable) amount.

The Experiments on Air.

The results of the first experiments* on air effected in the differential calorimeter, were to ascribe to it a somewhat higher specific heat than subsequent experiments have revealed. The air in these experiments was compressed in contact with oil, and hence moisture and impurities in the air are probably the source of difference. These experiments, too, appeared to indicate invariability of the specific heat over a considerable range of pressure. Subsequent experiments, which from the very fact of the existence of the discrepancy have been multiplied in number, disprove this result. There is an undoubted diminution of the specific heat with decreasing pressure or density. A plausible explanation of the first result exists in the very certain presence of moisture in the gas, taken in conjunction with the course of procedure adopted.

The lower pressures were obtained in fact by releasing a certain quantity of the gas between the first and second series of experiments. Now if moisture be present, the effect of this procedure is to concentrate the impurity at the lower pressure, for on the moment of release there is an intense chilling of the vessel, which will have the effect of precipitating any moisture in suspension in the gas. A small thing will conceal the decrement of thermal capacity, as will be judged from the succeeding experiments. I may point out here that over and above the quantity of moisture (calculated to be some 2 milligrammes) necessary to saturate the volume of 160 c.c. of air at the normal temperature, t_1, the effect of moisture is to cause very serious error. Water, in the form of a liquid, within the sphere must, in fact, be conceived as adding its latent heat of vaporisation to intensify the error. For every milligramme of water evaporating within, there will be thrown down externally a corresponding milligramme of steam. I have already described the results of observations which showed that even with the use of a much safer apparatus for effecting the compression of the gas, a considerable weight of water was, in some way, taken up by the gas in its passage through the pump; so that it was found essential to re-dry the gas immediately before its entrance into the sphere. The presence of moisture in the preliminary experiments is therefore very probable.

Not at first fully recognising the probability of error in the first results, it was some time before I recognised the possibility of a variation of the specific heat with pressure. The ordinary contention that from REGNAULT'S experiments on air at constant pressure the conclusion is legitimate that at constant volume there is no variation of specific heat, was so (apparently) forcible, that any other result was deemed improbable. In truth, however, the assumption of perfectness for air is at variance with the facts, and it is probable that the variation of specific heat is an outcome of its imperfectness. Later on, the experiments on carbon dioxide, a typically

* "On the Specific Heats of Gases at Constant Volume (Preliminary Note)," *loc. cit.*

imperfect gas, revealed a much more marked variation of specific heat with density, which by analogy tends to confirm the results on air. A still more remarkable evidence is afforded by the behaviour of hydrogen, as will be seen later.

The first two series of experiments on air succeeding those published, effected on separate samples of air, brought down the specific heat by some 2 per cent. (The results in the Preliminary Note are not corrected for errors of expansion, &c.), and revealed a variation following the density. With the improved conditions under which my experiments are now carried on, the more recent results are again somewhat at variance with these two, the first effected with the use of the improved pump. I think it best, in the case of each gas, to give the experiments in order of date.

To effect the drying and purifying of the air in these and the succeeding experiments, it is passed through a train of U-tubes, consisting of a tube containing pumice soaked in strong caustic potash, a tube containing pumice soaked in strong sulphuric acid, and then through three tubes packed with dried asbestos fibres, separated by layers of phosphoric anhydride. These tubes are closed with ground stoppers, are some 2 cms in diameter, and afford an effective length of 40 cms. each.

Finally, between receiver and sphere, the high-pressure drying tube, having a length of 40 cms., and a capacity of 31 c.c., also containing phosphoric anhydride, removes what moisture may be taken up in the receiver.

Regarding the mode of tabulating the results of the experiments, it should be mentioned for clearness, that the column headed ω gives the total apparent weight in grammes of precipitation observed in each experiment. The amounts of the corrections, in grammes, to be applied in each case, are given below the columns, as well as the mean of the column ω adjusted for all corrections. This is designated ϖ. On this number, on W (the weight of gas), on the mean of the column λ (the latent heat of steam), and on the mean range, the final result is calculated, which is given at the foot of the table. The number given as the specific heat is the specific heat compared weight for weight with water.

The last column, $\omega \div (t_2 - t_1)$, is added for the purpose of affording a means of judging of the agreement of the experiments *inter se*.

AIR I.—W = 4·3527 grammes. Mean Absolute Density, 0·028406. Mean Pressure, 26·45 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	13·42	99·82	536·6	86·40	0·1332	15417
2	13·60	99·76	536·7	86·16	0·1335	15494
3	13·04	99·36	536·9	86·32	0 1333	15442
	13·35	99·64	536·7	86·29	0·13333	15451

Rate of correction for spheres = − 0·0106 for 86°·52. Sphere B.
Corrections for thermal expansion, &c. = − 0·00090. $\varpi = 0{\cdot}12185$.

Deduced specific heat = 0·17411.

At the conclusion of this series, the air enclosed was passed out through a weighed drying tube, and assurance of the freedom (sensibly) from moisture of the air thereby obtained, as previously described.

Three experiments on the spheres at normal pressures were made after the release of the air.

CORRECTION for Spheres (Air I.).

No.	t_1.	t_2.	ω.
1	13·11	99·22	− 0·0109
2	12·37	99·14	− 0·0106
3	12·72	99·41	− 0·0103

Mean range, 86·52.
Mean ω = − 0·0106.

In the table the sign of ω indicates whether the excess precipitation is on the active or on the idle sphere; when in the former it is marked —, being then subtractive as a correction.

AIR II.—W = 4·8064 grammes. Mean Absolute Density, 0·031201. Mean Pressure, 29·586 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	11·70	99·56	536·8	87·86	0·1356	15434
2	11·27	99·88	536·6	88·61	0·1360	15348
3	12·30	99·83	536·6	87·53	0·1340	15309
4	11·60	99·64	536·7	88·04	0·1358	15425
	11·72	99·73	536·7	88·01	0·13535	15379

Rate of correction for spheres = + 0·0031 for 87°·92. Sphere B.
Corrections for thermal expansion, &c. = − 0·00100. ϖ = 0·13745.

Deduced specific heat = 0·17439.

Some alterations having been made to the spheres, a fresh set of observations on their state of thermal equilibrium was necessitated for obtaining the correction for the last experiments.

CORRECTION for Spheres (Air II.).

No.	t_1.	t_2.	ω.
1	12·39	99·98	+ 0·0028
2	12·30	99·98	+ 0·0029
3	11·71	99·70	+ 0·0032
4	11·37	99·79	+ 0·0035

Mean range 87·92.
Mean ω = + 0·0031.

These two series of experiments will be found plotted as the highest points on the Air Diagram, Plate 6. They show a fall of specific heat with diminishing density, but from after results it appears the rate of decrement is somewhat exaggerated. In subsequent experiments the number of observations in each series was increased.

An attempt was now made to fix points in the line of specific heat at lower pressures. The sphere A was taken into use for the purpose. The number obtained in series III. for the specific heat agrees fairly with the later observations. The succeeding series, IV. and V., are still more in harmony with the best observations made some months later. Although, on the small masses of air used in these series, very definite conclusions as to the exact position of the true line cannot be drawn ;

yet, from the number of experiments made and their agreement among themselves, it is probable that they are a fair approximation to the truth.

A glance at the diagram shows that what is very probably the true line passes fairly well through the mean of IV. and V.

In the series III., IV., and V., the air was separately compressed for each series, and in the case of series III. it was examined for the presence of moisture in the manner before described.

AIR III.—W = 2·3334 grammes. Mean Absolute Density, 0·015654. Mean Pressure, 14·58 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	9·80	100·32	536·3	90·52	0·0686	75784
2	9·20	100·18	536·4	90·98	0·0702	77160
3	10·70	100·07	536·4	89·37	0·0691	77319
4	10·43	99·76	536·6	89·33	0·0688	77018
5	11 51	99·74	536·7	88·23	0·0685	77638
	10·33	100·01	536·5	89·68	0·06903	76974

Rate of correction for spheres = − 0·0012 for 90°·77. Sphere A.
Corrections for thermal expansion, &c. = − 0·00053. ϖ = 0·06729.

Deduced specific heat = 0·17252.

CORRECTION for Spheres (Air III.).

No.	t_1.	t_2.	ω.
1	10·25	99·41	− 0·0012
2	8·60	99·82	− 0·0012
3	8·54	99·96	− 0·0009
4	9·17	99·85	− 0·0016
5	8·48	99·85	− 0·0012

Mean range = 90°·77.
Mean ω = − 0·00122.

AIR IV.—W = 1·4728 grammes. Mean Absolute Density, 0·010335. Mean Pressure, 9·56 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	8·97	100·35	536·2	91·38	0·0446	48807
2	9·44	100·41	536·2	90·97	0·0440	48368
3	9·95	100·41	536·2	90·46	0·0451	49856
4	10·42	100·29	536·3	89·87	0·0442	49182
5	11·26	100·29	536·3	89·03	0·0444	49871
6	12·16	100·29	536·3	88·13	0·0439	49813
7	9·36	99·73	536·7	90·37	0·0454	50238
8	9·63	99·70	536·7	90·07	0·0451	50072
	10·15	100·18	536·36	90·03	0·04459	49528

Rate of correction for spheres = − 0·00196 for 90°·36. Sphere A.
Corrections for thermal expansion, &c. = − 0·00033. ϖ = 0·04230.

Deduced specific heat = 0·17111.

CORRECTION for Spheres (Air IV.).

No.	t_1.	t_2.	ω.
1	9·30	99·97	− 0·0010
2	9·90	99·91	− 0·0019
3	10·50	100·09	− 0·0031
4	9·18	100·49	− 0·0019
5	10·10	100·49	− 0·0020
6	10·37	100·44	− 0·0022
7	9·94	100·44	− 0·0016

Mean range = 90°·36.
Mean ω = − 0·00196.

Air V.—W = 0·9766 gramme. Mean Absolute Density, 0·00728. Mean Pressure, 6·81 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	9·40	99·40	536·9	90·00	0·0302	33555
2	9·60	99·45	536·9	89·85	0·03075	34224
3	8·96	99·87	536.5	90·91	0·0301	33110
4	9·40	99·86	536·5	90·46	0·0300	33164
5	9·74	100·03	536·5	90·29	0·0304	33669
6	10·00	100·06	536·4	90·06	0·03075	34144
7	8·90	99·81	536·6	90·91	0·0311	34210
8	9·51	99·67	536·7	90·16	0·0308	34161
9	8·60	100·07	536·4	91·47	0·0304	33235
	9·34	99·80	536·6	90·46	0·03050	33717

Rate of correction for spheres = − 0·0019 for 89°·32. Sphere A.
Corrections for thermal expansion, &c. = − 0·000217. ϖ = 0·02832.

Deduced specific heat = 0·17202.

Correction for Spheres (Air V.).

No.	t_1.	t_2.	ω.
1	10·16	100·03	− 0·0017
2	10·40	99·96	− 0·0019
3	10·65	99·71	− 0·0026
4	10·27	99·89	− 0·0025
5	11·10	99·83	− 0·0014
6	10·30	99·78	− 0·0016
7	10·50	99·72	− 0·0021
8	10·51	99·93	− 0·0020
9	9·60	99·29	− 0·0017

Mean range 89°·32.
Mean ω = − 0·0019.

Some months after the completion of the foregoing observations, when under more favourable conditions (see *Introduction*) the results of experiments were visibly improved in consistency, and more experience had been obtained, I resumed the observations on air. The series VI., VII., VIII., and IX. were carried out. In these the sphere B alone was used. VI. and VII. are on the one sample of air; so are VIII. and IX. That is, the air for the low density series was in each case provided from that used at higher pressure by suffering part of the latter to escape. As great precautions were taken

(by the use of freshly filled tubes, &c.) to secure effective drying of the air, this procedure was probably not productive of error (see p. 90). In the interval between the series VII. and VIII., experiments on the equilibrium of the vessels were made.

AIR VI.—W = 3·7463 grammes. Mean Absolute Density, 0·024591. Mean Pressure, 23·35 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	14·29	100·30	536·3	86·01	0·1401	16289
2	14·58	100·31	536·3	85·73	0·1393	16249
3	13·74	100·40	536·2	86·66	0·1402	16178
4	14·44	100·40	536·2	85·96	0·1399	16275
5	15·03	100·37	536·3	85·34	0·1392	16311
	14·41	100·35	536·26	85·94	0·13974	16260

Rate of correction for spheres = − 0·0353 for 85°·36. Sphere B.
Corrections for thermal expansion, &c. = − 0·00078. ϖ = 0·10340.

Deduced specific heat = 0·17223.

AIR VII.—W = 2·0923. Mean Absolute Density, 0·014281. Mean Pressure, 13·563 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$
						0·00
1	13·96	100·32	536·3	86·36	0·0937	10850
2	14·78	100·25	536·3	85·47	0·0936	10951
3	14·12	100·30	536·3	86·18	0·0939	10896
4	14·46	100·31	536·3	85·85	0·0932	10856
5	14·45	100·34	536·3	85·89	0·0936	10898
6	13·51	100·40	536·2	86·89	0·0949	10922
	14·21	100·32	536·3	86·11	0·09382	10895

Rate of correction for spheres = − 0·0353 for 85°·36. Sphere B.
Correction for thermal expansion, &c. = − 0·000439. ϖ = 0·05776.

Deduced specific heat = 0·17193.

Correction for Spheres (Air VI., VII., VIII., IX.).

No.	t_1.	t_2.	ω.
1	14·40	100·35	− 0·0358
2	13·87	100·20	− 0·0358
3	14·78	100·17	− 0·0356
4	15·20	100·01	− 0·0348
5	14·60	99·90	− 0·0352
6	15·38	99·78	− 0·0347

Mean range = 85°·36.
Mean ω = − 0·0353.

Air VIII.—W = 4·2854 grammes. Mean Absolute Density, 0·027944. Mean Pressure, 26·62 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	15·60	99·93	536·5	84·33	0·1513	17941
2	14·96	100·03	536·5	85·07	0·1532	18009
3	15·82	100·05	536·5	84·23	0·1522	18070
4	16·50	100·02	536·5	83·52	0·1510	18080
5	15·01	100·18	536·4	85·17	0·1525	17906
6	15·48	100·29	536·3	84·81	0·1528	18016
	15·56	100·08	536·45	84·52	0·15217	18004

Rate of correction for spheres = − 0·0353 for 85°·36. Sphere B.
Correction for thermal expansion, &c. = − 0·000902. ϖ = 0·11629.

Deduced specific heat = 0·17225.

AIR IX.—W = 2·2558 grammes. Mean Absolute Density, 0·015297.
Mean Pressure, 14·53 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	14·78	100·36	536·3	85·58	0·0977	11416
2	14·40	100·55	536·1	86·15	0·0980	11376
3	15·40	100·57	536·1	85·17	0·0982	11530
4	15·40	100·55	536·1	85·15	0·0977	11474
5	14·44	100·30	536·3	85·86	0·0979	11402
6	15·60	100·13	536·4	84·53	0·0962	11381
7	15·64	100·10	536·4	84·46	0·0964	11414
8	15·16	100·14	536·4	84·98	0·0969	11403
	15·10	100·33	536·26	85·23	0·097375	11425

Rate of correction for spheres = − 0·0353 for 85°·36. Sphere B.
Correction for thermal expansion, &c. = − 0·000474. $\varpi = 0{\cdot}06164$.

Deduced specific heat = 0·17192.

Omitting the first two experiments as anomalous, and therefore, probably containing some error or inaccuracy, the results for air stand thus :—

No.	C_v.	Pressure in atmospheres.	Absolute density.
III.	0·17252	14·58	0·015654
IV.	0·17111	9·56	0·010335
V.	0·17202	6·81	0·00728
VI.	0·17223	23·35	0·024591
VII.	0·17193	13·56	0·01428
VIII.	0·17225	26·62	0·027944
IX.	0·17192	14·53	0·015297
Means .	0·171997	15·57	0·016483

From these it is, I think, well established that the mean specific heat at the mean pressure of 15 atmospheres has the value to three figures very closely of 0·172. Taking the means of the last four—by far the most reliable results—*the specific heat of air at constant volume to four figures is* 0·1721, *for the mean absolute density* 0·0205, *and mean pressure* 19·51 *atmospheres.* The close agreement of the experimental numbers is a strong reason for accepting these last four results as reliable.

If it be asked how far this agrees with theory, it may be answered that theory has more than one value to ascribe to the specific heat, for the data with which theory has to deal are not invariable.

Taking the value of the specific heat at constant pressure as 0·2375 (Regnault), and the value of R/E (Clausius, 'Mechanical Theory of Heat') as 0·0691, the number 0·1684 is obtained; assuming C_p to be 0·2389 (Wiedemann), the number becomes 0·1698. But further, in the fraction R/E the exact value of E (the mechanical equivalent of heat) is still in question. Clausius assumes 423·55, but a recent discussion of the results of Rowland* and others appears to indicate more nearly a value 426·8.† This brings the fraction R/E to 0·06858; which, applied to Regnault's result gives 0·1698 as the value of C_v; applied to Wiedemann's result (the most reliable), gives 0·1712 as C_v. These are supposed to be the values at 760 mm. pressure.

Basing speculations on the value of γ and Wiedemann's result for C_p, the number 0·1692 is found for C_v. Finally it may be calculated on the values γ, R, and E only, and if the last is taken as 426·8, C_v is found to be 0·1689.

Of the above results that value which appears most directly based on the most reliable experimental results assigns 0·1712 as the value of C_v at the pressure of one atmosphere. Now, assuming that the specific heat is a linear function of the pressure, the experimental data, VI., VII., VIII., and IX., afford the value 0·17154 for C_v at the pressure of one atmosphere.

On the reality of the variation of the specific heat with change of pressure I may observe that the experiments free of error are almost unanimous in showing such a variation. The coincidence (I might say the *identity*) of the slope of VI. and VII. with the slope of VIII. to IX. is remarkable. It may be objected that the difference of the actual numbers is very small. I find that the "probable error" of the specific heat (so far as the series of values of $\omega \div (t_2 - t_1)$ indicates) is $\pm$ 0·00018 in the case of IX., and $\pm$ 0·00019 in the case of series VIII. If the coincidence of a + error on the lower result with a — error on the higher result for the specific heat be assumed, the numbers just coincide. The lower becomes 0·17210, the higher 0·17206. But the same coincidence has to be assumed for the series VI. and VII. There is, further, the analogy with carbon dioxide, in which case there is no room for doubt as to the variability of the specific heat. In the case of hydrogen it will subsequently be seen there is also very certain variability of the specific heat, but in the *opposite direction*; that is, while in the case of air and carbon dioxide the specific heat increases with increasing density, in the case of hydrogen it diminishes. Now this for all three gases agrees with their behaviour as regards Boyle's law at the pressures covered by the experiments. It is well known that in the case of air and carbon dioxide the imperfectness of the gases is revealed by departures from Boyle's law, small in the one case, large in the other, but in the *opposite direction* to that which is apparent in the case of hydrogen, whose deviation from the law is, as in the case of air, small.

These facts, taken together, are, I think, a strong confirmation of the reality of the

* "On the Mechanical Equivalent of Heat," 'Amer. Acad. Proc.,' 1880.

† 'Thermodynamics,' by De Volson Wood, New York, 1889, p. 25 *et seq.*

variation observed in the experiments on air. For if there is no corresponding reality to the variation shown by nearly all the experiments on air, it is necessary to suppose some source of error varying in the one direction with the pressure. Considered now in connection with the other two gases it is necessary to suppose this error *increasing* in intensity for the more imperfect gas and *reversed in direction* for the super-perfect gas.

It appears therefore that:—

The specific heat of air at constant volume is not constant under variation of pressure, but increases with increase of pressure, between the limits of pressure 10 *to* 30 *atmospheres (which the experiments sufficiently cover) at the mean temperature of* 50° *C.*

If we assume that the function connecting density and specific heat over the range of observation is a linear one, the equation

$$C_v = \rho \times 0{\cdot}02788 + 0{\cdot}17151 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

affords the specific heat as a function of the absolute density ρ, in agreement with the experiments VI. to IX.

The Experiments on Carbon Dioxide.

The gas used in these experiments was generated by the action of "pure" dilute sulphuric acid on bicarbonate of soda. It was stored in a large glass holder under a head of about 50 cms. of water. From the holder it passed through two U-tubes containing sulphuric acid on pumice, and three U-tubes filled with phosphoric anhydride packed with asbestos. Through these the receiver drew the gas as required, and passed it on into the sphere through the high-pressure drying-tube filled with phosphoric anhydride. I give the results in the order in which they were obtained.

It is my intention shortly to endeavour to trace the specific heat of this gas more minutely throughout, and up to its state as a liquid. The spheres for this purpose have indeed already been completed.

Carbon Dioxide I.—W = 2·9283 grammes. Mean Absolute Density, 0·019950. Mean Pressure, 12·2 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	8·30	100·13	536·4	91·86	0·0902	98193
2	9·05	100·03	536·5	91·02	0·0896	98440
3	8·97	99·96	536·5	90·99	0·0885	97263
4	9·73	99·97	536·5	90·24	0·0887	98293
5	9·70	99·97	536·5	90·27	0·0886	98150
6	9·80	99·74	536·7	89·94	0·0875	97287
	9·26	99·97	536·5	90·72	0·08885	97939

Rate of correction for spheres = − 0·00395 for 89°·84. Sphere A used.
Correction for thermal expansion, &c. = − 0·00043. ϖ = 0·08439.

Deduced specific heat = 0·17054.

Corrections for Spheres (Carbon Dioxide I.–III.).

No.	t_1.	t_2.	ω.
1	10·32	99·60	− 0·0026
2	9·24	99·40	− 0·0034
3	9·62	99·38	− 0·0037
4	9·10	100·18	− 0·0041
5	10·70	100·12	− 0·0040
6	10·72	99·90	− 0·0045
7	10·41	99·77	− 0·0040

Mean range = 89°·84.
Mean ω = − 0·00395.

This series of experiments on the spheres also applies to the ensuing two series, II. and III. The same sample of gas is used in II. and III.

CARBON Dioxide II.—W = 2·9016 grammes. Mean Absolute Density, 0·019786. Mean Pressure, 12·1 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	10·80	99·33	537·0	88·53	0·0843	95222
2	11·35	99·33	537·0	89·98	0·0852	96840
3	9·20	99·28	537·0	90·08	0·0870	96581
4	10·40	99·22	537·0	88·82	0·0858	95600
	10·25	99·33	537·0	88·85	0·08558	96320

Rate of correction for spheres = − 0·00395 for 89°·84. Sphere A used.
Correction for thermal expansion, &c. = − 0·00043. ϖ = 0·08115.

Deduced specific heat = 0·16922.

CARBON Dioxide III.—W = 1·5622 grammes. Mean Absolute Density, 0·011530. Mean Pressure, 7·2 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	10·37	99·49	536·9	89·12	0·0475	53299
2	10·40	99 50	536·8	89·10	0·0483	54209
3	9·70	99·50	536·9	89·80	0·0476	53007
4	10·86	99·50	536·9	88·82	0·04785	53873
5	10·66	99·52	536·8	88·86	0·0474	53342
6	9·86	99·86	536·6	90·00	0·0479	53222
7	11·16	99·87	536·6	88·71	0·0476	53658
8	10·84	99·87	536·6	89·03	0·0481	54027
	10·46	99·64	536·7	89·18	0·04786	53667

Rate of correction for spheres = − 0·00395 for 89°·84. Sphere A used.
Correction for thermal expansion, &c. = − 0·00022. ϖ = 0·04367.

Deduced specific heat = 0·16841.

In the next three series the high-pressure sphere is taken into use. The experiments on the thermal equilibrium of the spheres, applying to all three, were made after IV. The gas in V. and VI. is from the same sample.

Carbon Dioxide IV.—W = 5·5524 grammes. Mean Absolute Density, 0·036529. Mean Pressure, 20·90 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	10·00	99·76	536·7	89·76	0·2010	22393
2	10·86	99·76	536·7	88·90	0·1986	22340
3	10·64	99·80	536·6	89·16	0·1998	22409
4	10·55	99·70	536·7	89·15	0·1988	22300
5	9·70	99·76	536·7	90·06	0·2010	22319
6	9·80	99·76	536·7	89·96	0·2004	22277
	10·26	99·76	536·7	89·50	0·19993	22339

Rate of correction for spheres = − 0·0390 for 89°·35. Sphere B used.
Correction for thermal expansion, &c. = − 0·000637. ϖ = 0·16023.

Deduced specific heat = 0·17305.

Correction for Spheres (Carbon Dioxide IV.–VI.).

No.	t_1.	t_2.	ω.
1	10·22	99·13	− 0·0379
2	10·58	99·10	− 0·0401
3	9·40	99·51	− 0·0390
4	10·00	99·60	− 0·0389
5	10·35	99·63	− 0·0381
6	10·16	99·86	− 0·0389

Mean range = 89°·35.
Mean ω = − 0·0390.

CARBON Dioxide V.—W = 5·757 grammes. Mean Absolute Density, 0·037802. Mean Pressure, 21·66 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	10·51	99·96	536·5	89·45	0·2066	23097
2	10·79	100·01	536·5	89·22	0·2056	23044
3	11·00	100·00	536·5	89·00	0·2057	23112
4	11·10	100·00	536·5	88·90	6·2056	23127
5	10·77	100·08	536·5	89·31	0·2064	23110
	10·83	100·01	536·5	89·18	0·20598	23097

Rate of correction for spheres = − 0·0390 for 89°·35. Sphere B used.
Correction for thermal expansion, &c. = − 0·000661. ϖ = 0·16638.

Deduced specific heat = 0·17386.

CARBON Dioxide VI.—W = 4·2654 grammes. Mean Absolute Density, 0·028498. Mean Pressure, 16·87 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	11·16	100·08	536·5	88·92	0·1605	18050
2	10·47	100·11	536·5	89·64	0·1615	18017
3	10·93	100·10	536·5	89·17	0·1610	18055
4	11·00	100·13	536·5	89·13	0·1609	18052
5	10·52	100·40	536·2	89·88	0·1624	18068
	10·82	100·16	536·5	89·34	0·16126	18050

Rate of correction for spheres = − 0·0390 for 89°·35. Sphere B used.
Correction for thermal expansion, &c. = − 0·000510. ϖ = 0·12175.

Deduced specific heat = 0·17141.

The foregoing experiments are plotted in the diagram for carbon dioxide, Plate 6. On the purity of the gas I may observe that, in the course of compressing the gas for one of the experiments, I drew off some of the gas and tested its purity by absorption with caustic potash. A *very small* bubble of air (presumably) remained over. It was impossible, in the tube used, to ascertain its volume; but as this was the impurity in 80 c.c., and as from the approximation of specific heats and, indeed, of densities, a small impurity would not affect the result, I did not think it necessary to delay over its exact estimation.

Connecting, on the diagram, the mean of Experiments I. and II. with the mean of Experiments IV. and V. gives a line probably expressing closely the true locus of the specific heat. From this line the equation for the specific heat as a function of the density is found to be

$$C_v = \rho \times 0{\cdot}2064 + 0{\cdot}16577 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking again the means of Experiments I. and II., IV. and V., as affording reliable results, which seems justifiable, the specific heat of carbon dioxide at constant volume is 0·16988 at the pressure 16·71 atmospheres and density 0·01987, and 0·17345 at the pressure 21·28 atmospheres and density 0·03716; both at the mean temperature of 55° C.

These numbers differ by so considerable an amount as to render it most improbable that the experiments express other than a real variation of the specific heat of carbon dioxide. And the conclusion seems justified that *the specific heat of carbon dioxide is not constant, but increases very appreciably with increase of density throughout the range of pressure* 10 *to* 25 *atmospheres at the mean temperature of* 55° *C.*

I add for convenience a table of the experiments.

Experiments on Carbon Dioxide.

No.	C_v.	Pressure in atmospheres.	Absolute Density.
I.	0·17054	12·20	0·019950
II.	0·16922	12·10	0·019786
III.	0·16841	7·20	0·011530
IV.	0·17305	20·90	0·036529
V.	0·17386	21·66	0·037802
VI.	0·17141	16·87	0·028498

The specific heat at the pressure of one atmosphere is by calculation 0·16552. The number obtained by Clausius from the experiments of Regnault is 0·172. But apparently Clausius' result applies to Regnault's experiments in C_p over the range 11° to 214°. If Clausius' formula* be applied to Regnault's results over the range 15° to 100°, a result departing from the experimental value of C_v in the opposite direction is obtained, *i.e.*, 0·157. The amendment of Clausius' formula for the value of E raises the last number by a trifling amount. The cause of the discrepancy of these results is not apparent. It is suggestive that in the case of the more perfect gases, air and hydrogen, the deduction of the one specific heat from the other is attended by no such wide discrepancy.

* 'Mechanical Theory of Heat,' p. 55. London, 1879.

The Experiments on Hydrogen.

The difficulties in the way of obtaining this gas in a sufficiently pure state proved so great, that what can only be considered as *qualitative* results have been obtained from the experiments up to the present. It is true that as the procedure of generating and storing the gas was refined upon, results closely agreeing with the theoretical value of the specific heat were obtained, but always so far differing from each other as to show *some* difference in the purity of the samples of gas dealt with. It has, hence, been impossible to assign with certainty a definite experimental value to the specific heat.

The earlier experiments are rejected as having been effected on very impure hydrogen, affording values entirely too low. A very small impurity by volume will seriously affect the result of experiment. The error due to an admixture with another gas is, in fact, very nearly proportional to the percentage of impurity by weight.

The hydrogen was generated from granulated distilled zinc, under the action of dilute "pure" sulphuric acid. It was purified, in the case of the immediately succeeding experiments, by passing it through a tube containing pumice and a solution of silver nitrate, and then dried by passage through a tube containing sulphuric acid on pumice, and three tubes of phosphoric anhydride, passing through a fourth tube of the same nature between receiver and sphere. From the generating bottle it passed into a glass gas-holder under a head of about half a metre of water for the supply of the pump.

At the conclusion of the Series I. a portion was allowed to escape and a second series of experiments made on the residue. The gas released from the sphere was in part passed into a eudiometer having a capacity of 50 c.c. and burned with an excess of electrolytically prepared oxygen. The residual oxygen was then treated first with caustic potash (which revealed no carbon dioxide) and then with pyrogallic acid. A small quantity, estimated later (by weighing a bead of mercury occupying the same volume in the tube) as 0·11 c.c., remained. The error by volume was estimated as 1 in 350, or, if the impurity was nitrogen, 1 in 25 by weight. This will be nearly the maximum error to be ascribed to the thermal result. The eudiometric experiment, however, is not satisfactory, as there is, I am informed, a liability to the evolution of carbon monoxide from the alkaline pyrogallol. It was, indeed, frequently found, in subsequent tests of like nature, that it was possible to obtain a second ignition on passing a spark when the absorption had reached a certain stage and apparently ceased. I decided in consequence to discontinue these tests, and had hoped to substitute determinations of density as tests of purity in future experiments, taking Lord Rayleigh's recent researches for comparison in judging of the degree of purity. The method I had intended to use for the determination of density only necessitated the use of a large volumenometer into which the gas from the sphere

might be passed, the weight of the gas being determined by successive weighings of the sphere. This method is described in a forthcoming number of the 'Proceedings of the Royal Dublin Society.* It surmounts or reduces many of the sources of error arising in the application of REGNAULT'S method. However, so far as the present experiments are concerned, I regret to say I have not been able to apply it, as the construction of the volumenometer was not completed in time. Some preliminary experiments on the application of this method appear to show that it is capable of a high degree of accuracy.

The degree of purity of the gas in the experiments from III. onwards must only be inferred from the increasing precautions taken to guard against the entrance of impurities. That higher degrees of purity were obtained as the experiments advanced may, indeed, partly be inferred from the results of the experiments themselves by the observed rise in the deduced specific heats. My subsequent attempts to test for impurities by absorption were so unsatisfactory and contradictory as to be valueless.

HYDROGEN I.—W = 0·3312 gramme. Mean Absolute Density, 0·0021514. Mean Pressure, 29·12 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	10·76	100·19	536·4	89·43	0·1680	18786
2	10·01	100·20	536·4	90·19	0·1691	18750
3	11·43	100·20	536·4	88·77	0·1666	18768
4	10·41	100·06	536·5	89·65	0·1675	18684
5	11·41	100·10	536·4	88·69	0·1656	18672
	10·80	100·15	536·4	89·35	0·16736	18731

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000990. ϖ = 0·12997.

Deduced specific heat = 2·3559.

* 'Roy. Dublin Soc. Proc.,' vol. 6, p. 534.

HYDROGEN II.—W = 0·2370 gramme. Mean Absolute Density, 0·0015646.
Mean Pressure, 21·15 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_1 - t_1)$.
						0·00
1	10·67	100·00	536·5	89·33	0·1295	14497
2	11·99	100·04	536·5	88·05	0·1286	14605
3	10·50	100·12	536·4	89·62	0·1297	14472
4	11·90	100·10	536·4	88·20	0·1288	14603
	11·26	100·06	536·45	88·80	0·12915	14544

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000709. ϖ = 0·09224.

Deduced specific heat = 2·3512.

HYDROGEN III.—W = 0·3274 gramme. Mean Absolute Density, 0·0021271.
Mean Pressure, 28·79 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	10·80	99·86	536·6	89·06	0·1661	18650
2	11·81	99·84	536·6	88·03	0·1638	18607
3	11·30	99·84	536·6	88·54	0·1643	18557
4	11·19	99·78	536·7	88·59	0·1652	18647
5	10·87	99·69	536·7	88·82	0·1653	18611
6	11·70	99·67	536·7	87·97	0·1654	18575
	11·28	99·78	536·65	88·50	0·16468	18608

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000980. ϖ = 0·12760.

Deduced specific heat = 2·3633.

HYDROGEN IV.—W = 0·2191 gramme. Mean Absolute Density, 0·0014525. Mean Pressure, 19·670 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_2 - t_1)$.
						0·00
1	11·51	99·68	536·7	88·17	0·1206	13678
2	10·69	100·01	536·5	89·32	0·1228	13748
3	11·87	100·05	536·5	88·18	0·1223	13869
4	11·60	100·10	536·4	88·50	0·1229	13887
5	10·62	100·26	536·3	89·64	0·1244	13878
6	12·10	100·22	536·4	88·12	0·1213	13765
7	10·50	100·12	536·4	89·62	0·1239	13825
8	11·45	100·08	536·5	88·63	0·1217	13731
	11·29	100·06	536·5	88·77	0·12249	13799

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000656. $\varpi = 0{\cdot}08564$.

Deduced specific heat = 2·3623.

CORRECTION for Spheres (Hydrogen I.–VII.).

No.	t_1.	t_2.	w.
1	12·03	100·00	− 0·0359
2	10·97	99·95	− 0·0361
3	12·40	99·97	− 0·0351
4	12·76	100·00	− 0·0360
5	11·68	100·09	− 0·0360
6	11·70	100·10	− 0·0366

Mean range = 88°·10.
Mean w = − 0·0359.

As seen by the estimation of the nitrogen present in the gas used in I. and II., a maximum error of 5 per cent. may affect these results. But this is very probably excessive. For the succeeding series, III. and IV., further precautions were taken in dealing with the hydrogen. Glass tubes were substituted for rubber as far as possible, the tubes being brought end to end and connected by a thick rubber, varnished externally. A test for the presence of sulphuretted hydrogen, by passing the gas into a tube containing a filter paper soaked in acetate of lead, revealed no trace of that impurity. The next two series, V. and VI., are upon the one sample of gas.

HYDROGEN V.—W = 0·2094 gramme. Mean Absolute Density, 0·0013919. Mean Pressure, 18·82 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·00
1	11·74	100·00	536·5	88·26	0·1191	13494
2	12·20	100·00	536·5	87·80	0·1174	13371
3	12·28	100·00	536·5	87·72	0·1182	13475
4	11·50	99·91	536·5	88·41	0·1183	13381
	11·93	99·98	536·5	88·05	0·11825	13430

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000628. ϖ = 0·08172.

Deduced specific heat = 2·3779.

HYDROGEN VI.—W = 0·1007 gramme. Mean Absolute Density, 0·00071402. Mean Pressure, 9·66 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	12·41	99·89	536·6	87·48	0·0752	85962
2	12·86	99·86	536·6	87·00	0·0753	86552
3	11·92	99·86	536·6	87·94	0·0751	85399
4	13·02	99·85	536·6	86·83	0·0749	86261
5	12·80	99·85	536·6	87·05	0·0754	86617
6	11·80	99·81	536·6	88·01	0·0762	86581
7	12·53	99·80	536·6	87·27	0·0748	85711
	12·48	99·85	536·6	87·37	0·07527	86151

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B. 0·03937.
Correction for thermal expansion, &c. = − 0·000302. ϖ = 0·03937.

Deduced specific heat = 2·4012.

The Series V. and VI. show a fall in the specific heat with increase of density. The specific heat, too, is raised above that obtained in the preceding series. Undoubtedly purer hydrogen had been obtained. What greatly conduced to this was an addition to the arrangements whereby the connection of the gas-holder with the generating bottle could be maintained throughout the operation of pumping. Previously the coupling with the bottle had to be transferred to the high-pressure receiver. The new

arrangement providing for a supply of gas not limited to the capacity of the gas-holder, enabled a much greater quantity to be used as a wash-out in the first instance.

The difficulty of obtaining concordant observations on different samples of hydrogen having become very apparent, I resolved to continue the system of making two observations upon the one sample of gas. It appeared to me, indeed, that a point of far higher interest than the actual value to be ascribed to the specific heat of the gas was the question as to the nature of its variation with change of pressure.

In the very next series, VII., however, my intention of duplicating the experiments was frustrated. In releasing the gas, in fact, at the conclusion of the series, I detected an odour of nitrous fumes, which induced me to seek the source of the impurity and start a fresh series. The source of the impurity turned out to be the tube containing the nitrate of silver, which was probably becoming affected by acid carried by the gas. The tube was suppressed, and a tube containing copper potassium tartrate, carried on pumice, substituted. The use of this tube entirely removed the odour of H_2S which the gas quitting the generating bottle had to some extent, and, in fact, the gas passed on quite free from smell.

It is improbable that VII. is affected appreciably by the amount of oxide of nitrogen present. A very small quantity would have sufficed to impart the odour.

HYDROGEN VII.—W = 0·2350 gramme. Mean Absolute Density, 0·0015513. Mean Pressure, 20·99 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_2 - t_1)$.
						0·00
1	12·01	99·82	536·6	87·81	0·1282	14600
2	12·69	99·81	536·6	87·12	0·1270	14578
3	13·00	99·78	536·7	87·78	0·1268	14612
4	13·00	99·80	536·6	86·80	0·1280	14747
5	13·28	99·80	536·6	86·52	0·1273	14713
6	12·12	99·90	536·6	87·78	0·1288	14673
	12·68	99·82	536·6	87·14	0·1277	14655

Rate of correction for spheres = − 0·0359 for 88°·10. Sphere B.
Correction for thermal expansion, &c. = − 0·000705. ϖ = 0·0915.

Deduced Specific Heat = 2·3975.

The next two series, VIII. and IX., are on the one sample of gas. This would appear to have been the purest sample of gas prepared. The specific heat has risen quite to the theoretic value, and the fall of specific heat with increasing density is well marked.

Series X., XI. and XII. are all on the one sample. The extremes X. and XII. agree with the testimony of all the latter experiments as to the nature of the variation of the specific heat with density, but the central series is discordant.

HYDROGEN VIII.—W = 0·2979 gramme. Mean Absolute Density, 0·0019432. Mean Pressure, 26·412 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_2 - t_1)$.
						0·00
1	13·50	99·74	536·7	86·24	0·1522	17648
2	12·22	99·86	536·6	87·64	0·1544	17618
3	12·97	99·81	536·6	86·84	0·1535	17676
4	13·46	99·78	536·7	86·32	0·1510	17493
5	12·20	99 95	536·5	87·75	0·1536	17504
6	13·08	99·98	536·5	86·90	0·1527	17572
	12·90	99·85	536·6	86·95	0·1529	17585

Rate of correction for spheres = − 0·0352 for 86°·09. Sphere B.
Correction for thermal expansion, &c. = − 0·000897. ϖ = 0·11646.

Deduced specific heat = 2·4120.

HYDROGEN IX.—W = 0·1737 gramme. Mean Absolute Density, 0·0011693. Mean Pressure, 15·876 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_1 - t_1)$.
						0·00
1	12·90	100·00	536·5	87·10	0·1040	11940
2	12·98	100·04	536·5	87·06	0·1045	12003
3	12·50	100·04	536·5	87·54	0·1044	11926
4	13·20	100·04	536·5	86·84	0·1045	12034
5	13·67	100·05	536·5	86·38	0·1050	12156
6	12·98	100·08	536·5	87·10	0·1049	12044
	13·04	100·04	536·5	87·00	0·10455	12017

Rate of correction for spheres = − 0·0352 for 86°·09. Sphere B.
Correction for thermal expansion, &c. = − 0·000523. ϖ = 0·06849.

Deduced specific heat = 2·4320.

HYDROGEN X.—W = 0·3050 gramme. Mean Absolute Density, 0·0019874. Mean Pressure, 26·848 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_2 - t_1)$.
						0·00
1	12·93	99·70	536·7	86·77	0·1546	17817
2	13·13	99·74	536·7	86·61	0·1543	17815
3	13·50	99·72	536·7	86·22	0·1533	17780
4	12·85	99·82	536·6	86·97	0·1544	17753
	13·10	99·74	536·6	86·64	0·15415	17792

Rate of correction for spheres = − 0·0352 for 86°·09. Sphere B.
Correction for thermal expansion, &c. = − 0·000914. $\varpi = 0{\cdot}11784$.

Deduced specific heat = 2·3928.

HYDROGEN XI.—W = 0·2100 gramme. Mean Absolute Density, 0·0013955. Mean Pressure, 18·938 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t_1$.	w.	$w \div (t_2 - t_1)$.
						0·00
1	12·61	99·97	536·5	87·36	0·1176	13462
2	13·17	99·96	536·5	86·79	0·1178	13573
3	13·40	99·95	536·5	86·55	0·1168	13495
4	13·24	99·78	536·7	86·54	0·1169	13508
5	13·66	99·78	536·7	86·12	0·1166	13539
6	13·94	99·78	536·7	85·84	0·1167	13595
7	13·20	100·09	536·5	86·89	0·1171	13477
	13·32	99·90	536·6	86·58	0·11707	13522

Rate of correction for spheres = − 0·0352 for 86°·09. Sphere B.
Correction for thermal expansion, &c. = − 0·000632. $\varpi = 0{\cdot}08104$.

Deduced specific heat = 2·3917.

HYDROGEN XII.—W = 0·1087 gramme. Mean Absolute Density, 0·0007632.
Mean Pressure, 10·396 Atmospheres.

No.	t_1.	t_2.	λ.	$t_2 - t$.	ω.	$\omega \div (t_2 - t_1)$.
						0·000
1	14·10	100·14	536·4	86·04	0·0768	89261
2	13·34	100·40	536·2	87·06	0·0776	89134
3	14·18	100·40	536·2	86·22	0·0774	89770
4	14·56	100·40	536·2	85·84	0·0780	90867
5	13·86	100·38	536·2	86·52	0·0773	89344
6	14·59	100·35	536·3	85·76	0·0780	90951
7	14·75	100·32	536·3	85·57	0·0773	90335
8	14·69	100·31	536·3	85·62	0·0776	90633
9	13·82	100·27	536·3	86·45	0·0779	90110
	14·22	100·33	536·3	86·11	0·07754	90048

Rate of correction for spheres = − 0·0352 for 86°·09. Sphere B.
Correction for thermal expansion, &c., = − 0·000329. ϖ = 0·04201.

Deduced specific heat = 2·4135.

CORRECTION for Spheres (Hydrogen VIII.–XII.).

No.	t_1.	t_2.	ω.
1	12·30	99·91	− 0·0359
2	12·68	99·80	− 0·0360
3	14·50	100·21	− 0·0355
4	14·78	100·20	− 0·0339
5	14·20	100·15	− 0·0354
6	15·10	100·13	− 0·0349
7	14·70	100·14	− 0·0350
8	13·90	100·23	− 0·0347

Mean range = 86°·09.
Mean ω = − 0·0352.

The diagram for Hydrogen, Plate 6, is a sufficient résumé of the foregoing experiments. It will be understood that the lines throughout the diagram connect experiments effected on gas of the same purity.

The two lines low down on the diagram depict the first four series in which there was very certainly a greater degree of impurity than in subsequent series. The impurity consisting almost certainly of air, its effect will be to conceal the true varia-

tion of the specific heat with density; and, in fact, the impurity has done so and even reversed it; less in the cases of III. and IV., than in I. and II.

The next two series are higher on the diagram, and a well marked (probably excessive) downward slope of the line is apparent. The lines connecting VI. and V., X. and XII., agree as regards the negative variation of the specific heat with increased density. X. and XI. are discordant.

Nearly all the experiments on the purer gas being at one upon this point, it is, I think, allowable to assume that these experiments prove that *the specific heat of hydrogen is not constant, but varies with the density, diminishing with increase of density, and vice versâ.*

As already remarked, this seems in harmony with the nature of the departure of this gas from BOYLE's law.

The theoretic value assigned by CLAUSIUS to the specific heat of hydrogen is 2·411. Experiment VIII. is just this number, IX. rises above it. The rest lie less or more below it.

Conclusion.

The principal result of these first experiments on gases is the broad fact of the departure of the specific heat at constant volume from constancy. Some such result might have been, perhaps, anticipated from the known imperfectness of all gases. But the invariability of the two specific heats is in text books most generally insisted on. It is probable that a gas whose molecules were without action upon one another would show invariability of specific heat. As it is, it appears that the mutual influence of the molecules of some gases being of a positive nature, that of others (hydrogen molecules at least) may be of a negative nature. But the subject is only touched upon in this first paper. Many questions remain to be answered. Is the specific heat dependent on temperature limits? If the perfectness of a gas depends upon its temperature, how will this affect the nature of the relation between specific heat and density? Will the rate of variation alter? Presumably it will. It is not improbable that at very high temperatures the slope of the line for air (or even for carbon dioxide), after approaching more and more nearly to parallelism with the axis of density, will take up the position of a perfect gas, and even assume the negative inclination of the super-perfect gas. Similarly with rise of temperature the slope of the line for hydrogen may become intensified; or, if we suppose the range to be sinking, the opposite may be the course of events, hydrogen ultimately falling into agreement with the behaviour of the other gases.

There may be for each gas a temperature, such that for it the gas behaves in all respects as a truly perfect gas, in which neither a positive nor a negative latent heat would become apparent at high densities. But, again, it may be that a truly perfect gas is not possible in nature, that is, with molecules of the one kind. For

t is evident that a gas undistinguishable in its thermal behaviour from a perfect gas could be compounded out of a mixture, in the proper proportions, of say air and hydrogen.

For dealing, at least to some extent, with the question of the influence of temperature on the specific heat, arrangements are being made, and I am in hopes that reliable quantitative measurements will be found attainable.

Other questions—as that of the mutual influence, thermally, of different intermixed gases, will suggest themselves, probably, as fields of research in which the present method may find application.

IV. *On the Determination of the Boiling-Point of Sulphur, and on a Method of Standardising Platinum Resistance Thermometers by reference to it.—Experiments made at the Cavendish Laboratory, Cambridge.*

By H. L. Callendar, *M.A., Fellow of Trinity College, Cambridge, and*
E. H. Griffiths, *M.A., Sidney Sussex College, Cambridge.*

Communicated by Professor J. J. Thomson, *F.R.S.*

Received November 29,—Read December 18, 1890.

[Plates 7, 8.]

General Summary of the Work.

The results of many careful experiments by different observers in the course of the past few years have shown that electrical resistance thermometers afford the most accurate and convenient method of measuring temperature through a very wide range.

The object of the following paper is to illustrate a method of reducing the indications of such thermometers to the absolute scale of temperature by means of a single observation at some known fixed point in addition to 0° and 100° C.

The boiling-point of sulphur happens to be the most convenient to choose for this purpose. We have therefore made a very careful determination of this point by reference to the normal air thermometer, and have given a description of the method and apparatus which we have found most suitable for standardising resistance thermometers by means of it.

In a communication* which one of us had the honour to make to the Royal Society in June, 1887, "On the Practical Measurement of Temperature," it was shown that, if t be the temperature by normal air thermometer, and pt be the temperature by platinum-resistance thermometer, the difference between them is very closely represented from 0° to 700° C. by the formula,

$$t - pt = \delta \{t/100^2 - t/100\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d).$$

The value of the constant δ for a particular specimen of wire was found to be 1·570

* Callendar, 'Phil. Trans.,' A, 1887, p. 161.

by direct comparison with the air thermometer, the platinum spiral used in the experiments being placed inside the bulb of the air thermometer itself.

In a recent communication* "On the Determination of some Boiling and Freezing Points by means of the Platinum Thermometer," some experiments are described in which the thermometers used were standardised by assuming the boiling points of naphthalene (218°·06 C.), benzophenone (306°·08 C.), and sulphur (448°·34 C.), as determined by CRAFTS and REGNAULT.

Curves were drawn with these data for each thermometer giving the difference, $d = t - pt$, in terms of pt. It was found that the upper portions of these curves differed materially (see Plate 8) from the parabolic formula (d) above given.

In order to compare the results thus obtained with those deduced from the δ-formula, a thermometer (L) was constructed out of a piece of the spiral used in the air thermometer experiments of 1887, for which the value of the constant δ was known to be 1·570.

Observations were taken with this thermometer in sulphur vapour in the same apparatus which had been used for standardising the other thermometers. The results gave $pt = 418{\cdot}5$ as the platinum-temperature attained by the thermometer. This gives $t = 442{\cdot}3$, assuming $\delta = 1{\cdot}570$.

The value thus found was nearly six degrees lower than that given by REGNAULT. This showed either that the value of δ was wrong, or that the thermometer did not attain the temperature of the sulphur vapour, or else that the number given by REGNAULT was too high.

But when the number thus found was used to calculate the appropriate values of δ for each of the other thermometers it appeared that the values of t for all the other boiling- and freezing-points, deduced by means of formula (d) from the observations with all the different thermometers, were in remarkably close agreement.

This suggests a very simple method of standardising such thermometers. Assuming that the (d) formula holds good, it is only necessary to observe the platinum-temperature of sulphur vapour with each thermometer, in order to find the appropriate value of δ, and to reduce all the platinum-temperatures to the air standard.

It was felt, however, that the value of δ assumed for the standard wire required verification. The wire had been very roughly treated, and used for all sorts of purposes since the experiments of 1887, and might well have altered in the interval. Two new thermometers, M_1 and M_2, were therefore constructed out of the remainder of the old spiral, and were very carefully compared with the air thermometer at a temperature very near the boiling-point of sulphur. Owing to the importance of this point, the experiments are very fully described in the following pages, Part I. The result agrees perfectly with that found in 1887, and shows that the value of the δ-coefficient has not altered appreciably in the interval.

The apparatus used for this comparison, although useful as a constant high-

* GRIFFITHS, *supra*, p. 43.

temperature bath, was not very well suited for determining the actual temperature of the sulphur vapour. Another series of experiments was therefore undertaken in a Meyer tube, which proved to be more convenient for the purpose. The platinum thermometers, which had been standardised as above mentioned, were used for this determination with various screens and other precautions to guard against errors due either to super-heating of the vapour or to loss of heat by radiation and condensation.

The experiments with the Meyer tube are described in Part II. They show conclusively that errors due to radiation, &c., are of serious magnitude at this temperature, and that, in order to find the true boiling-point, it is not sufficient merely to insert a thermometer in the vapour, as we find has generally been the practice among previous observers.

According to our experiments with carefully screened platinum thermometers, the temperature by normal air thermometer at constant pressure, of the saturated vapour of sulphur, boiling freely under a pressure of 760 mm. of mercury at 0° C. and $g = 980{\cdot}61$ C.G.S. (sea-level in latitude 45°), is

$$t = 444^{\circ}{\cdot}53 \text{ C.}$$

We have taken great pains to verify all the work of reducing the observations, and have carefully tested all the apparatus employed. We believe that the above result may be relied on to the tenth of a degree.

The number given by Regnault is nearly four degrees higher than this. His results have been so widely quoted and adopted that we have thought it necessary to add a short criticism of his method, and to suggest an explanation of the discrepancy.

A table is also given containing the boiling- and freezing-points of several substances, reduced from the previous observations with the other thermometers.*

There can be no doubt that, even at low temperatures, the platinum thermometer is a far more accurate and sensitive instrument than the mercury thermometer. With suitable apparatus it is easy to read to 0°·001 C., and there is no trouble about shifting of zero. We have made a careful comparison of the platinum and air thermometers between 0° and 100° C. The experiments are described in Part III., and we find that the δ-formula holds accurately between these limits.

In setting up the apparatus and taking the observations, we have received great assistance from Mr. G. M. Clark, B.A., of Sidney Sussex College. Our best thanks are due to him for many valuable suggestions.

* Griffiths, 'Phil. Trans.,' A, 1891.

Part I.

Comparison of Platinum and Air Thermometers at the Boiling-Point of Sulphur.

When we found that the preliminary experiments with thermometer L made the boiling-point of sulphur so much lower than the value given by Regnault, we at first endeavoured to account for it by supposing that the wire had altered in the four years' interval, or that the previous air thermometer experiments were faulty. In any case they were felt to require verification.

An iron-tube apparatus was therefore constructed, into which an air thermometer could be introduced, along with two platinum thermometers. The apparatus was so arranged that it could be kept at a steady temperature by a continuous flow of sulphur vapour for any length of time. Two new platinum thermometers, M_1 and M_2, were made out of the remainder of the wire of the old spiral, and were compared with the air thermometer in the iron-tube apparatus on two different days.

Description of the Air Thermometer.

The form of instrument employed is shown in Plate 7, fig. 1. It was similar in every respect to that described and figured in the paper* above referred to, except that the bulb B was formed of ordinary German glass, and not of hard glass.

A piece of tubing, about $2\frac{1}{2}$ cm. in diameter, was taken, and cleaned in the usual way. One end was closed in the blowpipe, and the other was drawn out into a fine capillary tube, a metre long. This was passed through a small brass tube, the end of which was split into four for a length of 15 cm., and bound round the bulb with wire, to serve as a support and to protect the capillary tube. The brass tube was firmly fastened to a wooden stand carrying the volume- and pressure-gauge.

The volume of the bulb was determined by water calibration up to a marked zero-point O on the capillary tube. In using the thermometer care was taken to subject it up to the point O to the temperature to be measured. A short distance beyond this point the brass tube was cooled by winding closely round it several turns of thin lead tubing, through which water was made to circulate. The volume of the capillary tube, for some distance on either side of the point O, was only ·00225 c.c. per cm. The uncertainty of a centimetre or two in the position of the neutral point would not therefore introduce appreciable errors. As a matter of precaution a platinum wire was originally inserted in the brass tube, to take its mean temperature, but it was found that the corrections were always so small that they could be neglected.

A length of about 30 cm. of the capillary tube was bent into the form of a narrow U, shown at C. This served to give the end free play, and to protect the tube from strain.

* 'Phil. Trans.,' A., 1887, p. 167.

The end of the capillary tube had been previously cut off, and ground flat to fit the end of the gauge-tube. A short double funnel of glass was drawn to fit them both closely, and sealing-wax was run in from either side. This form of joint was generally used by REGNAULT. It is perfectly air-tight if properly made, and is certainly quite unobjectionable in the case of a constant-pressure air thermometer.

Linear Expansion of German Glass.

The expansion of the glass forming the bulb was determined by the method described in the previous paper,* using the remainder of the piece of glass tubing from which the bulb had been drawn.

The glass tube was enclosed in two coaxial tubes of brass, separated from each other by strips of asbestos cloth, and was heated in a thick iron tube in a long gas furnace. The whole apparatus was supported on levelling-screws for focussing. The gas furnace was screened with bright tin plate, and all the precautions were taken which have been previously described.

The mean temperature of the portion heated was determined by means of a known platinum wire (the same piece as was used in thermometer L), extending down the axis of the tube, and supported at intervals by mica plates.

The ends of the tube were drawn down to 5 mm. diameter, to diminish conduction of heat, and to imitate as closely as possible the actual condition of the bulb. They projected on either side for a distance of about 15 cm. beyond the gas furnace.

A large and fairly rigid brass tube with end pieces projecting at right angles was supported, so that reference marks at either end of it could be brought into focus under the reading microscopes close to the ends of the glass tube. Its temperature, which was always very nearly that of the surrounding air, was taken by means of a delicate mercury thermometer placed inside it. Corrections were applied for the small variations in its temperature, and the changes in the length of the glass tube were determined by direct comparison with it.

Five series of observations were taken with this apparatus on different days. The highest temperature reached in each series was a little over 400° C. It was always taken first to avoid the permanent contraction. After the first series, the tube was found to have contracted by ·056 per cent. of its length; but towards the end of the experiments it appeared to reach a steady state.

The whole of the observations on the linear expansion of this glass were very closely represented by the formula

$$l/l_0 = 1 + \cdot00000960\, t + \cdot0000000057\, t^2.$$

This gives for the cubical expansion between 0° and 100° C. the factor

$$v_1/v_0 = 1\cdot003051.$$

* 'Phil. Trans.,' A., 1887, p. 206.

This factor appeared to be unusually high even for this kind of glass. By way of verification, the bulb of the air thermometer itself was used as a mercury weight thermometer, after it had been dismounted, at the conclusion of the whole series of experiments. It was very carefully and completely filled with pure mercury, and placed in melting ice for an hour and a half. It was then heated in steam at 99°·89 C., till some time after the mercury ceased to drop out. The mercury which had run out was very carefully weighed and found to be 12·0007 grms. The weight of the mercury filling the bulb at 0° up to the zero point (deducting the weight filling the known volume of the capillary tube exposed to the air temperature), reduced to *vacuo*, was found to be 809·023 grms., whence V_0, the volume of the bulb at 0° C., is 59·505 c.c., assuming that the density of mercury at 0° C. is 13·5960 grms. per c.c.

If V_0, V_1 be the volumes of the bulb at 0° and 99°·89 C., and ρ_0, ρ_1 the corresponding values of the densities of mercury, we have the following equation

$$V_1 \rho_1 + 12{\cdot}0007 = V_0 \rho_0.$$

The linear expansion experiments give

$$V_1/V_0 = 1{\cdot}0030476\,;$$

whence

$$\rho_0/\rho_1 = 1{\cdot}018154.$$

Regnault's experiments on the absolute expansion of mercury give for the expansion between 0° and 99°·89 the value $\rho_0/\rho_1 = 1{\cdot}018133$. The value found by assuming the linear expansion is too large, therefore, by one part in a thousand only. This degree of accuracy is amply sufficient for our purpose, as the whole correction for the expansion of the bulb only amounts to 11°·5 at the temperature of boiling sulphur.

Changes in the Volume of the Bulb.

The volume of the bulb was determined from time to time during the course of the experiments. It is interesting to trace the changes which it underwent.

Before mounting, the volume of the bulb was found by direct calibration to be 59·710 c.c. At the close of the experiments, after repeatedly boiling in sulphur, it was found to be 59·505 c.c. The volume at intermediate times was determined by observations in ice and steam in the following manner:—

Shortly after mounting, the bulb was heated in a gas furnace, and dried by exhaustion at a temperature of about 250° C. Observations taken with it next day in ice and steam, assuming the volume of the bulb to be $V_0 = 59{\cdot}710$, gave the value $\theta_0 = 272{\cdot}12$ as the absolute temperature of melting ice by air thermometer at

constant pressure. This gives the value 1·36749* for the dilatation between 0° and 100° C., which is approximately that found by Balfour Stewart, Chappuis, and the most careful of recent observers.

At the close of the experiments the bulb was repeatedly exhausted while in boiling sulphur, and filled afresh with dry air. Observations taken with it next day in ice and steam, before it was dismounted (assuming the volume of the bulb to be 59·505 c.c.), gave the value $\theta_0 = 272{\cdot}10$, which is practically identical with that previously found.

We may, therefore, reasonably assume the same value of this constant in all cases. We can then use the observations in ice and steam to calculate the volume of the bulb on intermediate days.

This is the most accurate way in which the volume of the bulb can be determined without dismounting it. All the conditions are the same as those under which the value of the constant θ_0 is determined. Any error in the value of V_0 calculated by this method will not affect the temperatures deduced to any greater extent than an equal error of observation in determining θ_0.

In a trial experiment in the iron tube apparatus on Tuesday, September 9, in which the sulphur distilled over too fast to allow good observations to be taken, the bulb was exposed to a temperature of 450° for nearly a quarter of an hour. Next day its volume was found to have diminished from $V_0 = 59{\cdot}710$ to $V_0 = 59{\cdot}574$. Showing that, although it had been subjected to no stress whatever, the contraction was very rapid at first.

On Friday, September 12, the thermometer was kept in sulphur vapour for nearly two hours. The temperature appeared to rise slightly owing to the contraction of the bulb, for a short time after the other thermometers had become steady. After the experiment the volume was found to have diminished from $V_0 = 59{\cdot}574$ c.c. to $V_0 = 59{\cdot}518$ c.c.

On Saturday, September 13, the thermometer was again heated in sulphur vapour. Several observations were taken, which gave no indications of change. The bulb was then exhausted fifteen times, and dry air readmitted. In spite of the great stress to which it was thus subjected, at a temperature not far below its softening point, its volume was afterwards found to have diminished very little, namely, from $V_0 = 59{\cdot}518$ to $V_0 = 59{\cdot}505$ c.c.

This seems to indicate that the bulbs of mercury thermometers may be practically freed from rise of zero by a preliminary boiling in sulphur before filling them with mercury. The sulphur can be readily boiled for hours in a common test-tube or

* Regnault gives for air at constant pressure 1·3670, for air at constant volume 1·3665. Balfour Stewart ('Phil. Trans.,' 1863) finds 1·36733 for air at constant volume. Chappuis ('Archives des Sciences,' Genève, vol. 20, 1888, pp. 5–36) finds for nitrogen at constant pressure the value 1·36747. The two latter observers appear to have used more accurate methods and to have taken greater precautions than Regnault.

beaker without requiring any attention, provided that the top is covered with a piece of asbestos to prevent its catching fire. The method of boiling in mercury, which is often adopted, takes a very long time, and would probably give less perfect results, at least in the case of thermometers intended for use at temperatures above 300°.

The Volume- and Pressure-Gauge.

The gauge used with this thermometer for measuring the variations in the volume and pressure of the enclosed air, was the same as that described and figured in the paper before referred to.* The figure is reproduced for convenience in Plate 7, fig. 2. The U-tube FHJK represents the sulphuric acid pressure-gauge for indicating the small differences between the external pressure and the pressure of the air enclosed. The bulb A, together with the rest of the tubes and taps, represents the volume-gauge for measuring the dilatation at constant pressure.

The use of the sulphuric acid pressure-gauge is particularly advantageous in connection with a constant-pressure air thermometer. Its employment makes it possible to take a series of readings very rapidly when the temperature is varying slightly. The reading of the gauge can be instantly recorded at any desired moment, and the other observations, such as weighing the mercury and reading the barometer, may be made comparatively at leisure. These and other advantages have been already explained in the previous paper. They have since been very fully confirmed and illustrated by Bottomley† ('Phil. Mag.,' Aug., 1888, p. 149).

The gauge for measuring the variations of volume of the enclosed air consists essentially (see Plate 7, fig. 2) of a large bulb A containing mercury, and communicating with the bulb of the air thermometer by the way ED. It is surrounded by a water bath (shown in fig. 1), whose temperature can be accurately observed by means of a delicate thermometer. Mercury can be either introduced into, or withdrawn from this bulb in weighed quantities, by means of the taps T_2 and T_3, from weighed beakers of mercury, into which the tubes open at P_2, P_3. Thus, the volume, temperature, and pressure of the air in the bulb A at any moment can be accurately determined.

* 'Phil. Trans.,' A, 1887, Plate 12.

† The reader who wishes to use an apparatus of this kind should certainly refer to his paper, as the explanations given are very full and detailed; but in our opinion the form of pressure-gauge which he describes is neither so simple nor convenient as that shown in Plate 7, fig. 2. The single three-way tap, T_1, can be made to serve all the purposes of the two taps shown in his figure (*loc. cit.*, p. 152). It can also be turned *in the middle of the acid*, so that it cannot move either way, whatever the variations of pressure. This is a most essential provision of great practical convenience, the absence of which is a serious defect in his form of gauge. The addition of the volume-gauge (Bulb A, &c., Plate 7, fig. 2), which makes it possible to use the air thermometer at constant pressure as well as at constant volume, is also a great advantage, and renders the apparatus much more complete.

Method of taking Observations.

In taking a series of observations at constant pressure with this instrument, the bulb A is first filled from the reservoir Q, till the level of the mercury stands at some point in the tube DE. The level x' is recorded by means of the scale S. The temperature t'_0 of the water-bath is also taken at the same time. These two numbers, as will presently appear, give the zero point from which the volumes are calculated. They remain the same throughout any one series of observations.

Mercury is then allowed to run out by the tap T_3 into a weighed beaker at P_3, until the sulphuric acid gauge FHJK shows that the pressure of the air in the bulb is nearly equal to that of the atmosphere. The water-bath surrounding the bulb A is well stirred; its temperature t' is recorded; and also t'' the temperature of the air. The reading x of the acid in the pressure-gauge FG, is taken at the same instant as the resistance of the platinum thermometer. The height H of the barometer is immediately observed, and the weight w of the mercury which has been allowed to run out, is taken at leisure.

The pressure-gauge FG has a range of about 7°, so that, if the temperature is varying slowly, a considerable number of observations may be taken for one weighing of the mercury.

Formulæ.

The temperature is calculated by means of the following formulæ :—

If p be the pressure of the air enclosed, V the volume of the bulb, and θ its absolute temperature, and if v be the volume of any portion of the air contained in the capillary tube, or other parts of the apparatus, at a temperature θ', the formula of the air thermometer may be stated as follows :—

$$p\{V/\theta + \Sigma v/\theta'\} = mk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where mk is a constant depending on the mass of air enclosed, and Σ implies that the summation is extended to all parts of the apparatus which it occupies.

To find θ we deduce from formula (1),

$$\theta = pV/\{mk - \Sigma pv/\theta'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The values of the quantities included in this formula, are deduced from the quantities observed as follows :—

Reduction of the Pressure, p.

The observed height H of the barometer is expressed in centimetres of mercury at 0° C. at sea-level in latitude 45°. A correction of + ·005 cm. is added to allow for

capillarity and vapour tension of mercury at 15°–20° C. The reading is also corrected for scale errors, for the error of the attached thermometer, and for any difference of level between the cistern and the air thermometer.

The barometer used is one of the Fortin pattern by CASELLA, with a scale very clearly divided in inches, and a vernier reading to ·001 inch. It was carefully compared with the Cavendish Laboratory standard metre scale, which has been verified at the Board of Trade Standards' Department. The scale errors were found to be exceedingly small, but the ivory point was set ·085 cm. too high.

To find the pressure of the air in the bulb, a small correction for the difference of level of the acid in either limb of the pressure-gauge has to be added. A preliminary experiment gave $x = 2{\cdot}50$ as the reading of the gauge when the pressure was the same on either limb. The factor for reduction to mercury was found by comparison with a mercury manometer to be 0·136. If H_0 be the corrected height of the barometer we have, therefore,

$$p = H_0 + {\cdot}136\,(x - 2{\cdot}50) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Calculation of the Volume.

The volume V of the bulb at the temperature t is found from the equation already given,

$$V/V_0 = 1 + {\cdot}00002880\,t + {\cdot}0000000171\,t^2 \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The volume of that part of the capillary tube which was exposed to the temperature of the air, was found by mercury calibration to be 0·164 c.c.

The volume of the gauge-tubes CDE, DF, from the end of the capillary tube to the zero point of the scales at F and E, was found be 0·483 c.c.

The volume per cm. of the gauge-tube FG was 0·0232 c.c. If x be the reading of the gauge, the total volume at the air-temperature t'' is, therefore,

$$v'' = 0{\cdot}647 + {\cdot}0232\,x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The volume of air v' in the bulb A, at the temperature t' of the water-bath, is found as follows. Let w_0 be the total weight of mercury filling the bulb A up to the zero point of the scale S, and let ρ_0 be its density at the temperature t'_0 of the bath, when the reading x' of the scale S is taken, before running out the mercury.

Let w be the weight of mercury run out, and let ρ be its density at the temperature t' of the water-bath at the time of observation. Then, if m be the coefficient of expansion of mercury, we have $\rho / \rho_0 = 1 - m\,(t' - t'_0)$.

The volume of the mercury remaining in the bulb will be $(w_0 - w)/\rho$. The volume per cm. of the tube DE is 0·0127 c.c. If the mercury stood at x' before running out,

a small correction $(19 - x')\ \cdot 0127$ has to be added to the above expression, the point 19·0 of the scale S being the zero from which the volume of the bulb A is reckoned.

The total volume of the bulb up to the zero point is w_0 / ρ_0 at the temperature t'_0. If g be the coefficient of expansion of glass, its volume at t' will be $\{1 + g(t' - t'_0)\}\ w_0 / \rho_0$.

The volume v' of the air in the bulb A at the temperature t', is the difference between the volume of the glass envelope and the volume of the mercury remaining in it at the time of observation. It is given by the expression

$$\{1 + g(t' - t'_0)\}\ w_0 / \rho_0 - (w_0 - w) / \rho - (19 - x')\ \cdot 0127,$$

which reduces to

$$w / \rho - (m - g)(t' - t'_0)\ w_0 / \rho_0 - (19 - x')\ \cdot 0127 \quad . \quad . \quad . \quad . \quad (6).$$

The value of the coefficient $(m - g)$ is ·000151, so that the second term in the above expression is always very small. The volume w_0 / ρ_0 of the bulb A is approximately 48 c.c. The whole correction may be neglected, provided that the rise in temperature $(t' - t'_0)$ of the water-bath is less than a tenth of a degree. It is not, as a rule, worth while to reduce the weight of the mercury to *vacuo*, since the correction amounts to only 1 in 20,000.

Sources of Error.

The great advantage of using the air thermometer at constant pressure is that we can dispense with the mercury manometer. The apparatus it involves is generally cumbrous, the observations are slow and laborious, and it is difficult to rely on the result beyond the nearest tenth-millimetre.

In reading the barometer, on the other hand, we are measuring the height of a mercury column under much more favourable conditions. A greater degree of accuracy can be attained; a kathetometer is not required; and it is comparatively easy to verify the scale.

The operations of weighing the mercury and of reading the volume-gauge are capable of being performed with very great accuracy. An error of a decigram in the weight w would produce an error of only + 0°·004 in the resulting value of t at 50° C., and of + 0°·021 at 445° C.

The mercury thermometer used for taking the temperature of the water-bath could be relied on to at least a fiftieth of a degree, and the mean of several observations would probably be correct to 0°·01. It had been carefully compared with the platinum thermometer H,* and its readings were by this means corrected to the air standard. An error of + 0°·01 would produce an error in t of only − 0°·003 at 50° C., and of − 0°·041 at 445° C.

An error of + ·01 cm. in reading the sulphuric acid pressure-gauge would produce an error of + 0°·0086 at 50° C., and of + 0°·039 at 445° C.

* Griffiths, 'Brit. Assoc. Report,' 1890.

The barometer readings were found to be the chief source of uncertainty, owing to the constant variations of the atmospheric pressure. An error of + 0·01 cm. in reading the barometer would give an error of + 0°·053 at 50° C., and of + 0°·24 at 445° C. The plan adopted was to read the barometer at frequent intervals, noting the time of each observation. By constructing a time-curve on a very large scale, the most probable value of the atmospheric pressure at any time could be deduced. The comparison of the platinum and air thermometers between 0° and 100° (p. 152), in which the mean deviation of the observations from the parabola is less than one-hundredth of a degree, shows that this method is capable of giving very good results.

Variations in the Constant, mk.

The observations in ice and steam give the equations

$$\left.\begin{aligned} p_1\{V_1/(\theta_0+100)+\Sigma v_1/\theta'\} &= mk. \\ p_0\{V_0/\theta_0+\Sigma v_0/\theta'\} &= mk. \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

These may be used, as above explained, to calculate θ_0, if V_0 is known; or to calculate V_0, if θ_0 is known. They cannot be used to calculate both θ_0 and V_0, because the variations in mk may amount, under certain circumstances, to nearly one part in a thousand in the course of a few days.

The variation is generally most marked for a short period after the bulb and volume-gauge have been exhausted. If the value of mk is determined immediately, and again after an interval of two or three days, the value found on the second occasion is always slightly the lower, as if some of the air had been either absorbed by the sulphuric acid or by the surface of the glass, or included between the glass surface and the mercury in bulb A. It is therefore desirable to allow the whole apparatus to settle for some time, before observations are taken with it.

In the present case a whole week was allowed to elapse between the first filling and the use of the thermometer to determine the boiling-point of sulphur.

The value of mk observed on Thursday, September 11, the day before the first boiling in sulphur, was 17·0594. The value found on Friday, September 12, immediately after boiling in sulphur for two hours, was 17·0581. These differ by less than one part in ten thousand, and we may therefore assume that the apparatus had reached a fairly steady state.

In the series of observations between 0° and 100° given on p. 151, which were taken within a day of exhausting and refilling the air thermometer, a diminution in the value of mk from 17·8628 to 17·8565, amounting to one part in three thousand, was observed. In order to allow for this, it was necessary to assume that the diminution took place uniformly, and the value of mk at intermediate times was deduced by means of the time observations which were taken throughout the series.

Method of Calculation.

The whole set of formulæ are collected here for reference.

$$\theta = pV/\{mk - \Sigma\, pv/\theta'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where

$$p = H_0 + \cdot 136\,(x - 2\cdot 50) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

$$V = V_0\,\{1 + \cdot 00002880\,t + \cdot 00000000171\,t^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

$$v'' = \cdot 647 + \cdot 0232\,x, \text{ at } \theta'' = 272\cdot 10 + t'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$$v' = w/\rho - \cdot 0073\,(t' - t'_0) - \cdot 0127\,(19 - x'), \text{ at } \theta' = 272\cdot 10 + t' \quad . \quad (6).$$

$$\left.\begin{aligned} mk &= p\,\{V_0/\theta_0 + \Sigma\, v_0/\theta'\} \text{ in ice} \\ &= p\,\{V_0/(\theta_0 + 100) + \Sigma\, v_1/\theta'\} \text{ in steam} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The values of the small correction terms were worked on a small slide-rule, with the exception of the barometer correction. The latter was taken from a large scale diagram giving the whole correction for temperature, scale errors, gravity, &c., for any height between 70 and 80 cm.

The multiplications and divisions involved in calculating $\Sigma\, pv/\theta'$ were generally performed on a FULLER'S Spiral Slide-Rule, giving results correct to one part in twenty or thirty thousand.

Logarithms, as a rule, were used only in the final division.

The whole work of calculating a single observation is shown in the following table. The limit of accuracy in a single reading of the barometric pressure was about ·001 inch, corresponding to an error of 0°·01 at 50° C. The results cannot, therefore, be relied on much beyond this point, but the calculations were carried to the next figure to avoid inaccuracy in the hundredths.

TABLE I.—Showing Method of Calculation.

	Barometer.	Pressure-gauge.	Volume-gauge.	Expansion of bulb.
Quantities observed	30·271 inches at 69°·5 F.	$x = 2{\cdot}30$ cm. $t'' = 21°{\cdot}7$ C.	$x' = 17{\cdot}65$ cm., $t'_0 = 18°{\cdot}61$ C. $w = 183{\cdot}132$ gm., $t = 21°{\cdot}18$ C.	$t = 50{\cdot}3$ approximately
Corrections and reductions	$= 20°{\cdot}83$ C.	$\theta_0 = 272°{\cdot}1$ C.	$\rho = 13{\cdot}5442$, $\theta' = 293{\cdot}28$	$t^2 = 2530$
	76·887 cm. − ·279	$\theta' = 293°{\cdot}8$ C.	$w/\rho = 13{\cdot}5203$ c.c.	$at = {\cdot}001448$
	$H_0 = 76{\cdot}608$ $q = -{\cdot}027$	$v_0 = {\cdot}647$ c.c. $v_x = {\cdot}053$ c.c.	·0127 $(19 - x') = {\cdot}0171$ c.c. ·0073 $(t' - t'_0) = {\cdot}0186$ c.c.	$\beta t^2 = {\cdot}000044$
	$p = 76{\cdot}581$ cm.	$v'' = {\cdot}700$ c.c.	$v' = 13{\cdot}4846$ c.c.	$gt = {\cdot}001492$
Calculation of θ	$mk = 17{\cdot}8592$ $pv''/\theta'' = {\cdot}1825$ $pv'/\theta' = 3{\cdot}5211$		$\log V_0 = 1{\cdot}7745535$ $\log(1 + gt) = {\cdot}0006475$ $\log p = 1{\cdot}8841210$	
	$mk - \Sigma = 14{\cdot}1556$		$\log(mk - \Sigma) = 1{\cdot}1509284$	
	$\log\theta = \log 322{\cdot}398 = 2{\cdot}5083936$			
Result	$t = \theta - \theta_0 = 50{\cdot}298$			

The Platinum Thermometers.

Four platinum thermometers of different patterns were used in this investigation. They are designated by the letters E, L, M_1, and M_2 respectively.

Thermometer E has been previously described.* The platinum coil was wound on calcined asbestos, and the resistance of the connections was kept constant by a flow of tap water.

Thermometer L was made on a somewhat similar pattern. The platinum wire (length 53 cm., diameter ·013 cm.) was a piece of the spiral used in the hard glass air thermometers, Nos. IV. and V., in the experiments of 1887. It was wound on calcined asbestos round a small tube of hard glass. Its ends were then soldered with silver to long electrodes of copper wire (B.W.G. 22), one of which was double. The electrodes were insulated with thin asbestos board bound round them with fine wire. The whole was calcined and slipped into a thin tube of hard glass about 30 cm. long and ·5 cm. in diameter. The tube was heated nearly red hot, and air drawn through it for a quarter of an hour to dry the asbestos thoroughly. The lower end was then

* GRIFFITHS, *supra*, pp. 56–57.

drawn off in the blow-pipe, and the upper end was closed with sealing-wax to keep out the damp.

The ends of the three electrodes were amalgamated and dipped into ebonite mercury cups Nos. 1, 2, and 3, the single electrode being in No. 2. In taking observations the thick copper leads from the resistance box were first placed in cups 1 and 2, to measure the resistance of the platinum spiral *plus* that of the connecting wires; then in cups 1 and 3 to measure that of the connecting wires alone; and, lastly, in Nos. 2 and 3 to measure the spiral and connections again. Deducting the value found in the second observation from the mean of the first and third, we obtain the most probable value of the resistance of the spiral itself at the time corresponding to observation 2.

This form of thermometer is extremely easy to make, and gives very accurate results, provided that a good resistance-box is used.

Some preliminary experiments made with a double coil of this kind showed that, if it was kept dry, the insulation was amply sufficient up to a temperature of at least 500° C., but that it fell off beyond this point.

Thermometers M_1 and M_2 were also made from the same wire as that used for L. They were intended for use at higher temperatures with a special form of wire bridge, and were arranged and insulated in a rather different manner, as shown in Plate 7, fig. 3.

The two wires were interwound in a double screw thread, passing through holes in a thin mica plate AB, which sufficed to keep the wires from touching each other and to preserve the form of the spirals.

Each spiral was furnished with a double and single electrode symmetrically arranged. The six electrodes were insulated and kept in place by passing through holes in thin mica wads, one of which is shown at C, which were cut to fit the glass tube containing the thermometer.

In the earlier experiments in the iron tube apparatus electrodes of copper were used. These were subsequently replaced by electrodes of platinum to which the ends of the spirals were fused in the oxyhydrogen blow-pipe.

This form of thermometer is rather difficult to make, but it has the advantage of very perfect insulation even at high temperatures. Moreover, by using it with a suitable wire bridge, the resistance of the connecting wires can be automatically eliminated, and the platinum temperature can be read off on a scale of equal parts without the necessity of any calculation.

The wire bridge intended for this thermometer could not be made in time for this series of experiments. The resistances of the two spirals M_1 and M_2 were therefore observed separately with a resistance-box in the same way as that of thermometer L.

Measurement of the Resistance.

The resistance-box used in the present case was a very good Post Office box of the dial pattern made by Messrs. Elliott Bros. The coils were of German silver, and had a mean temperature coefficient of ·00038 per 1° C. It was kept in a large copper water-bath to ensure steadiness and uniformity of temperature. The temperature of the coils was taken by a delicate mercury thermometer placed inside a dummy coil in the centre of the box. The resistances observed were corrected to 17° C. The temperature of the box never varied far from this point. Resistances of 10 and 1000 units were taken out in the arms of the bridge (except when determining small resistances, such as those of connections, &c., when 10 and 10,000 were used). The resistance to be measured was generally about 5–20 ohms, and it was found that this arrangement gave the best results.

A battery of two Leclanché cells* was generally used. The battery circuit connected the junction of the two lower to that of the two higher resistances. Thus the cells had to work through a resistance of about 500 ohms. The current through the thermometer was generally less than one three-hundredth of an ampère. The current-heating of the wire must therefore have been negligible.

A sensitive mirror galvanometer was used having a resistance of about 14 ohms. The sensitiveness was generally adjusted, so that one scale division corresponded to about one part in a hundred thousand of the resistance to be measured. The last two figures in each resistance measurement were determined by observing galvanometer throws and interpolating. To facilitate readings, the galvanometer was adjusted so as to be very nearly dead-beat.

A mercury commutator was used in the battery circuit, and the swings were observed by reversing the battery. This method has the advantage of doubling the sensitiveness, and of eliminating all possible errors arising from thermo-electric effects, and from the shifting of the zero of the spot of light on the scale.

The resistance of each thermometer was repeatedly observed in ice and steam. If R_0, R_1 be the corrected values of the resistances at 0° and 100° C, and R be the resistance at any other temperature, the corresponding value of the platinum temperature pt is deduced by the formula

$$pt = 100\,(R - R_0)/(R_1 - R_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Sources of Error.

The sources of error to be avoided in using platinum resistance thermometers have been very fully discussed in a previous communication, but it will be convenient to recapitulate them briefly here.

* In some cases (in the iron tube observations, and with thermometer H), six cells were used with 100 ohms in the battery circuit.

A good resistance-box is necessary for accurate work. The dial pattern is much to be preferred. Any change in the plug or screw contacts in the low resistance arms of the bridge must be carefully avoided. In the high resistance arms they are less material; but even here they are sometimes serious. For instance, in using a P.O. box of the ordinary pattern, it was found that the change from 499 to 500, involving a sudden change of ten in the number of plugs, produced an error of a tenth of a degree in the resulting platinum-temperature.

Thermo-electric effects are the most serious source of error. It is important to arrange the platinum-copper junctions close together. Any small residual thermo-effects in other parts of the circuit are not likely to give any trouble, and can be easily eliminated.

Variations in the resistance of the connecting wires may become a serious source of error, if not properly allowed for. In our experience, the safest method of avoiding such errors is to use double electrodes, and to measure their resistance at each observation, as above described (p. 133). The chief objection to this method is that (unless a wire bridge of special pattern is used) a complete observation of the resistance of the spiral and its connections takes two or three minutes; but this is immaterial, provided that the temperature is fairly steady.

If a resistance-box is used, and *rapidity* of observation is important, the best plan is to use connections of low resistance, as in thermometer E, the variations of which may be disregarded. The disadvantages of this method are, that a water cooler must be used to keep the upper part of the stem at constant temperature, and that the stem of the thermometer must always be immersed to nearly the same extent. Thick electrodes have also the drawback, that they increase the loss of heat by conduction along the stem, and that, unless they are similar and are symmetrically arranged, they are likely to give rise to large thermo-electric effects by unequal cooling of the junctions.

If double electrodes are used, the connections to the resistance-box must be made by mercury cups. The wires should be freshly amalgamated at frequent intervals to ensure good contacts.

Changes of Zero.

The changes of zero to which platinum thermometers are liable, when used at temperatures below 500° C., very rarely amount to so much as 1 in 10,000.

The following Table contains all the observations of the fixed points of M_1 and M_2 taken after they had been fitted with platinum electrodes. The observations were taken on different days, at different heights of the barometer, and at different temperatures of the resistance-box. We are here working up to the limit of accuracy of the resistance measurements, and it is certain that the small discrepancies shown in the Table are partly due to errors of observation.

Fortuitous errors of this kind produce the greatest effect on the platinum-temperatures deduced when they alter the value of the divisor $(R_1 - R_0)$. We have found it best, therefore, in calculating the platinum-temperatures, to take the mean value of

this constant deduced from a large number of observations, and not to admit the small apparent variations from day to day.

The mean values of the constants assumed for thermometers M_1 and M_2 in reducing all the later observations were as follows :—

	R_0.	$R_1 - R_0$.	R_1 / R_0.
M_1	4·2267	1·4290	1·33805
M_2	4·1732	1·4120	1·33832

The mean deviations of the values of R_0 shown in the Table of fixed points are, for thermometer M_1, about 1 in 20,000 ; for thermometer M_2, about 1 in 30,000. The variations of the coefficient R_1 / R_0 are very much smaller.

TABLE II.—Observations on the Fixed Points of M_1 and M_2.

Date.		R_1.	R_0.	$R_1 - R_0$.	R_1 / R_0.
Saturday, Sept. 27.	M_1	5·6554	4·2268	1·4286	1·33796
	M_2	5·5848	4·1730	1·4118	1·33831
	Heated red-hot for ten minutes to anneal electrodes.				
	M_1	5·6554	R_0 not observed.		
	M_2	5·5849			
	Boiled in sulphur twice for an hour, and cooled rapidly.				
Tuesday, Sept. 30.	M_1	5·6556	4·2265	1·4291	1·33808
	M_2	5·5851	4·1731	1·4120	1·33834
	Boiled in sulphur for an hour.				
Wednesday, Oct. 1.	M_1	5·6552	4·2262	1·4290	1·33810
	M_2	5·5849	4·1730	1·4119	1·33832
Thursday, Oct. 2	M_1	5·6554	In a hypsometer of different pattern.		
	M_2	5·5850			
	Thermometers taken out of tube, and spirals re-arranged. Wire perhaps slightly strained. Boiled in mercury for one hour, and cooled very slowly.				
Saturday, Oct. 4.	M_1	5·6562	4·2270	1·4292	1·33810
	M_2	5·5853	4·1733	1·4120	1·33832
	Boiled in sulphur for an hour and a half, and allowed to cool very slowly.				
Tuesday, Oct. 7.	M_1	5·6555	R_0 not observed.		
	M_2	5·5854			
	Boiled in sulphur for an hour, and taken out as soon as possible.				
Wednesday, Oct. 8.	M_1	5·6555	4·2269	1·4286	1·33795
	M_2	5·5853	4·1733	1·4120	1·33832

[Note by H. L. CALLENDAR, added March, 1891.—Subsequent experiments with this thermometer make it almost certain that the small apparent changes of zero shown in the above table were due to changes in the mercury cup and screw connections or to errors of the resistance-box. I have since completed and compensated the resistance coils for the wire bridge mentioned on p. 133. With this apparatus I find it comparatively easy to read to a thousandth of a degree, and the thermometer no longer shows any changes of zero. I have made many other thermometers of this pattern with mica insulation, and I find them all remarkably constant even when used at temperatures as high as 1300° C.]

Description of the Iron Tube Apparatus.

In designing this piece of apparatus* we were chiefly guided by a study of the experiments of REGNAULT on the vapour pressures of sulphur. The difficulties which he encountered are very fully described by him in his paper in the 'Mémoires de l'Institut,' vol. 26, 1862, p. 526. A short description of the apparatus he finally adopted may not be out of place.

It consisted essentially of two coaxial cylinders of thick wrought iron, closed below, and bolted together at the top. The outer cylinder was heated in a furnace. The space between the cylinders was half filled with sulphur. The inner cylinder was open to the air. The air thermometer was made to fit the inner cylinder, and was not itself in actual contact with the sulphur vapour. A small iron tube connected the apparatus with a wider tube, in which the sulphur condensed.

REGNAULT states that he experienced great difficulties with this apparatus, owing to the super-heating of the vapour. This arose partly from the clogging of the small exit tube with viscous sulphur, but chiefly from the conduction of heat along the thick iron walls of the cylinder.

We have endeavoured to avoid these sources of error in our experiments by giving the apparatus a somewhat different form, shown in Plate 8, fig. 1.

Lengths A S, S L of two-inch gas-pipe were screwed together in the form of an inverted L by means of a four-way piece. The ends of the tubes at A and L were fitted with caps of the ordinary pattern.

The whole of the vertical limb S L was filled with sulphur up to the level S. This required about a kilogramme of sulphur, occupying, when melted, a volume of 500 c.c.

The lower cap F G was heated in a Fletcher's circular gas-furnace, shown at K, by means of a powerful blowpipe. The top of the furnace was covered with a piece of asbestos card, and the whole of the remainder of the apparatus was clothed with a double layer of asbestos board and yarn. The level of the sulphur stood at 15 cm. above the part of the tube exposed to the flame. The vapour had, therefore, to rise

* We also received great assistance from Mr. THOMAS, of Jesus Lane, Cambridge, who personally superintended the construction of the apparatus.

through a long column of liquid before reaching the thermometers. It appeared probable that the chance of super-heating would thus be minimised.

The horizontal limb A B was about 30 cm. in length, and was provided with openings at both ends for the insertion of the platinum and air thermometers. The end B, through which the air thermometer was inserted, was closed by a split cork made by nailing together pieces of asbestos board. Some trouble and annoyance was caused at first by the escape of burning sulphur at the various joints and openings, till it was found that asbestos board, when soaked in water, formed an adhesive putty, which, if applied wet, at once stopped the escape.

With this apparatus a preliminary experiment was tried, in which the opening at H was closed with a screw-plug, and the sulphur was allowed to distil over through a tube near the end of the horizontal limb.

This arrangement, however, did not prove satisfactory. The whole of the sulphur distilled over in about a quarter of an hour, which did not suffice for the attainment of a steady and uniform temperature.

The apparatus was, therefore, reconstructed, with the addition of a condenser, as shown in Plate 8, fig. 1. The sulphur vapour was made to pass round by A and E to the condenser H G, which was fitted with a wide glass chimney a metre long, in which the condensation of the vapour could be easily observed. The condensed sulphur flowed back to the boiler by the small tube H L, which was bent in such a way as not to interfere with the insertion of the air thermometer. The tubes A E H, leading to the condenser, were of $1\frac{1}{4}$-inch bore, and so ran little risk of clogging. By way of precaution, however, a small blow-pipe flame was made to play on the joint E, until the circulation was fully established.

Observations and Results.

Observations were taken with this apparatus on two separate days with very concordant results.

Within half an hour or less of lighting the furnace on each occasion, the heavy brown sulphur vapour made its appearance in the glass tube condenser, and gradually rose with a well-defined surface until the condensation balanced the supply. The air-blast was regulated each day so as to maintain the vapour at a constant level of 40 cm. above B, thus ensuring a very uniform and steady circulation through the whole apparatus. The quantities observed, and the results of each observation, are given in Table III. The readings of the platinum and air thermometers were taken simultaneously by different observers. The fractions given in the columns headed "Resistances" represent the observed swings of the galvanometer, and show the sensitiveness of the apparatus.

TABLE III.—Comparison of Platinum and Air Thermometers in Iron tube Apparatus, at B.-P. of Sulphur.

Time.		Observed resistances.		Mercury thermometers corrected.			Barometer.		Pressure gauge.	R — connections corrected.	Results.	
		Thermometer.	Connections.	Box.	Air.	Water.	Reading.	Temp.			*pt.*	*t.*
hr	min.						inches.	° F.				
Friday, September 12, 1890. Volume-gauge $x' = 19{\cdot}20$ cm., $t_0' = 20°{\cdot}41$ C., $w = 539{\cdot}972$ grms.												
4	52	$M_1\ 10{\cdot}43\frac{4}{18}$	$\cdot 100\frac{20}{58}$	18·95	21·7	21·38	30·282	72·2	2·60	10·3395	420·70	444·52
5	2	$M_1\ 10{\cdot}43\frac{15}{58}$	$\cdot 100\frac{3}{43}$	19·00	21·8	21·70	,,	,,	2·91	10·3403	420·75	444·90
5	6	$M_2\ 10{\cdot}35\frac{22}{23}$	$\cdot 100\frac{8}{30}$	19·00	21·8	21·77	30·280	72·1	3·05	10·2642	420·63	444·68
5	18	$M_1\ 10{\cdot}43\frac{8}{32}$	..	19·00	21·8	21·93	,,	,,	3·23	10·3401	420·74	444·77
5	21	$M_2\ 10{\cdot}35\frac{25}{32}$	..	19·00	21·8	21·96	30·281	72·2	3·28	10·2653	420·71	444·79
5	26	$M_1\ 10{\cdot}43\frac{20}{123}$	$\cdot 100\frac{10}{122}$	19·05	21·8	21·98	,,	,,	3·34	10·3392	420·68	444·94
5	30	$M_2\ 10{\cdot}35\frac{91}{123}$	$\cdot 100\frac{24}{115}$	19·05	21·8	22·03	,,	,,	3·40	10·2652	420·70	444·98
Saturday, September 13, 1890. Volume-gauge $x' = 19{\cdot}0$ cm., $t_0' = 19{\cdot}07°$ C., $w = 537{\cdot}739$.												
8	25	$M_1\ 10{\cdot}43\frac{9}{46}$	$\cdot 100\frac{26}{44}$	18·95	21·8	19·67	30·298	67·8	1·75	10·3391	420·67	444·51
8	28	$M_2\ 10{\cdot}35\frac{40}{46}$	$\cdot 100\frac{22}{43}$	18·97	21·9	19·71	30·298	67·8	1·79	10·2658	420·74	444·51
8	38	E 6·9180	..	18·98	22·0	19·94	30·300	67·7	2·00	6·9233	417·65	444·52
8	42	E 6·9180	..	19·00	22·1	20·04	30·300	67·7	2·11	6·9233	417·65	444·53
8	48	$M_1\ 10{\cdot}43\frac{81}{182}$	$\cdot 101\frac{97}{292}$	19·00	22·2	20·21	30·301	67·5	2·28	10·3407	420·78	444·54
8	50	$M_2\ 10{\cdot}36\frac{16}{236}$	$\cdot 101\frac{136}{287}$	19·00	22·2	20·26	30·302	67·5	2·31	10·2671	420·82	444·52
9	7	E $6{\cdot}918\frac{2}{5}$		19·00	22·3	20·57	30·304	67·5	2·62	6·9236	417·68	444·61
10	11	$M_2\ 10{\cdot}3600$	$\cdot 101\frac{17}{40}$	19·05	..	..	30·312	68·4	..	10·2664	420·78	..

Air Thermometer Constants . . . . . $V_0 = 59{\cdot}518$. $\theta_0 = 272{\cdot}10$. $mk = 17{\cdot}0581$.

Platinum Thermometer Constants . . . . . M_1 $R_0 = 4{\cdot}2700$. $R_1 - R_0 = 1{\cdot}4427$.
M_2 $R_0 = 4{\cdot}2375$. $R_1 - R_0 = 1{\cdot}4328$.
E $R_0 = 3{\cdot}0618$. $R_1 - R_0 = 0{\cdot}9246$.

The roll-sulphur used in these experiments was said to be prepared from H_2S. It smelt strongly of that gas when freshly crushed. It appears that sulphur prepared in this way is quite pure enough for the purpose, and that the presence of a small percentage of impurities does not materially affect the temperature of the vapour.

The sulphur used on the Friday was poured out of the apparatus before it solidified. It was found to be much discoloured, owing to the presence of various impurities due to residues of oil, red-lead, &c., used in fitting the iron tubes together. Fresh sulphur was used on the Saturday, and this, on examination, appeared to have suffered hardly any change.

The mean values of the platinum-temperatures, observed with thermometers M_1 and M_2, were :—

On Friday, $pt = 420{\cdot}70$,
On Saturday, $pt = 420{\cdot}75$.

The small difference is exactly accounted for by the difference of the barometric pressures on the two days. This would seem to show that the temperature of the vapour was practically unaffected by the impurities present in the sulphur used on the Friday.

The observations of the barometer taken during the course of the experiments on Friday, failed to show with certainty any variation amounting to more than ·001 inch. The mean value of all the readings was therefore taken and the same value was assumed in the reduction of all the observations.

It is probable that the slight apparent oscillations of the temperature, shown by the air thermometer, are partly due to real oscillations of the barometer, of which no account could be taken.

The observations of the barometer on Saturday showed a very steady and continuous rise. A time-chart was therefore constructed from which the most probable value of the barometric pressure, corresponding to the time of each observation, was deduced. The very close agreement in the values of t is certainly, in part, fortuitous ; but the observations must, at least, be regarded as proving that the air thermometer had reached a very steady state, and that the volume of the bulb was not undergoing any further change. It is almost certain that the further diminution of volume from $V_0 = 59{\cdot}518$ c.c. to $V_0 = 59{\cdot}505$ c.c., took place when the bulb was subsequently exhausted while still exposed to the high temperature. The value $V_0 = 59{\cdot}518$ is therefore used in the reduction of the observations taken on Saturday, as well as of those taken on Friday.

The mean values of t deduced from the observations with the platinum thermometers M_1 and M_2, by assuming the value $\delta = 1{\cdot}570$ in formula (d), are :—

On Friday, $t = 444°{\cdot}78$ C.
On Saturday, $t = 444°{\cdot}84$ C.

The mean values of t deduced from the simultaneous observations with the air thermometer are :—

On Friday, $t = 444°{\cdot}80$ C.
On Saturday, $t = 444°{\cdot}52$ C.

The value found on Friday is seen to agree perfectly with that deduced from the observations with the platinum thermometers. The value of t deduced from Saturday's observations, is $0°{\cdot}32$ too low; but it is probable that the value of mk, used in reducing the observations taken on that day, is a little too great.

The *same values* of the air thermometer constants were used in the reduction of both sets of observations. They were determined by readings in ice and steam, taken on *Friday* evening shortly after the boiling in sulphur. It was, unfortunately, impossible to re-determine mk after the experiments on Saturday, because the thermometer had been filled with a fresh lot of air. A diminution of only 1 in 6000 in the value of mk would suffice to account for the small discrepancy observed, and it is certain, from the measurements taken on preceding days, that mk was diminishing at about this rate *per diem*. We may, therefore, conclude that these experiments, so far as they go, are a complete verification of the value of δ found in 1887, and show that the platinum wire has not altered appreciably in the interval.

Part II.

Experiments in the Meyer Tube.

The iron tube apparatus above described gave very good results as far as steadiness of temperature was concerned, but it was liable to several defects. It was cumbrous and difficult to set up, and, being very massive, it took some time to reach a steady state, and required a considerable blast to heat it.

The success of the glass-tube condenser suggested a much simpler form of apparatus. A wide glass tube, AB (Plate 7, fig. 4), was used, having a diameter of about 4 cm., and a spherical bulb at the lower end. Tubes of this kind are commonly employed to heat Victor Meyer's vapour-density apparatus. For the sake of brevity, therefore, we shall refer to this piece of apparatus as a "Meyer" tube. It was cut off to a suitable length (about 40 cm.) for the insertion of the thermometers. The lower half was padded externally with asbestos. The upper half was left bare to act as a condenser, and the top was covered with a piece of asbestos card, shown at A, to prevent the sulphur catching fire.

The bulb was filled with sulphur, and made to stand in a circular hole cut in a piece of asbestos board, CD. A preliminary experiment was tried, in which the tube was heated with a blow-pipe. No difficulty was found in driving the sulphur

vapour up nearly to the top of the tube, and in keeping its level steady for any length of time.

Thermometer M was inserted, and some readings of its resistance were taken with the sulphur vapour at different levels in the condenser. They showed that any increase in the blast merely sent the sulphur vapour higher up the tube till its effect was compensated by the increased area of condensation, and did not appreciably alter the temperature observed in the lower portion of the tube protected by the asbestos.

The result of the observations reduced to a pressure 760, gave $pt = 418{\cdot}60$, $t = 442{\cdot}38$; which is almost identical with the value found in the preliminary experiment with thermometer L (p. 120).

To render the observations in this series comparable with each other, all the platinum-temperatures have been reduced to a standard pressure of 760 mm. of mercury at 0° C. in lat. 45°, and the corresponding air-temperature, t, by formula-(d) is also given in each case.

The correction for pressure to be applied to the air-temperature, t, of the boiling-point of sulphur to reduce to 760, is

$$\frac{dt}{dp} = 0°{\cdot}082 \text{ per mm.}^*$$

The pressure correction to be applied to the *platinum*-temperature, pt, may be deduced from the formula

$$t - pt = \delta \{t/100|^2 - t/100\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d).$$

By differentiating we obtain

$$\frac{dpt}{dp} = \frac{dpt}{dt} \cdot \frac{dt}{dp} = {\cdot}082 \left\{1 - \delta \frac{2t - 100}{10000}\right\},$$

whence, if $\delta = 1{\cdot}57$, and $t = 444°{\cdot}5$ C., we obtain $dpt/dp = 0°{\cdot}072$ C. per mm.

The blowpipe was soon found to be unnecessary. By padding the tube with asbestos more carefully and to a greater height, the sulphur could be made to boil quite readily with a Bunsen burner (Fletcher, No. 48). By simply adjusting the gas, the vapour could be maintained at a fixed level near the top of the tube. The arrangement finally adopted for padding and supporting the tube is shown in Plate 7, fig. 4. The bulb projects through a circular hole in a piece of asbestos board CD which fits it tightly. The tube is filled with sulphur some 3 or 4 cm. above the level of the board. The padding of asbestos wool is kept in place by an outer tube of asbestos board bound with wire. A cone of asbestos board surrounds the whole. This gives the arrangement greater stability, and diminishes the rate of cooling. The method and precautions to be observed in working the apparatus are as follows.

* Deduced from REGNAULT'S observations. See GRIFFITHS, *supra*.

If the sulphur is allowed to solidify in the bulb after each experiment, it must be carefully remelted *from the top*, otherwise it would burst the bulb. When the sulphur is all melted, the apparatus is placed on a tripod stand, the thermometers and leads are adjusted, and the gas is turned full on till the level of the vapour rises halfway up the condenser. This generally takes from two to three quarters of an hour. The time depends greatly on the thinness of the glass of the bulb—the thinner the better—and on the length of condenser-tube left bare above the asbestos packing. When the level of the vapour reaches the desired point, the gas is turned down till it just suffices to maintain the level steady.

Some observations taken with thermometer E in this apparatus gave the temperature

$$\text{E. } pt = 416{\cdot}19\,; \quad t = 443{\cdot}31.$$

With this arrangement there are evidently two sources of error: (1) The thermometer is cooled by conduction along the stem, so that the sulphur condenses on the upper part of the stem and runs down it, cooling the bulb slightly below the temperature of the vapour surrounding it; (2) The sulphur condensed on the sides of the upper part of the tube exposed to the air acts in a similar way, and the thermometer loses heat by radiation to the walls of the tube.

To test this some readings were taken with the spiral of thermometer M inserted to various depths in the tube, with the following results :—

With the spiral just immersed below the level of the asbestos padding, and about 3 cm. below the top of the vapour, the temperature found was

$$pt = 418{\cdot}45\,; \quad t = 442{\cdot}22.$$

With the spiral about halfway down

$$pt = 418{\cdot}67\,; \quad t = 442{\cdot}47.$$

With the spiral partly immersed in the boiling sulphur

$$pt = 419{\cdot}06\,; \quad t = 442{\cdot}91.$$

These results plainly showed that the sources of error suggested above were of serious magnitude, and that, in order to find the temperature of the vapour, it was not sufficient merely to insert a thermometer in the vapour of the liquid boiling freely in a glass tube, even if the tube were very carefully protected by thick asbestos wrappings.

The difficulties above described were overcome in the following manner.

To protect the thermometer from the condensed sulphur which ran down the stem, and to shield it from radiation to the upper parts of the tube, an asbestos umbrella E, Plate 7, fig. 4, was made to fit the stem, and was bound on to it with wire a short distance above the spiral.

To protect it from cooling by radiation to the sides of the tube, a glass tube screen, FG, coaxial with the thermometer, was hung on this umbrella by three wires.

To protect it from radiation from the liquid sulphur and from the flame beneath, which might possibly tend to raise its temperature above that of the surrounding vapour, two circular screens of asbestos H were hung on to the lower end of the glass-tube screen. They fitted the outer tube very closely, and were each perforated with three holes, so arranged as to give free passage to the sulphur vapour, while not allowing any radiation to pass. The glass-tube screen was protected by the umbrella from drippings of condensed sulphur, and must have attained very nearly the temperature of the vapour. The thermometer and screens were kept in position in the centre of the tube by means of three-legged distance-pieces of wire twisted round them, some of which are indicated in the figure.

The observations indicated, as might have been expected, a slightly higher temperature than those in which the thermometer was exposed to radiation from the sides of the tube. The gas was adjusted to keep the level of the vapour near the top of the tube. The sulphur was boiling very steadily and quietly, and the thermometer did not show a variation of more than a hundredth of a degree for about half an hour.

The result found was

$$pt = 419{\cdot}81\,; \quad t = 443{\cdot}77\,;$$

showing a rise of 0°·86 C. in the value of t.

In order to test the amount of cooling produced by the condensed sulphur which ran down the stem of the thermometer, as compared with that due to radiation to the sides of the tube, a small cap of asbestos paper was fitted to the stem of thermometer E just above the spiral, to divert the drippings. The temperature observed was

$$pt_e = 416{\cdot}43\,; \quad t = 443{\cdot}59\,;$$

showing a rise of 0°·28 C.

E was then tried with the umbrella and glass-tube screen, arranged as above described. The observations gave

$$pt = 416{\cdot}86\,; \quad t = 444{\cdot}08.$$

The effect of the glass-tube screen in stopping radiation is therefore to raise the observed temperature by 0°·49 C., and the effect of diverting the dripping down the stem is to raise it by 0°·28 C. Both causes together produce a rise of 0°·77 C., which is nearly the same as the rise found in the case of thermometer M.

Seeing that the effect of radiation was so large, another experiment was tried with thermometer L, in which *two* coaxial glass-tube screens were used. It gave the result

$$pt = 420{\cdot}01\,; \quad t = 444{\cdot}00.$$

To test whether L agreed with thermometer M, an experiment was also tried with a single glass-tube screen, which gave

$$pt = 419{\cdot}83\,; \quad t = 443{\cdot}79.$$

a number practically identical with that found for M when singly screened.

The further rise caused by using the double screen seemed to indicate that *glass* tubing, though probably opaque to the radiation from the glass stem of the thermometer, might not be sufficiently opaque to the heat radiation from the platinum wire itself at this temperature. An experiment was therefore tried with thermometer M, in which the inner glass screen was replaced by a tube of brass. This was found to cause a still further rise, the result being

Thermometer M_1. $pt = 420{\cdot}36$; $t = 444{\cdot}39$.
„ M_2. $pt = 420{\cdot}34$; $t = 444{\cdot}37$.

After the experiment the brass tube was found to be rather thickly coated with sulphide. It was therefore replaced by a tube of platinum foil on glass. The naked portion of the Meyer tube at the top was cut off to a length of 3 cm., to minimise the condensation. It was felt that this arrangement would probably give the nearest approach to the actual temperature of the vapour, as a very small flame was sufficient to maintain the ebullition. All four thermometers were used, and their fixed points were re-determined after the experiment. The results found were

Thermometer E. $pt = 417{\cdot}24$. Thermometer M_1. $pt = 420{\cdot}48$.
„ L. $pt = 420{\cdot}46$. „ M_2. $pt = 420{\cdot}48$.

Assuming $\delta = 1{\cdot}570$ for thermometers L, M_1, and M_2, we find for the corresponding air-temperature the value

$$t = \mathbf{444^{\circ}{\cdot}53}\ \text{C.}$$

We believe this to be within $0^{\circ}{\cdot}1$ of the true temperature of the vapour of sulphur boiling freely under a pressure of 760 mm.

Results of Previous Observers.

We have not been able to find that any previous observers, with the exception of Regnault, have seriously attempted the accurate determination of the boiling-point of sulphur by reference to the air thermometer.

The numbers given in Carnelley's tables are either deduced from Regnault's observations,* or are rough experiments made to test the working of some vapour-density apparatus by comparison with his results.

Of Regnault's eight observations on the pressures of sulphur vapour† at various temperatures, only two were taken in the neighbourhood of the atmospheric pressure. They are

* *E.g.*, the number $444^{\circ}{\cdot}41$ C. attributed to V. Meyer corresponds to a pressure of 725 mm. at Zurich, and is deduced by interpolation from two of Regnault's observations. ('Berichte Deutsch. Chem. Ges.,' vol. 9, p. 1225.)

† 'Mémoires de l'Institut,' vol. 26, 1862, p. 527.

$t = 440°{\cdot}30$ C. $p = 679{\cdot}97$ mm.
$t = 447°{\cdot}71$ C. $p = 763{\cdot}04$ mm.

From the latter we obtain $t = 447°{\cdot}48$ C., corrected to 760 mm. at 0° in lat. 45°.

The table calculated from Regnault's empirical formula by Weinhold* gives $t = 448°{\cdot}4$, at 760 mm. The table re-calculated and smoothed by Ramsay† from Regnault's observations gives $t = 448°{\cdot}34$. These values are nearly a degree higher than that found from the observation at 760. The reason of this discrepancy is, possibly, that the empirical formula was calculated to satisfy the observations at higher temperatures, which were more likely to be vitiated by superheating.

It was part of Regnault's programme to determine the vapour-pressures of sulphur through a wide range of temperature. He was therefore compelled to work with a very strong and thick iron cylinder, heated in a charcoal furnace. The air thermometer itself was not in direct contact with the vapour, but was enclosed in a closely-fitting and massive iron tube, firmly bolted to the top of the cylinder, and coaxial with it.

Regnault describes, at some length, the difficulties which he experienced with this apparatus from the super-heating of the vapour, and it is plain from the language he uses that he had no great confidence in his results.‡

In the iron tube apparatus which we ourselves used, the possibility of superheating was reduced to a minimum by making the vapour rise through a long column of liquid sulphur which was not directly heated. Want of space prevented the introduction of screens to guard against the cooling of the thermometers by radiation to the sides of the tube. Thus, although the temperature attained by the thermometers was steady to $0°{\cdot}1$ C. for more than an hour, and although it was the same (allowing for the difference of pressure) on both days, we cannot assume that it was the actual temperature of the sulphur vapour. All that we can safely assume is that since all the thermometers had glass envelopes, the loss by radiation was very nearly the same for each. The experiment may therefore be taken as affording an accurate *comparison* between the platinum and air thermometers, and a satisfactory verification of those performed in 1887.

The experiments in the Meyer-tube illustrate very forcibly the necessity of suitably screening the thermometers from loss of heat by radiation. They show that errors from this source may readily amount to 2° at 450°. In the last series of experiments, however, in which the thermometers were doubly screened and the condensation was reduced to a minimum, it is probable that the results obtained are within $0°{\cdot}1$ C. of the true temperature of the vapour.

We therefore propose to assume the value $t = 444{\cdot}53$ for standardising platinum

* 'Poggendorff, Annalen,' vol. 149, 1873, p. 231.

† 'Phil. Mag.,' vol. 20, 1885, p. 524.

‡ See also Ramsay, 'Chem. Soc. Journ.,' vol. 49, p. 37 (1886).

thermometers. We may observe that, even if this is not the actual temperature of the vapour, it is probable that it will be the temperature *attained by the thermometer* in an apparatus similar to ours.

Any other known boiling-point, such as that of mercury, might very well be used for this purpose; but that of sulphur presents special advantages. The material is cheap, and readily procured in a state of sufficient purity. It boils at a convenient temperature, and can be kept boiling in glass vessels for any length of time without alteration. It is more suited for the purpose than mercury, because it boils more quietly and at a higher temperature. It is also less likely to crack the glass vessels, and its fumes are not poisonous.

Method of Standardising Platinum Thermometers.

The method which we recommend for standardising platinum thermometers is briefly as follows. Observe the platinum-temperature pt of sulphur vapour in an apparatus similar to that shown in Plate 7, fig. 4. Calculate the temperature t of the vapour at the observed barometric pressure, assuming t at 760 mm. to be 444°·53 C., and adding 0°·082 C. for each mm. of pressure above the standard. The value of the constant δ may then be found by substituting the observed value of the difference $(t - pt)$ in formula (d).

For instance, the observed difference in the case of thermometer E is $t - pt = 27°{\cdot}29$ C. at 444°·53 C. This gives $\delta = 1{\cdot}783$. The value of the ratio R_1/R_0 for this thermometer is 1·3024.

Another thermometer, which was made in a similar way to L, with double electrodes, but of different wire, having a coefficient $R_1/R_0 = 1{\cdot}3253$, gave the value $pt = 415{\cdot}73$ at t 442·38, whence $\delta = 1{\cdot}760$.

The value of R_1/R_0 for each of these thermometers is much lower than in the case of L, M_1, and M_2. It appears, generally, that the lower the value of R_1/R_0, the higher the value of δ.

Another wire (used in air thermometers 2 and 3, 'Phil. Trans.' A, 1887) gives the values $R_1/R_0 = 1{\cdot}3460$, $\delta = 1{\cdot}46$.

For many purposes an accuracy of the order of 1° C. at 450° is sufficient. In this case, the trouble of making a special experiment with sulphur to find δ, may be spared. It appears likely that for values of R_1/R_0 in the neighbourhood of 1·340 the value of δ may be assumed to be 1·60 without much risk of making an error of more than half a degree at 450° C., and that for thermometers of the L type the value of δ may be approximately found from the empirical formula

$$\delta = 1{\cdot}57 + 15\,(1{\cdot}3383 - R_1/R_0).$$

Boiling-Point of Mercury.

A determination of the boiling-point of mercury was also made in a similar apparatus with thermometers L, M_1, and M_2. The result found was $t = 356°{\cdot}60$ C. The result deduced from previous experiments with thermometer E, in a different apparatus, is $t = 356°{\cdot}76$ C. The closeness of the agreement is an indirect verification of the δ-formula. The agreement would probably have been closer, but the Meyer-tube cracked near the bottom in the course of the experiments, so that we have reason to believe that the result found with L and M is a small fraction of a degree too low.

Regnault gives seven observations on the boiling-point of mercury at atmospheric pressure. They range from $358°{\cdot}8$ to $354°{\cdot}7$. The mean is $356°{\cdot}9$, which agrees as closely as could be expected with the number found above.

Boiling- and Freezing-Points of some other Substances.

The results of an investigation into certain boiling- and freezing-points have been given in a previous paper,* which is referred to as "Paper (G)" throughout this Section. It was therein pointed out that the values obtained depended on the following "fixed points":—

B.-P. Naphthalene	(760 mm.)	=	$218°{\cdot}06$ (Crafts)
B.-P. Benzophenone	„	=	$306°{\cdot}08$ „
B.-P. Sulphur	„	=	$448°{\cdot}34$ (Regnault)

and sufficient data for the correction of the deduced temperatures were given, in case these "fixed points" were found to be inaccurate.

The conditions under which the experiments described in Paper (G) were performed, closely resembled those prevailing in the Meyer-tube when the thermometer was unscreened. This conclusion is borne out by the numbers given on p. 120 of this Paper, where the platinum-temperature of sulphur vapour in the *same apparatus* as that used in the previous experiments was found to be $418°{\cdot}5$, a number almost identical with that given by a specimen of the same wire when unscreened in the Meyer-tube (see p. 142).

Our experiments have shown that the temperature of the unscreened thermometer, when placed in sulphur vapour in the Meyer-tube, is too low; at the same time they enable us to apply the small correction thus rendered necessary.

The values of R_1 and R_0 in both thermometers E and G have shown signs of change in the interval which has elapsed since the spring of the year, but by assuming the old values of pt when in the vapour of sulphur under 760 mm. to be *equivalent* to $t = 444°{\cdot}53$, and calculating the values of δ for each thermometer, we shall avoid the

* Griffiths, *supra*.

subsequent changes of zero, and it is probable that any small errors due to radiation, &c., will be eliminated, since all the determinations of boiling-points were made with similar apparatus and under similar circumstances. A small correction has first to be made in the numbers given in Paper (G), owing to a scale error in the Sidney College barometer, which had previously escaped detection. A careful comparison with the CAVENDISH Standard Barometer has shown a constant error of + 0·50 mm. The consequent correction is given in column C of Table IV. Column I. gives the corrected values of pt for each thermometer, deduced from the numbers given in Paper (G).

Applying a similar correction to the values of the platinum-temperatures of sulphur vapour in the old apparatus, we get

	pt.			δ.
E . . .	417·18	Hence the values of δ under the old conditions are	E . . .	1·786
F . . .	414·21		F . . .	1·980
G . . .	416·06		G . . .	1·859

The values of d, given in column II. of the Table of Boiling-Points, are deduced from the above values of δ.

The circumstances under which the freezing-points of the metals were determined, preclude the possibility of loss of heat by radiation, &c. In reducing these observations, therefore, the *corrected* value of the platinum-temperature of sulphur vapour under 760 mm. should be assumed, in order to obtain the value of δ.

Our experiments have shown that the difference in the platinum-temperatures given by thermometer E when unscreened and when fully protected is 1°·05 (see p. 144). We may assume, therefore, that, had all the precautions been observed which we have since found to be necessary, the values of pt would have been raised by a like amount.

The resulting values of δ are

E . . . 1·718; F . . . 1·912; G . . . 1·792;

and the numbers given in column II. of the Table of Freezing-Points are deduced from these values of δ.

Column III. gives the result for each thermometer, re-calculated and corrected, as above described. The close agreement of the resulting numbers is an indication of the validity of the method of reduction adopted. We believe that the results are within a tenth of a degree of the real values of t.

The vacant spaces under thermometers F and G indicate that no observations were taken with them at those points.

TABLE IV.—Boiling-Points.

Nature of experiment.	C.	Thermometer E.			Thermometer F.			Thermometer G.			Mean t by E, F, & G.	Other observers.
		I. *pt.*	II. *d.*	III. *t.*	I. *pt.*	II. *d.*	III. *t.*	I. *pt.*	II. *d.*	III. *t.*		
B.-P. Aniline (760 mm.) .	−·032+·023	181·33	2·78	184·11	181·07	3·06	184·13	181·27	2·87	184·14	184·13	RAMSAY, 184·41 THORPE, 183·7
„ Naphthalene . . .	−·038+·026	213·26	4·62	217·88	212·88	5·08	217·96	213·21	4·77	217·98	217·94	CRAFTS, 218·06
„ Methyl salicylate .	−·039+·026	218·07	4·90	222·98	217·68	5·40	223·08	..	..	..	223·03	RAMSAY, 222·88 CAHOURS, 222
„ Benzophenone . .	−·054+·028	294·55	11·27	305·82	293·42	12·45	305·87	294·08	11·70	305·78	305·82	CRAFTS, 306·08
„ Triphenyl methane (770·3 mm.)	−·062	340·12	16·35	356·47	..	..	..	339·41	17·00	356·41	356·44	KEKULÉ, 355 (760) CRAFTS, 358 (754)
„ Mercury (760 mm.) .	−·062+·033	340·37	16·37	356·74	338·70	18·12	356·82	339·69	17·02	356·71	356·76	REGNAULT, 357·25 RAMSAY, 358·2
Freezing-Points.												
Freezing-point of Sn . .	−·041	226·41	5·25	231·66	225·83	5·83	231·66	226·26	5·47	231·73	231·68	REIMSDYK, 228·5 KUPFFER, 230 PERSON, 232·7 CRICHTON, 238
„ „ Bi . .	−·048	261·34	7·84	269·18	..	..	..	261·10	8·15	269·25	269·22	PERSON, 270·5 REIMSDYK, 268·3
„ „ Cd . .	−·055	308·53	12·17	320·70	..	..	..	307·98	12·68	320·66	320·68	PERSON, 320·7 REIMSDYK, 320 VAN DER WYDE, 325
„ „ Pb . .	−·057	314·85	12·81	327·66	..	..	..	314·34	13·37	327·71	327·69	PERSON, 326·2 KUPFFER, 334 QUINCKE, 330
„ „ Zn . .	−·072	394·77	22·78	417·55	..	..	..	393·83	23·76	417·59	417·57	REIMSDYK, 420 WRIGHT and LUFF, 420 PERSON, 433·3 Other observers range from 342 (DANIELL) to 450 (BOUSSINGAULT)

NOTE.—The freezing-points of the metals were determined by the limit of the rise observed during surfusion.

The dotted curve (Plate 8) shows the path obtained by assuming CRAFT'S and REGNAULT'S numbers, and is a reduced copy of the one that was used in Paper (G) for reducing observations with thermometer E. The continuous line shows the path of the curve obtained from the parabola, the abscissa being $t - d = pt$, the ordinate d.

The scale on which these curves are plotted is about $\frac{1}{10}$th of the original.

It will be seen that the divergence up to a temperature of about 300° is very small, the rapid divergence above that temperature being due solely to the difference between REGNAULT'S and our determinations of the boiling-point of sulphur.

As pointed out in the previous paper, thermometers A to E* were to be regarded as trial instruments, and a summary only was given of the results obtained by means of them. Since they were of different patterns, and the spirals formed of different wires, it may be interesting to give the mean of the results obtained by applying a similar correction. Their values of (δ) range from 1·667 (Thermometer E*) to 2·158 (D). The experiments were not performed with the same care as those with thermometers E, F, and G, and their results are more irregular ; in one or two cases they differ from the mean by as much as 0°·4.

The mean values of t deduced from the five thermometers are as follows :—

Aniline	184°·13	
Naphthalene	217°·82	
Methyl salicylate	222°·81	(A, B, C only)
Benzophenone	305°·85	
Mercury	356°·74	

These thermometers were not used for the determination of any freezing-points.

[Note by E. H. GRIFFITHS, March 21st, 1891.—Since the communication of the preceding paper, I have constructed and graduated another platinum thermometer.

The method of insulation and construction is similar to that described on p. 133, and shown on Plate 7, fig. 3.

The thermometer contains two coaxial spirals, the smaller one being inserted within the larger. The stem is 18 inches in length, and both ends of each spiral are fused to double platinum electrodes. The walls of the surrounding hard glass tube are of egg-shell thinness.

The resistance of the spirals can be taken separately, in series, or in parallel arc. Four resistances differing greatly in magnitude can thus be determined at any temperature. The accuracy of the resistance-box is thus exposed to a severe test. In order to reduce the stem resistance as much as possible, the stout wires forming the electrodes are, when taking observations, connected in pairs. The method adopted is as follows :—

Let R_a be the resistance of the longer spiral, and r_1, r_2 the resistances of the two arms of one of its electrodes, and r_3, r_4, the resistances of the arms of the other electrode.

A connector is first used which causes the current to pass through r_1, r_2 in parallel arc, then through R_a and leave by r_3, r_4 in parallel arc. The connector is then replaced by a second one which causes the current to pass through r_1, r_3 in parallel arc and leave in the same manner by r_2, r_4.

The resistances r_1, r_2, r_3, and r_4, being very nearly equal, both ends of the spiral will, in the second case, be at the same potential; hence the second connector gives the total external resistance.

The value of R_a can thus be determined by two observations.* Similar connectors are used to determine R_b, R_{a+b}, and R_{ab} (R_{a+b} and R_{ab} being the resistance of the two spirals when in series and parallel arc respectively).

The thermometer has been standardised in the manner described on p. 147, and all the precautions observed which our experience has shown to be necessary.

The umbrella, coaxial tubes, &c. (see Plate 7, fig. 4), were invariably attached to the thermometer when it was placed in a vapour.

I was anxious to test the accuracy of the method adopted for correcting the results of my former determinations (see Table IV., p. 150), and I have, therefore, by means of this thermometer, re-determined the boiling-points of mercury and naphthalene with great care. The results may be summarised as follows :—

Spirals.	Mercury (760 mm.).		Naphthalene† (760 mm.).	
	pt.	*t.*	*pt.*	*t.*
$a + b$	343·39	356·74	214·04	218·04
a	343·19	356·72	214·00	218·03
b	343·33	356·73	214·03	218·04]

Part III.

Comparison of Platinum and Air Thermometers between 0° and 100° C.

Thermometer H was constructed with the view of measuring temperatures between 0° and 100°. It has already been described in a paper read before the British Association.‡ A short description, however, appears to be necessary.

* The validity of this method depends on the equality of r_1, r_2, &c. I, therefore, carefully determined their resistances separately at different temperatures. I found the limits of error to be within $\frac{3}{10000}^{\text{ths}}$ of the stem resistance, and since the stem resistance in no case exceeded $\frac{1}{25}^{\text{th}}$ of the total resistance, any inaccuracy thus introduced might be neglected.

† A fresh specimen of naphthalene gave $t = 218°{\cdot}16$ C.

‡ Griffiths, "On the Comparison of Platinum and Mercury Thermometers at Low Temperatures." 'Brit. Assoc. Rep.,' 1890.

In thermometer H there is no glass contact, and the stem of the instrument, after repeated exhaustion and refilling with dry air at a temperature of three to four hundred degrees, was filled with melted anthracene. This substance (melting-point 213°) was found to be as good an insulator as paraffin at temperatures below 130°. The coil, consisting of about 56 inches of platinum wire, having a diameter of 0″·005, rests on calcined asbestos, but has no contact with the glass envelope. The spiral and asbestos are thus perfectly dry, and contained in an almost vacuous space. The leads down the stem are of silver, with a resistance of less than ·001 ohm, and approach to within one inch of the spiral, and are connected with it by moderately thick platinum wires, thus any flow of heat from the spiral to the silver is diminished. The instrument is connected with the resistance box by two leads about 6 feet long, each consisting of 50 copper wires (B. W. G. 24) enclosed in an india-rubber tube. Changes in the external resistance, caused by changes in temperature of the leads, could therefore be neglected.

The fixed points were determined at various times with the following results :—

	R_1.	R_0.
* July 26.	18·2038	13·5219
„ 27.	18·2043	—
Aug. 12.	18·2038	13·5217
„ 13.	18·2044	13·5219
Sept. 11.	18·2041	13·5213
„ 12.	18·2042	13·5214

The first pair of observations were taken in the Sidney Laboratory, the second pair at the Cavendish, but on a different floor to the barometer, which had therefore to be corrected for difference in elevation. During the last pair of observations the barometer was placed by the hypsometer; these observations therefore are probably the most reliable. In any case the divergence of any one experiment from the mean would not affect $R_1 - R_0$ by more than 1 in 15,000.

Hence, mean $R_0 = 13{\cdot}5216$ and $R_1 - R_0 = 4{\cdot}6825$, and $R_1/R_0 = 1{\cdot}3463$.

The comparison of H and the air thermometer was conducted as follows. Both thermometers were placed in a nearly cubical iron tank, holding about 16 gallons of water. The air thermometer was inserted through a hole in the middle of one of the sides, the thermometer H through a hole in the centre of the lid. The two bulbs met in the centre of the tank, being in all but actual contact. The sides of the tank were covered with several wrappings of flannel, as we desired to keep the temperature nearly constant during each group of observations. The water was raised to the required temperature by a gas burner beneath the bath.

* The first four numbers have been slightly altered since their communication to the British Association, owing to the subsequent discovery of the constant scale errors in the barometers.

A large paddle, pivoted in one corner of the lid, and driven by a water motor, proved to be the most efficient form of stirrer. It both drove the water before it, and lifted it upwards during each stroke.

The readings of the two thermometers were taken simultaneously by the two observers. Three observations were taken of the galvanometer swings for each reading of the platinum thermometer, the mean of the 1st and 3rd on the same side giving the true swing to the left at the time of taking the 2nd swing which was to the right, or *vice versâ*. The reading of the air thermometer was taken while the second swing of the galvanometer was being read.

Observations were taken about every 5° between 0° and 100°, and five observations were taken about each point. The following group of five (selected at random) will illustrate the methods of observation.

TABLE V.

Values of the constants. $V_0 = 59{\cdot}505$ c.c. ; $\theta_0 = 272°{\cdot}10$; $mk = 17{\cdot}8592$.
Volume-gauge. $x' = 17{\cdot}65$ cm. ; $t'_0 = 18°{\cdot}61$; $w = 183{\cdot}132$ grms.
Barometer corrected. $H_0 = 76{\cdot}608$ cm.
Pt. thermometer. $R_1 - R_0 = 4{\cdot}6825$; $R_0 = 13{\cdot}5216$.

Observed R.	Swings.	Mercury thermometers corrected.			*x*.	*pt*.	*t*.	Difference.
		Box.	Air.	Water.				
1587	L. 48							
1588	R. 14	19·45	21·7	21·16	2·26	50·634	50·269	0·365
1587	L. 54							
1587	L. 57							
1588	R. 6	19·45	21·7	21·17	2·29	50·657	50·292	0·365
1587	L. 59							
1587	L. 59							
1588	R. 5	19·45	21·7	21·18	2·30	50·661	50·298	0·363
1587	L. 60							
1587	L. 61							
1588	R. 2	19·45	21·7	21·19	2·31	50·672	50·304	0·368
1587	L. 63							
1587	L. 62							
1588	R. 2	19·45	21·7	21·19	2·31	50·672	50·304	0·368
1587	L. 62							
				Means		50·659	50·293	0·366

The whole work of calculating the middle observation of this group is shown in Table I., p. 132.

A single observation of the platinum-temperature with this apparatus would probably be correct to 0°·003, nearly. The indications of the air thermometer cannot

be relied on much beyond 0°·01, which corresponds to the limit of accuracy attainable in a single reading of the barometer; but the last column shows that the small differences between consecutive observations, which are not affected by the uncertainty of the barometric reading, agree much more closely among themselves.

The following table gives the mean results of each group of five.

TABLE VI.—Comparison of Platinum Thermometer H with the Air Thermometer.

pt.	*t.*	Difference.	Calculated difference.
2·768	2·721	·047	·039
3 128	3·086	·042	·044
10·601	10·499	·102	·138
16·366	16·159	·207	·200
20·599	20·370	·229	·239
25·447	25·169	·278	·278
30·209	29·910	·299	·309
35·808	35·464	·344	·337
40·573	40·215	·358	·354
45·732	45·379	·353	·365
50·659	50·293	·366	·368
56·043	55·682	·361	·364
61·169	60·815	·354	·351
65·941	65·603	·338	·333
70·632	70·323	·309	·308
75·299	75·013	·286	·276
81·092	80·856	·236	·228
85·923	85·731	·192	·189
90·200	90·090	·110	·132
95·461	95·417	·044	·064

The differences given in the last column are calculated from the formula

$$d = t - pt = 1{\cdot}474\,\{t/100^2 - t/100\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (d).$$

Omitting the group of observations at 10°·5, where the discovery of a faulty connection led us to suspect the observations before their results were deduced, we see that the observed and calculated values in no case differ by more than 0°·012, except in the last two observations. These show an error of 0°·02, both in the same direction. In taking the two observations in question, it was unfortunately necessary to read the barometer by gaslight, when its temperature had just begun to fall. It is very likely that the temperature of the mercurial column itself would lag slightly behind that of the attached thermometer. This would account for the small discrepancy observed.

The results obtained by a comparison of H with REGNAULT'S mercury thermometer No. 7, and with the Kew standard, are given in the communication previously

referred to.* It was then pointed out that REGNAULT and Kew were in practical agreement. REGNAULT says that the differences between No. 7 and the air thermometer were so small that they could not be determined with precision.† His mercury thermometers generally showed a difference of 0°·2 to 0°·3 as compared with the air thermometer at 50°, but the observations are not consistent to 0°·1. The readings of thermometer No. 7 (which was made of ordinary glass) appear to approach the air readings more closely than is usually the case with mercury thermometers.

The comparison with the Kew standard was made by means of four thermometers, having a range of temperature from 0° to 34°, divided to 0·1 of a degree, and corrected at Kew.‡

A comparison of the (*pt* — *Kew*) difference-curve with that whose ordinates are given in the above Table, shows that the Kew standard reads 0°·10 C. lower than our air thermometer at 30° C.

CONCLUSIONS.

The experiments described in the preceding pages show conclusively that the platinum resistance thermometer is an extremely self-consistent and trustworthy instrument, and that it is remarkably free from change of zero and from secular alteration of its coefficients.

It appears certain that, by choosing some particular thermometer as the standard with which other thermometers could be directly compared, it would be possible to attain a very high degree of accuracy in the relative measurements of temperature on the arbitrary scale of the standard chosen.

We have found that, by using a good resistance-box (correct to 1 in 100,000 in the relative values of the coils), it is possible to read to 0°·003 between 0° and 100° C. By using a suitably-constructed and compensated wire bridge, a still higher degree of accuracy may be attained, and the sensitiveness of the thermometer may readily be increased so as to indicate a difference of one ten-thousandth of a degree at ordinary temperatures.

The reduction of the platinum-temperatures to the absolute scale is less certain, since it depends on the accuracy of the air thermometer experiments. But it appears probable that, by assuming the parabolic formula (*d*), and calculating the appropriate value of the constant δ for each thermometer by means of an observation at the boiling-point of sulphur, as above described (p. 147), we may attain a degree of accu-

* GRIFFITHS, "On Comparison of Platinum and Mercury Thermometers." 'Brit. Assoc. Rep.,' 1890. Also, see the 'Electrical Review' for Sept. 26, 1890.

† 'Mém. de l'Institut,' vol. 21, 1847, p. 220.

‡ Three of these thermometers belong to the B.A. Committee on Electrical Standards; see Appendix to their Report, Brit. Assoc., 1890.

racy of the order of 0°·01 between 0° and 100° C., or of 0°·1 at 450° C., on the scale f the standard air thermometer.

We believe that thermometers standardised in this way will be found to afford the most convenient and practical method of measuring temperature in all cases where accuracy is desired.

Explanation of Plates 7 and 8.

PLATE 7.

Fig. 1. General view of the arrangement adopted for supporting the bulb of the air thermometer, and connecting it to the gauge used for determining the variations in the pressure and volume of the enclosed air.

Fig. 2. Represents the gauge on a larger scale.—A detailed explanation of the various parts will be found on p. 126.

Fig. 3. A slightly enlarged view of the spiral of the platinum thermometers M_1 and M_2, showing the method of supporting and insulating the spiral and its electrodes by means of the mica plate AB, and the mica wads shown at A and C.

Fig. 4. Shows the method of screening the thermometer when placed in the Meyer-Tube Apparatus for determining the boiling-point of sulphur, as described in detail on pp. 142 and 143.

PLATE 8.

Fig. 1. Iron tube apparatus used in the comparison of the platinum and air thermometers at the boiling-point of sulphur. It could be maintained at a steady temperature by means of a continuous flow of sulphur vapour for any length of time. A full description of it is given on pp. 137 and 138.

The Curves in this plate illustrate graphically the effect of using the lower value, above found, of the boiling-point of sulphur, instead of that given by Regnault. See p. 151.

V. Bakerian Lecture.—*On Tidal Prediction.*

By G. H. Darwin, *F.R.S., Plumian Professor and Fellow of Trinity College, Cambridge.*

Received December 16, 1890—Read January 29, 1891.

Table of Contents.

PAGE.

Introduction 159
I. Analysis 162
II. Computation 192
III. Comparison and Discussion 225

Introduction.

At most places on the North Atlantic the prediction of high and low water is fairly easy, because there is hardly any diurnal tide. This abnormality makes it sufficient to have a table of the mean fortnightly inequality in the height and interval after lunar transit, supplemented by tables of corrections for the declinations and parallaxes of the disturbing bodies. But when there is a large diurnal inequality, as is commonly the case in other seas, the heights and intervals after the upper and lower lunar transits are widely different; the two halves of each lunation differ much in their characters, and the season of the year has great influence. Thus simple tables, such as are applicable in the absence of diurnal tide, are of no avail.

The tidal information supplied by the Admiralty for such places, consists of rough means of the rise and interval at spring and neap, modified by the important warning that the tide is affected by diurnal inequality. Information of this kind affords scarcely any indication of the time and height of high and low water on any given day, and must, I should think, be almost useless.

This is the present state of affairs at many ports of some importance, but at others a specially constructed tide-table for each day of each year is published in advance. A special tide-table is clearly the best sort of information for the sailor, but the heavy expense of prediction and publication is rarely incurred, except at ports of first rate commercial importance.

There is not, to my knowledge, any arithmetical method in use of computing a special tide-table, which does not involve much work and expense. The admirable

tide-predicting instrument of the Indian Government renders the prediction comparatively cheap, yet the instrument can hardly be deemed available for the whole world, and the cost of publication is so considerable, that the instrument cannot, or at least will not, be used for many minor ports at remote places. It is not impossible, too, that national pride may deter the naval authorities of other nations from sending to London for their predictions.*

The object then, of the present paper, is to show how a general tide-table, applicable for all time, may be given in such a form that any one, with an elementary knowledge of the 'Nautical Almanac,' may, in a few minutes, compute two or three tides for the days on which they are required. The tables will also be such that a special tide-table for any year may be computed with comparatively little trouble.

Any tide-table necessarily depends on the tidal constants of the particular port for which it is designed, and it is here supposed that the constants are given in the harmonic system, and are derived from the reduction of tidal observations. Where the observation has been by tide-gauge, the process of reduction is that explained in the Report to the British Association for 1883, but where the observations were only taken at high and low water a different process becomes necessary. I have given, in a previous paper, a scheme of reduction in these cases.†

At ports not of first rate commercial importance, tidal observation has rarely been by tide-gauge, and thus it is exactly at those ports where the method of this paper may prove most useful, that we are deprived of the ordinary methods of harmonic analysis. On this account I regard my previous paper as preliminary to the present one, although the two are logically independent of one another.

In the harmonic method the complete analytical expression for the height of water at any time consists of the sum of a number of heights, each multiplied by the cosine of an angle. Each of these angles or arguments involves some or all of the mean longitudes of moon, sun, lunar perigee, and solar perigee; there are, also, certain corrections depending on the longitude of the moon's nodes. The variability of height of water depends principally on the mean longitudes of moon and sun, and, to a subordinate degree, on the longitudes of lunar perigee and node—for the solar perigee is sensibly fixed. There are, therefore, two principal variables and two subordinate ones besides the time. This statement suggests the construction of a table of double entry for the variability of water due to the principal variables, and of correctional tables for the subordinate ones; and this is the plan developed below.

It would not be appropriate to retain the mean longitudes of moon and sun as principal variables, nor the mean longitude of lunar perigee as one of the subordinate variables. In the final table the principal variables are the time of moon's transit, and the time of year, or strictly speaking the sun's true longitude. As to the sub-

* The instrument may be used, I believe, on the payment of certain fees.

† "On the Harmonic Analysis of Tidal Observations of High and Low Water," 'Roy. Soc. Proc.,' vol. 48, 1890, p. 278.

ordinate variables, the moon's parallax is taken as one, whilst the longitude of the node is retained as the other.

Throughout the first part of the analytical development the moon's true longitude, measured in the orbit, is taken as one of the principal variables. Transition is then made to the time of transit of a fictitious satellite whose right ascension is equal to the moon's true longitude, and the final transition is to the transits of the real moon. This transposition of variables has necessitated a complete development of the tide-generating forces from the beginning, and thus the analysis of the paper is almost complete in itself, although reference is necessarily made to the harmonic method.

Such a piece of work as the present can only be deemed complete when an example has been worked out to test the accuracy of the tidal prediction, and when rules have been drawn up for the arithmetical processes, forming a complete code of instructions to the computer. The example below is intended to carry out these requirements.

I chose the port of Aden for the example, because its tides are more complex and apparently irregular than those of any other place, which, as far as I know, has been thoroughly treated. The tidal constants for Aden are well determined, and the annual tide-tables of the Indian Government afford the means of comparison between my predictions and those of the tide-predicting instrument.

The arithmetic of the example has been long, and the plan of marshalling the work has been rearranged many times. An ordinary computer is said to work best when he is ignorant of the meaning of his work, but in this kind of tentative work a satisfactory arrangement cannot be attained without a full comprehension of the reason of the method. I was, therefore, very fortunate in securing the enthusiastic assistance of Mr. J. W. F. Allnutt, and I owe him my warm thanks for the laborious computations he has carried out. After computing fully half of the original table, he made a comparison for the whole of 1889 between our predictions and those of the Indian Government.

I had hoped that tables, less elaborate than those exhibited below, might have sufficed. It appeared, however, that the changes during the lunation and with the time of year, which affect the height and interval, are so abrupt and so great, that short tables would give very inaccurate results, unless used with elaborate interpolation. We can clearly never expect sailors to use tables of double entry, in which interpolation to second and third differences is required; and indeed any interpolation is objectionable. It is proposed, therefore, that the tables shall be made so full that interpolation will be unnecessary. This plan, of course, throws the whole of the interpolation on to the computer, and although extensive interpolation—even when done graphically—is tedious, yet it is obviously best to have it done once for all, rather than piecemeal by the user of the tables.

The first part of the paper gives the analytical development, the second contains the numerical example and rules of computation, and in the third I give some account

of the comparison with other predictions and actuality, and suggestions for abridgements in cases where less accuracy shall be thought sufficient.

The analytical formulæ of the first part are much scattered, and it may not always be easy to see whither they tend ; but the second part virtually contains a summary of the first, and a comparison between these two parts will show both the reasons for the analysis and those for the arithmetical processes.

I have tried to make the instructions to the computer so complete that he need not be troubled with the reason for his operations ; but if there is, through inadvertence, any incompleteness, reference must be made to the analytical development for interpretation and instruction.

Part I.—Analysis.

§ 1. *Development of the Tide-Generating Potential.*

Let A, B, C (fig. 1) be axes fixed in the earth, AB being the equator and C the north pole.

Let r, ρ be respectively the radius-vectors of M the moon, and of P any other point. Let M_1, M_2, M_3 be the direction cosines of the moon, and ζ_1, ζ_2, ζ_3 those of P, both referred to the axes ABC.

Then by the usual theory the potential V, of the second order of harmonics, is

$$V = \tfrac{3}{2}\frac{\mu M}{r^3}\rho^2(\cos^2 \mathrm{PM} - \tfrac{1}{3}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

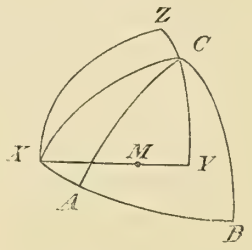
where M is the moon's mass, and μ the attractional constant.

But since

$$\cos \mathrm{PM} = \zeta_1 M_1 + \zeta_2 M_2 + \zeta_3 M_3,$$

$$\cos^2 \mathrm{PM} - \tfrac{1}{3} = 2\zeta_1\zeta_2 . M_1M_2 + 2 \cdot \frac{\zeta_1^2 - \zeta_2^2}{2} \cdot \frac{M_1^2 - M_2^2}{2} + 2\zeta_2\zeta_3 . M_2M_3 + 2\zeta_1\zeta_3 . M_1M_3$$
$$+ \tfrac{3}{2}(\tfrac{1}{3} - \zeta_3^2)(\tfrac{1}{3} - M_3^2) \quad . \quad . \quad . \quad . \quad (2).$$

Fig. 1.



Now let X, Y, Z, fig. 1, be a second set of rectangular axes, fixed in space ; let M be the projection of the moon in her orbit XY. Let $I = \mathrm{ZC}$, the obliquity of the

lunar orbit to the equator; let $\chi = \mathrm{AX} = \mathrm{BCY}$, and $\psi = \mathrm{MX}$, the moon's longitude measured in her orbit from the point X.

From the figure it is clear that

$$\left.\begin{aligned} M_1 &= \cos\psi\cos\chi + \sin\psi\sin\chi\cos I \\ &= \cos^2\tfrac{1}{2}I\cos(\chi-\psi) + \sin^2\tfrac{1}{2}I\cos(\chi+\psi), \\ M_2 &= -\cos\psi\sin\chi + \sin\psi\cos\chi\cos I \\ &= -\cos^2\tfrac{1}{2}I\sin(\chi-\psi) - \sin^2\tfrac{1}{2}I\sin(\chi+\psi), \\ M_3 &= \sin I\sin\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3).$$

Hence from (3)

$$\left.\begin{aligned} M_1^2 - M_2^2 &= \cos^4\tfrac{1}{2}I\cos 2(\chi-\psi) \\ &\quad + \tfrac{1}{2}\sin^2 I\cos 2\chi + \sin^4\tfrac{1}{2}I\cos 2(\chi+\psi), \\ -2M_1M_2 &= \cos^4\tfrac{1}{2}I\sin 2(\chi-\psi) \\ &\quad + \tfrac{1}{2}\sin^2 I\sin 2\chi + \sin^4\tfrac{1}{2}I\sin 2(\chi+\psi), \\ M_2M_3 &= -\sin\tfrac{1}{2}I\cos^3\tfrac{1}{2}I\cos(\chi-2\psi) \\ &\quad + \tfrac{1}{2}\sin I\cos I\cos\chi + \sin^3\tfrac{1}{2}I\cos\tfrac{1}{2}I\cos(\chi+2\psi), \\ M_1M_3 &= -\sin\tfrac{1}{2}I\cos^3\tfrac{1}{2}I\cos(\chi-2\psi) \\ &\quad + \tfrac{1}{2}\sin I\cos I\sin\chi + \sin^3\tfrac{1}{2}I\cos\tfrac{1}{2}I\sin(\chi+2\psi), \\ \tfrac{1}{3} - M_3^2 &= \tfrac{1}{3} - \tfrac{1}{2}\sin^2 I + \tfrac{1}{2}\sin^2 I\cos 2\psi, \end{aligned}\right\} \quad . \quad (4).$$

In order to simplify the result of the substitutions from (4) into (2) and (1), the axes fixed in the earth may be taken as follows:—

The axis A on the equator in the meridian of the place where the tides are observed, B 90° east of A, and C as already stated at the north pole.

Then if λ be the latitude of the place of observation

$$\zeta_1 = \cos\lambda, \qquad \zeta_2 = 0, \qquad \zeta_3 = \sin\lambda.$$

Hence (2) becomes

$$\cos^2\mathrm{PM} - \tfrac{1}{3} = \tfrac{1}{2}(M_1^2 - M_2^2)\cos^2\lambda + M_1M_3\sin 2\lambda + \tfrac{3}{2}(\tfrac{1}{3} - M_3^2)(\tfrac{1}{3} - \sin^2\lambda) \quad . \quad (5).$$

We shall therefore only require the first and the last two of (4), and these may be considerably simplified.

The angle I ranges between $23^\circ\ 27' \pm 5^\circ\ 9'$; hence the term in $M_1^2 - M_2^2$ which involves $\sin^4\frac{1}{2}I$ is negligible, and similarly that which involves $\sin^3\frac{1}{2}I\cos\frac{1}{2}I$ in M_1M_3 may be omitted.

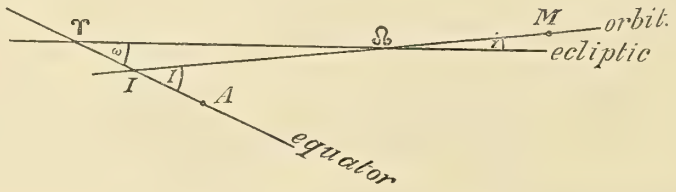
Further, it is only necessary to develop V in as far as it is variable; now as I ranges

from 18° 18′ to 28° 36′, and back again in the course of 19 years, $\sin^2 I$ oscillates slightly in value about a mean. The "tides of long period" corresponding to the term $\frac{1}{3}\sin^2 I \cos 2\psi$ in $\frac{1}{3} - M_3^2$, are, although sensible, so small that we shall take no account of them; *à fortiori* the variability of $\frac{1}{3} - \frac{1}{2}\sin^2 I$ is negligible. It would be easy, however, to take account of these terms if it were desirable to do so.

We need therefore only pay attention to the first and fourth of (4), and the last term in each may be omitted.

In fig. 2 let M be the moon in her orbit, and let A be the axis A of fig. 1 fixed in the earth.

Fig. 2.



Then let ν be the right ascension of the point I, and ξ its longitude measured in the moon's orbit.

Let ω be the obliquity of the ecliptic, i the inclination of the moon's orbit, ☊ the longitude of the moon's node.

Let t be the mean solar hour angle at the place and time of observation, h the sun's mean longitude, and l the moon's longitude measured in the orbit.

Then, by the definition of χ and ψ, we have

$$\chi = \mathrm{IA}, \qquad \psi = \mathrm{IM}.$$

Since $\nu = \Upsilon\mathrm{I}$, $\xi = \Upsilon\,☊ - ☊\,\mathrm{I}$, $\Upsilon\,☊ = ☊$, and $t + h$ the sidereal hour angle,

$$\chi = t + h - \nu, \qquad \psi = l - \xi.$$

Thus to the degree of approximation adopted, (4) becomes

$$\left.\begin{aligned} M_1^2 - M_2^2 &= \cos^4\tfrac{1}{2} I \cos 2\,(t + h - l - \nu + \xi) + \tfrac{1}{2}\sin^2 I \cos 2\,(t + h - \nu), \\ M_1 M_3 &= \cos^3\tfrac{1}{2} I \sin\tfrac{1}{2} I \cos\,(t + h - 2l + \tfrac{1}{2}\pi - \nu + 2\xi) \\ &\quad + \tfrac{1}{2}\sin I \cos I \cos\,(t + h - \tfrac{1}{2}\pi - \nu), \\ \tfrac{1}{3} - M_3^2 &= 0, \end{aligned}\right\} \quad (6).$$

We must now obtain approximate formulæ to express the functions of I, ν, and ξ in terms of ω, i, and ☊. The inclination i may be treated as so small that its square may be neglected.

From fig. 2 we have

$$\cos I = \cos i \cos\omega - \sin i \sin\omega \cos ☊,$$

whence

$$\left.\begin{aligned}
\cos I &= \cos\omega - i\sin\omega\cos\text{Ⓧ}\,,\\
\cos^2\tfrac{1}{2}I &= \cos^2\tfrac{1}{2}\omega\,(1 - i\tan\tfrac{1}{2}\omega\cos\text{Ⓧ}\,),\\
\sin^2\tfrac{1}{2}I &= \sin^2\tfrac{1}{2}\omega\,(1 + i\cot\tfrac{1}{2}\omega\cos\text{Ⓧ}\,),\\
\cos^4\tfrac{1}{2}I &= \cos^4\tfrac{1}{2}\omega\,(1 - 2i\tan\tfrac{1}{2}\omega\cos\text{Ⓧ}\,),\\
\sin^2 I &= \sin^2\omega\,(1 + 2i\cot\omega\cos\text{Ⓧ}\,),\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\left(1 + i\frac{2\cos\omega - 1}{\sin\omega}\cos\text{Ⓧ}\right),\\
\sin I\cos I &= \sin\omega\cos\omega\,(1 + 2i\cot 2\omega\cos\text{Ⓧ}\,),
\end{aligned}\right\} \quad . \quad . \quad (7).$$

Again from the figure

$$\frac{\sin\nu}{\sin i} = \frac{\sin\text{Ⓧ}}{\sin I} = \frac{\sin(\text{Ⓧ} - \xi)}{\sin\omega}\,.$$

Since ξ is small,

$$1 - \xi\cot\text{Ⓧ} = \frac{\sin\omega}{\sin I} = 1 - i\cot\omega\cos\text{Ⓧ}\,,$$

$$\nu = \frac{\sin i\sin\text{Ⓧ}}{\sin I}\,.$$

Therefore

$$\left.\begin{aligned}
\nu &= i\,\frac{\sin\text{Ⓧ}}{\sin\omega}\,,\\
\xi &= i\cot\omega\sin\text{Ⓧ}\,,\\
\nu - \xi &= i\tan\tfrac{1}{2}\omega\sin\text{Ⓧ}\,,\\
2\xi - \nu &= i\,\frac{2\cos\omega - 1}{\sin\omega}\sin\text{Ⓧ}\,,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence from (7) and (8)

$$\left.\begin{aligned}
\cos^4\tfrac{1}{2}I\cos 2(\nu - \xi) &= \cos^4\tfrac{1}{2}\omega\,(1 - 2i\tan\tfrac{1}{2}\omega\cos\text{Ⓧ}\,),\\
\cos^4\tfrac{1}{2}I\sin 2(\nu - \xi) &= \cos^4\tfrac{1}{2}\omega\,.\,2i\tan\tfrac{1}{2}\omega\sin\text{Ⓧ}\,,\\
\tfrac{1}{2}\sin^2 I\cos 2\nu &= \tfrac{1}{2}\sin^2\omega\,(1 + 2i\cot\omega\cos\text{Ⓧ}\,),\\
\tfrac{1}{2}\sin^2 I\sin 2\nu &= \tfrac{1}{2}\sin^2\omega\,.\,2i\operatorname{cosec}\omega\sin\text{Ⓧ}\,,\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I\cos(2\xi - \nu) &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,(1 + i\,(2\cos\omega - 1)\operatorname{cosec}\omega\cos\text{Ⓧ}\,),\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I\sin(2\xi - \nu) &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,.\,i\,(2\cos\omega - 1)\operatorname{cosec}\omega\sin\text{Ⓧ}\,,\\
\tfrac{1}{2}\sin I\cos I\cos\nu &= \tfrac{1}{2}\sin\omega\cos\omega\,(1 + 2i\cot 2\omega\cos\text{Ⓧ}\,),\\
\tfrac{1}{2}\sin I\cos I\sin\nu &= \tfrac{1}{2}\sin\omega\cos\omega\,.\,i\operatorname{cosec}\omega\sin\text{Ⓧ}\,,
\end{aligned}\right\} \quad (9).$$

Now $\omega = 23^\circ\ 27'{\cdot}3$, $\quad i = 5^\circ\ 8'{\cdot}8$; whence

$$2i\tan\tfrac{1}{2}\omega = {\cdot}0372,\ 2i\cot\omega = \frac{{\cdot}283}{{\cdot}683},\ 2i\operatorname{cosec}\omega = \frac{{\cdot}308}{{\cdot}683},\ i\,\frac{2\cos\omega - 1}{\sin\omega} = {\cdot}188,$$

$$2i\cot 2\omega = \frac{{\cdot}115}{{\cdot}683},\ i\operatorname{cosec}\omega = \frac{{\cdot}154}{{\cdot}683}\,.$$

Four of these are written with ·683 in the denominator for reasons given below. Then from (9) and (6) we have

$$\left.\begin{aligned}
M_1^2 - M_2^2 &= \cos^4\tfrac{1}{2}\omega\cos 2(t+h-l) + \tfrac{1}{2}\sin^2\omega\cos 2(t+h)\\
&\quad + \Big\{-\cos^4\tfrac{1}{2}\omega\,.\,{\cdot}0372\cos 2(t+h-l)\\
&\qquad + \tfrac{1}{2}\sin^2\omega\,.\,\frac{{\cdot}283}{{\cdot}683}\cos 2(t+h)\Big\}\cos\aleph\\
&\quad + \Big\{\cos^4\tfrac{1}{2}\omega\,.\,{\cdot}0372\sin 2(t+h-l)\\
&\qquad + \tfrac{1}{2}\sin^2\omega\,.\,\frac{{\cdot}308}{{\cdot}683}\sin 2(t+h)\Big\}\sin\aleph\,.\\
M_1M_3 &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\cos(t+h-2l+\tfrac{1}{2}\pi)\\
&\quad + \tfrac{1}{2}\sin\omega\cos\omega\cos(t+h-\tfrac{1}{2}\pi)\\
&\quad + \Big\{\cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,.\,{\cdot}188\cos(t+h-2l+\tfrac{1}{2}\pi)\\
&\qquad + \tfrac{1}{2}\sin\omega\cos\omega\,.\,\frac{{\cdot}115}{{\cdot}683}\cos(t+h-\tfrac{1}{2}\pi)\Big\}\cos\aleph\\
&\quad + \Big\{-\cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,.\,{\cdot}188\sin(t+h-2l+\tfrac{1}{2}\pi)\\
&\qquad + \tfrac{1}{2}\sin\omega\cos\omega\,.\,\frac{{\cdot}154}{{\cdot}683}\sin(t+h-\tfrac{1}{2}\pi)\Big\}\sin\aleph
\end{aligned}\right\} \quad (10).$$

Now let c be the moon's mean distance, and let $\tau = \frac{3}{2}\mu M/c^3$, and

$$1 + P = \frac{c^3}{r^3}\cdot$$

Then as far as concerns the moon

$$V = \tau\rho^2(1+P)\{\tfrac{1}{2}\cos^2\lambda(M_1^2 - M_2^2) + \sin 2\lambda\,.\,M_1M_3\} \quad . \quad . \quad . \quad (11),$$

where $M_1^2 - M_2^2$, M_1M_3 are given by (10).

In writing down the corresponding functions for the sun, we shall write a subscript accent to all the symbols, and accordingly V for the sun is given by

$$V_{,} = \tau_{,}\rho^2(1+P_{,})\{\tfrac{1}{2}\cos^2\lambda(M_{,1}^2 - M_{,2}^2) + \sin 2\lambda\,.\,M_{,1}M_{,3}\} \quad . \quad . \quad (12),$$

where, by symmetry with (10)

$$\left.\begin{aligned}
M_{,1}^2 - M_{,2}^2 &= \cos^4\tfrac{1}{2}\omega\cos 2(t+h-l_{,}) + \tfrac{1}{2}\sin^2\omega\cos 2(t+h),\\
M_{,1}M_{,3} &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\cos(t+h-2l_{,}+\tfrac{1}{2}\pi)\\
&\quad + \tfrac{1}{2}\sin\omega\cos\omega\cos(t+h-\tfrac{1}{2}\pi),
\end{aligned}\right\} \quad . \quad (13).$$

The masses and mean distances of the moon and sun are such that

$$\frac{\tau}{\tau+\tau_{,}} = \cdot 683, \quad \frac{\tau_{,}}{\tau+\tau_{,}} = \cdot 317 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Strictly speaking, τ denotes $\frac{3}{2}\mu M c^{-3}(1+\frac{3}{2}e^2)$, and $\tau_{,}$ denotes $\frac{3}{2}\mu M_{,} c_{,}^{-3}(1+\frac{3}{2}e_{,}^2)$; where e, $e_{,}$ are the eccentricities of the lunar and solar orbits. A comparison with the Report on Tides to the British Association for 1883, will show that, for the purpose in hand, it is correct to introduce these factors involving e^2 and $e_{,}^2$.

The whole potential may be collected from (10), (11), (12), (13), (14).

Since P is small the factor $(1+P)$ may be treated as unity in the small terms which involve $\cos \varpi$ and $\sin \varpi$, and we shall find it convenient to distinguish the several parts of V, denoting by V the principal part, by V_P the part involving P, by V_ϖ the part involving ϖ, and by $V_{P,}$ the part involving $P_{,}$.

For brevity write

$$K = \rho^2 . \tfrac{1}{2}\cos^2\lambda . \cos^4\tfrac{1}{2}\omega ; \qquad K_{,,} = \rho^2 . \tfrac{1}{2}\cos^2\lambda . \tfrac{1}{2}\sin^2\omega ;$$
$$K_o = \rho^2 . \sin 2\lambda . \cos^3\tfrac{1}{2}\omega \sin\tfrac{1}{2}\omega ; \qquad K_{,} = \rho^2 . \sin 2\lambda . \tfrac{1}{2}\sin\omega\cos\omega .$$

Then remarking that $\tau(1+P)+\tau_{,}(1+P_{,}) = (\tau+\tau_{,})(1+\cdot 683P)(1+\cdot 317P_{,})$ approximately, we have

$$\begin{aligned} V = {} & \tau K \cos 2(t+h-l) + (\tau+\tau_{,}) K_{,,} \cos 2(t+h) + \tau_{,} K \cos 2(t+h-l_{,}) \\ & + \tau K_o \cos(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,}) K_{,} \cos(t+h-\tfrac{1}{2}\pi) \\ & + \tau_{,} K_o \cos(t+h-2l_{,}+\tfrac{1}{2}\pi). \end{aligned}$$

$$\begin{aligned} V_P = {} & P\{\tau K \cos 2(t+h-l) + \cdot 683(\tau+\tau_{,}) K_{,,} \cos 2(t+h) \\ & + \tau K_o \cos(t+h-2l+\tfrac{1}{2}\pi) + \cdot 683(\tau+\tau_{,}) K_{,} \cos(t+h-\tfrac{1}{2}\pi)\}. \end{aligned}$$

$$\begin{aligned} V_\varpi = {} & \cos\varpi\{-\tau K \cdot 0372 \cos 2(t+h-l) + (\tau+\tau_{,}) K_{,,} \cdot 283 \cos 2(t+h) \\ & \quad + \tau K_o \cdot 188 \cos(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,}) K_{,} \cdot 115 \cos(t+h-\tfrac{1}{2}\pi\} \\ & + \sin\varpi\{+\tau K \cdot 0372 \sin 2(t+h-l) + (\tau+\tau_{,}) K_{,,} \cdot 308 \sin 2(t+h) \\ & \quad - \tau K_o \cdot 188 \sin(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,}) K_{,} \cdot 154 \sin(t+h-\tfrac{1}{2}\pi)\}. \end{aligned}$$

$$\begin{aligned} V_{P,} = {} & P_{,}\{\cdot 317(\tau+\tau_{,}) K_{,,} \cos 2(t+h) + \tau_{,} K \cos 2(t+h-l_{,}) \\ & + \cdot 317(\tau+\tau_{,}) K_{,} \cos(t+h-\tfrac{1}{2}\pi) + \tau_{,} K_o \cos(t+h-2l_{,}+\tfrac{1}{2}\pi)\}. \quad . \quad (15). \end{aligned}$$

The whole tide-generating potential is then $V + V_P + V_\varpi + V_{P,}$.

§ 2. *Formula for the Height of Tide.*

If we had been going to consider the complete expression for the tide in terms of a series of simple harmonic functions of the time, it would have been necessary to substitute for l, $l_{,}$, P, $P_{,}$ their values in terms of the mean longitudes s and h of the two bodies, and of the eccentricities e, $e_{,}$ of their orbits. When the potential is so expanded the principle of forced oscillations allows us to conclude that the oscillations of the sea will be of the same periods and types as the several terms of the potential, but with amplitudes and phases which can only be deduced from observation. The oscillations of the sea will not however be necessarily of the simple harmonic form, and accordingly "over-tides" of double and triple frequency have to be introduced in order to represent the motion according to FOURIER'S method.

This is the plan pursued in the "harmonic analysis of tidal observations," and each simple harmonic oscillation is known by an arbitrarily chosen initial letter.

It is found in fact, as is suggested by theory, that tides of approximately the same frequency or "speed" have amplitudes approximately proportional to their corresponding terms in the potential, and have their phases retarded by approximately the same amount.

The notation of harmonic analysis will be adopted here, because it is proposed to compute the tide table from the harmonic constants.

The mean longitudes of the sun, moon, and lunar perigee are denoted by h, s, p, and their hourly changes by σ, η, ϖ (from $\sigma\epsilon\lambda\acute{\eta}\nu\eta$, $\mathring{\eta}\lambda\iota\circ\varsigma$, perigee); the angular velocity of the earth's rotation is γ (from $\gamma\hat{\eta}$), so that $\gamma - \eta$ is 15° per hour.

The eccentricities of the lunar and solar orbits are e, $e_{,}$.

In the harmonic system the tides are denominated by the arbitrarily chosen initials M_2, M_4, S_2, S_4, K_2, K_1, N, L, ν, λ, μ, T, R, O, P, Q, J, Sa, Ssa, and the semi-ranges of these tides will be here denoted by the suffixes m, $2m$, s, $2s$, n, l, ν, λ, μ, t, r, o, p, q, j, sa, ssa to the symbol H, and the retardations of phase by the same suffixes to the symbol κ. But the semi-ranges and retardations of the tides K_2, K_1 are denoted by $H_{,,}$, $\kappa_{,,}$, $H_{,}$, $\kappa_{,}$ instead of conforming to the rest of the notation.*

Then all the H's and κ's are the immediate results of harmonic analysis, and are supposed to be given when the construction of a tide-table is contemplated.

In the development (15) of V, the first term corresponds with the principal lunar tide M_2, its over-tide M_4, parts of the elliptic tides N, L, parts of the evectional tides ν, λ, and part of the variational tide μ.

The first term of V_P contributes the remainder of N, L, ν, λ, μ.

The second term of V corresponds with the luni-solar semi-diurnal tide K_2, and the second term of V_P corresponds with small inequalities in K_2, which are neglected in the harmonic analysis.

* I adopted this originally for convenience in writing, and having got used to the notation shall retain it.

The third term of V corresponds with the principal solar tide S_2, its over-tide S_4, and parts of the elliptic tides T, R.

The fourth term of V corresponds with the lunar diurnal tide O, and parts of the elliptic tides M_1 and Q; the third term of V_P corresponds with another part of M_1 and the rest of Q.

The fifth term of V corresponds with the luni-solar diurnal tide K_1, and with part of the elliptic tides M_1 and J; its over-tide fuses with K_2. The fourth term of V_P corresponds with part of the elliptic tide M_1, and the rest of J.

The last term of V corresponds with the solar diurnal tide P, and with small inequalities neglected in the harmonic analysis.

The whole of $V_\mathfrak{S}$ corresponds with those factors of augmentation and small alterations of phase, which are denoted by f and functions of ν and ξ in harmonic analysis.

V_{P_1} corresponds with the remainder of the solar elliptic tides T, R and with other small inequalities usually neglected in the harmonic analysis.

The semi-annual (Ssa) and annual (Sa) tides are due to meteorological causes, and have to be introduced after we have done with astronomical considerations, so that they do not enter through the V's.

Since the terms V, V_P, $V_\mathfrak{S}$, V_{P_1} are approximately simple harmonics, it follows that each term will correspond to an approximately simple harmonic tide, with amplitude and phase the same as that given by harmonic analysis, and to its over-tides. If then we write

$$\Delta = 2t + 2h - 2l - \kappa_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

the height of water corresponding to V is,

$$\begin{aligned} h = {} & \mathrm{H}_m \cos \Delta + \mathrm{H}_{''} \cos (\Delta + \kappa_m + 2l - \kappa_{''}) + \mathrm{H}_s \cos (\Delta + \kappa_m + 2l - 2l_{'} - \kappa_s) \\ & + \mathrm{H}_o \cos \left[\tfrac{1}{2} (\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o\right] + \mathrm{H}_{'} \cos \left[\tfrac{1}{2} (\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{'}\right] \\ & + \mathrm{H}_p \cos \left[\tfrac{1}{2} (\Delta + \kappa_m) + l - 2l_{'} + \tfrac{1}{2}\pi - \kappa_p\right] + \mathrm{H}_{2m} \cos \left[2(\Delta + \kappa_m) - \kappa_{2m}\right] \\ & + \mathrm{H}_{2s} \cos \left[2 (\Delta + \kappa_m) + 4l - 4l_{'} - \kappa_{2s}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17). \end{aligned}$$

We shall, in the developments immediately following, leave V_{P_1} out of consideration, and shall resume the effects of the sun's parallax in § 11.

In order to represent, as far as possible, the elliptic tides, it is found to be necessary to introduce certain numerical factors α, β, and certain angles ${}_m\kappa$, ${}_o\kappa$, as shown in the next formula. It is also practically convenient that P should denote the value of c^3/r^3, not at the time corresponding to t, but some hours earlier.

Then the height of water corresponding to V_P is taken as

$$\begin{aligned} h_P = P \{ & \alpha \mathrm{H}_m \cos (\Delta + \kappa_m - {}_m\kappa) + \cdot 683\, \mathrm{H}_{''} \cos (\Delta + \kappa_m + 2l - \kappa_{''}) \\ & + \beta \mathrm{H}_o \cos \left[\tfrac{1}{2} (\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - {}_o\kappa\right] \\ & + \cdot 683\, \mathrm{H}_{'} \cos \left[\tfrac{1}{2} (\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{'}\right] \} \quad . \quad . \quad . \quad . \quad . \quad (18). \end{aligned}$$

The height corresponding to $V_{\aleph}$ is

$$\begin{aligned} h_{\aleph} = \cos\aleph \{ &- \mathrm{H}_m \cdot 0372 \cos\Delta + \mathrm{H}_{''} \cdot 283 \cos(\Delta + \kappa_m + 2l - \kappa_{''}) \\ &+ \mathrm{H}_o \cdot 188 \cos[\tfrac{1}{2}(\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o] \\ &+ \mathrm{H}_{'} \cdot 115 \cos[\tfrac{1}{2}(\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{'}]\} \\ + \sin\aleph \{ &\mathrm{H}_m \cdot 0372 \sin\Delta + \mathrm{H}_{''} \cdot 308 \sin(\Delta + \kappa_m + 2l - \kappa_{''}) \\ &- \mathrm{H}_o \cdot 188 \sin[\tfrac{1}{2}(\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o] \\ &+ \mathrm{H}_{'} \cdot 154 \sin[\tfrac{1}{2}(\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{'}]\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19). \end{aligned}$$

§ 3. *The Elliptic Tides.*

The constants α, β, ${}_m\kappa$, ${}_o\kappa$, are insufficient to fully represent the harmonic tides, N, L, ν, λ, μ, Q, J, and it would render the proposed method of computing a tide-table too cumbrous for practical use if additional constants were introduced. It is, therefore, necessary to adopt a compromise.

The tides N and L of the harmonic system are given by

$$\mathrm{H}_n \cos[2t + 2h - 2s - (s - p) - \kappa_n] + \mathrm{H}_l \cos[2t + 2h - 2s + (s - p) + \pi - \kappa_l].$$

In the present development the mean lunar tide M_2 and the elliptic tides are to be represented by

$$\mathrm{H}_m \cos[2t + 2h - 2l - \kappa_m] + \alpha\, \mathrm{H}_m P \cos[2t + 2h - 2l - {}_m\kappa].$$

It has been already remarked that it will be convenient that P should refer to a time earlier than t, and the time referred to is that of the moon's transit. Now H. W. occurs later than moon's transit by an interval whose mean value is $\kappa_m/2\,(\gamma - \sigma)$, and the mean value of the interval to the preceding L. W. is $(\kappa_m - \pi)/2\,(\gamma - \sigma)$. If, therefore, we put $\zeta = \dfrac{\sigma - \varpi}{2\,(\gamma - \sigma)}\kappa_m$ for H. W., and $\dfrac{\sigma - \varpi}{2\,(\gamma - \sigma)}(\kappa_m - \pi)$ for L. W., we have approximately

$$P = \frac{v^3}{r^3} - 1 = 3e \cos[s - p - \zeta],$$

also

$$2l = 2s + 4e \sin(s - p).$$

Hence the elliptic tides are represented in the present development by

$$\begin{aligned} &2e_m\mathrm{H} \cos(2t + 2h - 3s + p - \kappa_m) + \tfrac{3}{2}\alpha e \mathrm{H}_m \cos(2t + 2h - 3s + p - {}_m\kappa + \zeta) \\ &+ 2e\mathrm{H}_m \cos(2t + 2h - s - p + \pi - \kappa_m) - \tfrac{3}{2}\alpha e \mathrm{H}_m \cos(2t + 2h - s - p + \pi - {}_m\kappa - \zeta). \end{aligned}$$

In order that the first pair of these terms may give the N tide correctly we must have

$$H_n = 2eH_m \cos(\kappa_n - \kappa_m) + \tfrac{3}{2}\alpha e H_m \cos(\kappa_n - {}_m\kappa + \zeta),$$
$$0 = 2eH_m \sin(\kappa_n - \kappa_m) + \tfrac{3}{2}\alpha e H_m \sin(\kappa_n - {}_m\kappa + \zeta).$$

And the condition that the second pair may give the L tide is

$$H_l = 2eH_m \cos(\kappa_l - \kappa_m) - \tfrac{3}{2}\alpha e H_m \cos(\kappa_l - {}_m\kappa - \zeta),$$
$$0 = 2eH_m \sin(\kappa_l - \kappa_m) - \tfrac{3}{2}\alpha e H_m \sin(\kappa_l - {}_m\kappa - \zeta).$$

The condition for N may be written

$$\tan(\kappa_n - {}_m\kappa + \zeta) = \frac{2eH_m \sin(\kappa_m - \kappa_n)}{H_n - 2eH_m \cos(\kappa_m - \kappa_n)},$$
$$\tfrac{3}{2}\alpha e H_m = \{H_n - 2eH_m \cos(\kappa_m - \kappa_n)\} \sec(\kappa_n - {}_m\kappa + \zeta).$$

Then, since both $\kappa_n - \kappa_m$ and $\kappa_n - {}_m\kappa + \zeta$ are small angles, we have approximately

$${}_m\kappa = \kappa_n + \zeta - \frac{2eH_m(\kappa_m - \kappa_n)}{H_n - 2eH_m},$$
$$\alpha H_m = \frac{2}{3e}(H_n - 2eH_m).$$

Similarly the condition for L gives

$${}_m\kappa = \kappa_l - \zeta - \frac{2eH_m(\kappa_l - \kappa_m)}{2eH_m - H_l},$$
$$\alpha H_m = \frac{2}{3e}(2eH_m - H_l).$$

These conditions cannot be satisfied simultaneously and a compromise must be adopted.

If H_l were zero we should have $\alpha = \frac{4}{3}$. Hence if α be taken as greater than $\frac{4}{3}$ we are virtually making the L tide negative. Now, it is unadvisable to take the L tide as negative even if small, because, if so, we are representing the elliptic tide compounded from N and L as greatest when it should be smallest and *vice versâ*.

I therefore propose to use the condition for the L tide only for the purpose of putting a superior limit to α, and to the constants to be derived from the condition for the more important N tide.

In order to express the result in the form which will ultimately be required I write

$$p_n = \frac{2\gamma - 3\sigma + \varpi}{2\gamma - 2\sigma} \qquad p_m = \frac{2\gamma - 2\sigma}{2\gamma - 2\sigma} = 1,$$

so that p_n is the ratio of the "speed" of the N tide to that of M_2, and p_m is merely introduced for the sake of algebraic symmetry.

Then

$$\zeta = (p_m - p_n)\,\kappa_m \text{ for H. W. and } (p_m - p_n)\,(\kappa_m - \pi) \text{ for L. W.}$$

We thus have approximately, when referring to H. W.,

$$p_m\kappa_m - {}_m\kappa = p_n\kappa_m - \kappa_n + \frac{2e\mathrm{H}_m\,(\kappa_m - \kappa_n)}{\mathrm{H}_n - 2e\mathrm{H}_m},$$

and

$$\alpha\mathrm{H}_m \text{ is equal to the smaller of } \frac{2}{3e}\,(\mathrm{H}_n - 2e\mathrm{H}_m) \text{ and } \tfrac{4}{3}\,\mathrm{H}_m.$$

When referring to L. W. we have $(\kappa_m - \pi)$ in the above formula in the place of κ_m wherever κ_m occurs multiplied by a p.

A similar argument may be followed with regard to the diurnal tides, save that the smaller tide corresponding to L, has not usually been evaluated in the harmonic analysis. The tide corresponding to N bears the initial Q, and

$$p_q = \frac{\gamma - 3\sigma + \varpi}{2\gamma - 2\sigma}, \quad p_o = \frac{\gamma - 2\sigma}{2\gamma - 2\sigma},$$

so that p_q and p_o are the ratios of the "speeds" of the Q and O tides to that of M_2. Then

$$\frac{\sigma - \varpi}{2\,(\gamma - \sigma)} = p_o - p_q,$$

and we find for H. W.

$$p_o\kappa_m - {}_o\kappa = p_q\kappa_m - \kappa_q + \frac{2e\mathrm{H}_o\,(\kappa_o - \kappa_q)}{\mathrm{H}_q - 2e\mathrm{H}_o},$$

and

$$\beta\mathrm{H}_o \text{ is equal to the smaller of } \frac{2}{3e}\,(\mathrm{H}_q - 2e\mathrm{H}_o) \text{ and } \tfrac{4}{3}\,\mathrm{H}_o.$$

Since $e = \cdot0549$, $\frac{2}{3e} = 12\cdot144 = 19 \times \cdot639$ (the factor being introduced because we shall hereafter have to divide by 19); also $\frac{4}{3} = 19 \times \frac{4}{57} = 19 \times \cdot070$, and $\frac{1}{2e} = 9\cdot11$.

If we put then $\alpha' = \frac{1}{19}\,\alpha$, $\beta' = \frac{1}{19}\,\beta$, our formulæ for H. W. become

$$\left.\begin{aligned} p_m\kappa_m - {}_m\kappa &= p_n\kappa_m - \kappa_n + \frac{(\kappa_m - \kappa_n)}{9\cdot11\,\mathrm{H}_n/\mathrm{H}_m - 1}, \\ p_o\,\kappa_m - {}_o\kappa &= p_q\kappa_m - \kappa_q + \frac{(\kappa_o - \kappa_q)}{9.11\,\mathrm{H}_q/\mathrm{H}_o - 1}, \end{aligned}\right\} \quad \ldots\ldots \quad (20).$$

where $p_m = 1$, $p_n = \cdot 981$, $p_o = \cdot 481$, $p_q = \cdot 462$, and where for L. W. κ_m is to be replaced by $\kappa_m - \pi$ wherever it is multiplied by a p.

Also, if $36\cdot4\ H_n$ is less than $8H_m$,

$$\left.\begin{aligned} \alpha &= 12\cdot14\,\frac{H_n}{H_m} - \tfrac{4}{3} \\ \alpha' H_n &= \cdot639\,H_m - \cdot070\,H_m \end{aligned}\right\}$$

If $36\cdot4\ H_q$ is less than $8H_o$

$$\left.\begin{aligned} \beta &= 12\cdot14\,\frac{H_q}{H_o} - \tfrac{4}{3} \\ \beta' H_o &= \cdot639\,H_q - \cdot070\,H_o \end{aligned}\right\} \qquad \ldots\ldots\ldots \quad (21).$$

If $36\cdot4\ H_n$ is greater than $8\ H_m$, $\alpha = \frac{4}{3}$, and $\alpha' H_m = \cdot070\ H_m$.

If $36\cdot4\ H_q$ is greater than $8\ H_o$, $\beta = \frac{4}{3}$, and $\beta' H_o = \cdot070\ H_o$.

Lastly, if the elliptic tides are unknown, $\alpha = \beta = 1$, $\alpha' = \beta' = \frac{1}{19}$.

§ 4. *Reference to Transit of Fictitious Moon.*

Suppose that there is a fictitious satellite whose R. A. is equal to the Moon's longitude, and let $l^{(o)}$, $h^{(o)}$ be the R. A. of the fictitious moon and of the mean sun at noon of the day under consideration, and let τ be the mean solar time of the fictitious moon's transit.

Then at transit

$$t + h - l = q\pi,$$

where q is even for upper and odd for lower transit.

The rate of change of $t + h$ is γ, and if $\dot{l}$ be the rate of change of l, τ is given by

$$\tau = \frac{l^{(o)} - h^{(o)} + q\pi}{\gamma - \dot{l}}.$$

Then at time $\tau + T$

$$\begin{aligned} \Delta &= 2t + 2h - 2l - \kappa_m \\ &= 2q\pi + 2(\gamma - \dot{l})\,T - \kappa_m, \end{aligned}$$

and

$$\tfrac{1}{2}\Delta = q\pi + (\gamma - \dot{l})\,T - \tfrac{1}{2}\kappa_m.$$

Now Δ occurs in the arguments of all the semi-diurnal tides, $\frac{1}{2}\Delta$ in those of the diurnal, and 2Δ in those of the quater-diurnal.

Hence we may omit the $2q\pi$ in the expression for Δ, and write

$$\Delta = 2(\gamma - \dot{l})\,T - \kappa_m,$$

provided that when q is odd we change the signs of the diurnal terms.

Therefore we may write the diurnal terms with alternative signs $\pm$, and the upper sign will be appropriate when we refer to an upper transit, and the lower sign to a lower transit.

In the application of the formulæ $T + \tau$ will be the time for which the mean solar hour angle is t, and we shall have

$$T = \frac{\kappa_m}{2(\gamma - \dot{l})} + \frac{\Delta}{2(\gamma - \dot{l})} \quad \ldots\ldots\ldots \quad (22).$$

The first of these terms is obviously the interval from fictitious transit to the mean lunar H. W., and the second is the interval from lunar H. W. to the time t.

When we wish to discuss low waters it is convenient to put Δ equal to $\delta - \pi$, and in this case we shall have

$$T = \frac{\kappa_m - \pi}{2(\gamma - \dot{l})} + \frac{\delta}{2(\gamma - \dot{l})} \quad \ldots\ldots\ldots \quad (23).$$

The first of these terms is the interval from transit to the mean lunar L. W., which precedes the H. W. given by (22), and the second is the interval after lunar L. W. to the time t.

We shall proceed to consider H. W., and deduce therefrom the formulæ for L. W.

§ 5. *Reduction of Longitudes of Moon and Sun to time of Fictitious Transit.*

We have seen in (22) that the interval, from fictitious moon's transit to the time t, is

$$T = \frac{\Delta + \kappa_m}{2(\gamma - \dot{l})}.$$

Now the equation of conservation of areas for the moon's motion gives

$$r^2 \dot{l} = \sigma c^2 \sqrt{(1 - e^2)},$$

but since $c^3/r^3 = 1 + P$, and since P is small, we have

$$\dot{l} = \sigma (1 + \tfrac{2}{3} P), \text{ very nearly.}$$

Therefore

$$T = \frac{\Delta + \kappa_m}{2(\gamma - \sigma)} \left(1 + \tfrac{2}{3} \frac{P\sigma}{\gamma - \sigma}\right),$$

$$\dot{l}T = \frac{(\Delta + \kappa_m)\,\sigma}{2(\gamma - \sigma)} + \tfrac{2}{3} P \frac{\gamma\sigma}{2(\gamma - \sigma)^2} (\Delta + \kappa_m).$$

Let

$$p_m = \frac{2\gamma - 2\sigma}{2\gamma - 2\sigma}, \qquad p_{''} = \frac{2\gamma}{2\gamma - 2\sigma}, \qquad p_s = \frac{2\gamma - 2\eta}{2\gamma - 2\sigma},$$

$$p_o = \frac{\gamma - 2\sigma}{2\gamma - 2\sigma}, \qquad p_{'} = \frac{\gamma}{2\gamma - 2\sigma}, \qquad p_p = \frac{\gamma - 2\eta}{2\gamma - 2\sigma},$$

and let

$$\epsilon = \tfrac{2}{3}\frac{\gamma\sigma}{2(\gamma - \sigma)^2} = \tfrac{1}{3}p_{''}(p_{''} - 1) = \cdot 01311.$$

Then if l_o, $l_{o'}$ be the moon's and sun's longitudes at the time of fictitious transit, we have

$$l = l_o + \frac{\sigma}{2\gamma - 2\sigma}(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m),$$

$$l_{'} = l_{o'} + \frac{\eta}{2\gamma - 2\sigma}(\Delta + \kappa_m) + \epsilon\frac{\eta}{\gamma}P(\Delta + \kappa_m).$$

Then neglecting the term in P when multiplied by the small fraction η/γ, and introducing the p's, we have

$$\left.\begin{aligned}
2l &= 2l_o + (p_{''} - 1)(\Delta + \kappa_m) + 2\epsilon P(\Delta + \kappa_m),\\
2(l - l_{'}) &= 2(l_o - l_{o'}) + (p_s - 1)(\Delta + \kappa_m) + 2\epsilon P(\Delta + \kappa_m),\\
-l &= -l_o + (p_o - \tfrac{1}{2})(\Delta + \kappa_m) - \epsilon P(\Delta + \kappa_m),\\
l &= l_o + (p_{'} - \tfrac{1}{2})(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m),\\
l - 2l_{'} &= l_o - 2l_{o'} + (p_p - \tfrac{1}{2})(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m).
\end{aligned}\right\} \quad . \quad . \quad (24).$$

We must now take the several terms of the expressions for the tide in (17) (18) (19), and rearrange them by aid of (24).

(i.) $\cos(\Delta + \kappa_m + 2l - \kappa_{''}) + \cdot 683\, P \cos(\Delta + \kappa_m + 2l - \kappa_{''})$

$$= \cos[p_{''}(\Delta + \kappa_m) + 2l_o - \kappa_{''}] + \cdot 683\, P \cos[{}_{''}p(\Delta + \kappa_m) + 2l_o - \kappa_{''}],$$

where

$${}_{''}p = p_{''} + \frac{2\epsilon}{\cdot 683} = p_{''} + \cdot 0384 = 1\cdot 0379 + \cdot 0384 = 1\cdot 0763.$$

(ii.) $\cos(\Delta + \kappa_m + 2(l - l_{'}) - \kappa_s) = \cos[p_s(\Delta + \kappa_m) + 2(l_o - l_{o'}) - \kappa_s]$

$$- 2\epsilon P(\Delta + \kappa_m)\sin[p_s(\Delta + \kappa_m) + 2(l_o - l_{o'}) - \kappa_s].$$

Now the maximum value of P is about $\frac{1}{6}$, and $\Delta + \kappa_m$ will not differ largely from κ_m, because Δ will oscillate about the value zero, and κ_m might always be taken as less than 180°, either positively or negatively. Hence the coefficient of this second term cannot exceed $2 \times \cdot 013 \times \frac{1}{6} \times \pi$, or about $\cdot 013$.

Thus the second term cannot be more than about an eightieth of the first, and may be neglected. This term may also be safely neglected, even when κ_m is greater than 180°.

(iii.) $$\cos\left[\tfrac{1}{2}(\Delta+\kappa_m)-l+\tfrac{1}{2}\pi-\kappa_o\right]+P\beta\cos\left[\tfrac{1}{2}(\Delta+\kappa_m)-l+\tfrac{1}{2}\pi-{}_o\kappa\right]$$
$$=\cos\left[p_o(\Delta+\kappa_m)-l_o+\tfrac{1}{2}\pi-\kappa_o\right]+P\beta\cos\left[{}_op(\Delta+\kappa_m)-l_o+\tfrac{1}{2}\pi-{}_o\kappa\right],$$

where

$${}_op=p_o-\frac{\epsilon}{\beta}=\cdot4811-\frac{\cdot0131}{\beta}.$$

The reader who verifies the above formula will perceive the nature of the approximation adopted.

(iv.) $$\cos\left[\tfrac{1}{2}(\Delta+\kappa_m)+l-\tfrac{1}{2}\pi-\kappa_{,}\right]+\cdot683\,P\cos\left[\tfrac{1}{2}(\Delta+\kappa_m)+l-\tfrac{1}{2}\pi-\kappa_{,}\right]$$
$$=\cos\left[p_{,}(\Delta+\kappa_m)+l_o-\tfrac{1}{2}\pi-\kappa_{,}\right]+\cdot683\,P\cos\left[{}_{,}p(\Delta+\kappa_m)+l_o-\tfrac{1}{2}\pi-\kappa_{,}\right],$$

where

$${}_{,}p=p_{,}+\frac{\epsilon}{\cdot683}=p_{,}+\cdot0192=\cdot5189+\cdot0192=\cdot5381.$$

(v.) The P term may be written

$$\cos\left[p_p(\Delta+\kappa_m)+l_o-2l_{o,}+\tfrac{1}{2}\pi-\kappa_p\right],$$

with a similar neglect to that involved in (ii.), but, of course, less important.

Before using these transformations the following notation must be introduced

$$\left.\begin{aligned}\mathfrak{D}&=l_o-l_{o,},\\ \odot&=l_{o,}+5^\circ.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(25).$$

So that $l_o=\mathfrak{D}+\odot-5^\circ$, $l_{o,}=\odot-5^\circ$. We then write

$$\begin{aligned}
\vartheta_m&=p_m\kappa_m-\kappa_m,\\
{}_m\vartheta&=p_m\kappa_m-{}_m\kappa\ \text{(see (20), § 3)}\\
\vartheta_{,,}&=2(\mathfrak{D}+\odot-5^\circ)+p_{,,}\kappa_m-\kappa_{,,},\\
{}_{,,}\vartheta&=\vartheta_{,,}+({}_{,,}p-p_{,,})\kappa_m=\vartheta_{,,}+\cdot0384\,\kappa_m,\\
\vartheta_s&=2\mathfrak{D}+p_s\kappa_m-\kappa_s,\\
\vartheta_o&=-(\mathfrak{D}+\odot-5^\circ)+p_o\kappa_m-\kappa_o+90^\circ,\\
{}_o\vartheta&=-(\mathfrak{D}+\odot-5^\circ)+p_o\kappa_m-{}_o\kappa-\frac{\cdot01311}{\beta}\kappa_m+90^\circ\ \text{(see (20), § 3)},\\
\vartheta_{,}&=(\mathfrak{D}+\odot-5^\circ)+p_{,}\kappa_m-\kappa_{,}-90^\circ,\\
{}_{,}\vartheta&=\vartheta_{,}+({}_{,}p-p_{,})\kappa_m=\vartheta_{,}+\cdot0192\,\kappa_m,\\
\vartheta_p&=(\mathfrak{D}-\odot+5^\circ)+p_p\kappa_m-\kappa_p+90^\circ\quad\ldots\ldots\quad(26).
\end{aligned}$$

We also require for the quater-diurnal tides

$$\vartheta_{2m} = 2p_m\kappa_m - \kappa_{2m},$$
$$\vartheta_{2s} = 2p_s\kappa_m - \kappa_{2s} \quad \ldots\ldots\ldots\ldots (26),$$

and for the annual and semi-annual tides

$$\vartheta_{sa} = (\odot - 5^\circ) - \kappa_{sa},$$
$$\vartheta_{ssa} = 2(\odot - 5^\circ) - \kappa_{ssa} \quad \ldots\ldots\ldots\ldots (26).$$

The parts of the ϑ's which are independent of $\mathbb{D}$ and $\odot$ will, in the example, be written K, with appropriate suffix.

The moon's mean parallax is $57'$, then when its parallax is $57' + \Pi$,

$$1 + P = (1 + \tfrac{1}{57}\Pi)^3,$$

so that $P = \frac{1}{19}\Pi = \cdot0526\,\Pi$ approximately.

Π will be substituted for P, and the numerical coefficients in h_{P} will be divided by 19 or multiplied by $\cdot0526$.

We observe then that $\frac{1}{19} \times \cdot683 = \cdot0360$, and $\frac{1}{19}\alpha = \alpha'$, $\frac{1}{19}\beta = \beta'$.

We can, by aid of the transformations (i.) to (v.) and the formulæ (26), now re-write (17) (18) (19), but some part of what was originally included in h is transferred to h_{P}. The result is as follows :—

$$\begin{aligned} h = {} & \mathrm{H}_m \cos(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cos(p_{,,}\Delta + \vartheta_{,,}) + \mathrm{H}_s \cos(p_s\Delta + \vartheta_s) \\ & \pm \mathrm{H}_o \cos(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cos(p_{,}\Delta + \vartheta_{,}) \pm \mathrm{H}_p \cos(p_p\Delta + \vartheta_p) \\ & + \mathrm{H}_{2m} \cos(2p_m\Delta + \vartheta_{2m}) + \mathrm{H}_{2s} \cos(2p_s\Delta + \vartheta_{2s}) \quad \ldots\ldots (27). \end{aligned}$$

$$\begin{aligned} h_{\mathrm{P}} = {} & \Pi\{\alpha' \mathrm{H}_m \cos(p_m\Delta + {}_m\vartheta) + \cdot0360\,\mathrm{H}_{,,} \cos({}_{,,}p\Delta + {}_{,,}\vartheta) \\ & \pm \beta' \mathrm{H}_o \cos({}_op\Delta + {}_o\vartheta) \pm \cdot0360\,\mathrm{H}_{,} \cos({}_{,}p\Delta + {}_{,}\vartheta)\} \quad \ldots\ldots (28). \end{aligned}$$

$$\begin{aligned} h_{\aleph} = {} & \cos\aleph\ \{-\mathrm{H}_m \cdot037 \cos(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cdot283 \cos(p_{,,}\Delta + \vartheta_{,,}) \\ & \pm \mathrm{H}_o \cdot188 \cos(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cdot115 \cos(p_{,}\Delta + \vartheta_{,})\} \\ & + \sin\aleph\ \{\mathrm{H}_m \cdot037 \sin(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cdot308 \sin(p_{,,}\Delta + \vartheta_{,,}) \\ & \mp \mathrm{H}_o \cdot188 \sin(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cdot154 \sin(p_{,}\Delta + \vartheta_{,})\} \quad \ldots (29). \end{aligned}$$

$$\text{Annual and semi-annual tides} = \mathrm{H}_{sa} \cos\vartheta_{sa} + \mathrm{H}_{ssa} \cos\vartheta_{ssa} \quad \ldots\ldots (30).$$

Except for the solar parallactic portion to be considered later, this is the final expression for the tide. Apart from the slow variability of Π and $\aleph$ the only variable is Δ, for the ϑ's are not continuous variables, but change *per saltum* from one lunar transit to the next. Thus, in finding the maximum or H. W., we may treat Δ as the variable.

§ 6. *The Time and Height of High Water.*

Apart from parallactic and nodal corrections H. W. occurs when h in (27) is a maximum, and the value of Δ which satisfies that condition will give the time estimated from the epoch when Δ is zero. The value of Δ, when reduced to time in the manner shown below, gives the inequality in the interval after moon's transit, and will be called I. The time from moon's transit until Δ is zero is the mean interval, and will be called i. Then the interval from moon's transit is $i + I$. There is indeed a small parallactic correction to i, which is omitted in this statement, but is included below.

I begin by considering the mean interval.

Δ vanishes when $2(t + h - l) - \kappa_m = 0$, that is to say, at a time $\kappa_m/2\ (\gamma - \dot{l})$ after moon's transit.

Now, as shown in § 5,

$$\gamma - \dot{l} = (\gamma - \sigma)\left(1 + \tfrac{2}{3}\frac{P\sigma}{\gamma - \sigma}\right);$$

therefore

$$\frac{\kappa_m}{2(\gamma - l)} = \frac{\kappa_m}{2(\gamma - \sigma)} + \frac{\kappa_m}{2(\gamma - \sigma)}\tfrac{2}{3}\frac{P\sigma}{\gamma - \sigma}.$$

The first of these terms is denoted by i, so that

$$i = \frac{\kappa_m}{2(\gamma - \sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The second term is the parallactic correction to the mean interval, and will be considered in § 8, together with the other parallactic corrections.

If we denote by the suffixes 2, 1, 4 the semi-diurnal, diurnal, and quater-diurnal terms, (27) may be written

$$h = \Sigma\, \mathrm{H}_2 \cos(p_2 \Delta + \vartheta_2) + \Sigma\, \mathrm{H}_1 \cos(p_1 \Delta + \vartheta_1) + \Sigma\, \mathrm{H}_4 \cos(p_4 \Delta + \vartheta_4).$$

The condition for maximum is that $dh/d\Delta$ should vanish, and the equation consists of the sum of three pairs of terms of the form $\Sigma\,[\mathrm{H}p \cos\vartheta \sin p\Delta + \mathrm{H}p \sin\vartheta \cos p\Delta]$ equated to zero.

It is well known that

$$\frac{\sin k\alpha}{k \sin\alpha} = 1 + \frac{1 - k^2}{3!}\sin^2\alpha + \ldots$$

$$\frac{\cos k\alpha}{\cos\alpha} = 1 + \frac{1 - k^2}{2!}\sin^2\alpha + \ldots$$

Now all the p_2's are nearly unity, the p_1's nearly $\frac{1}{2}$, the p_4's nearly 2. Hence to a near approximation

$$\sin p_n\Delta = \frac{2p_n}{n}\sin\tfrac{1}{2}n\Delta\left[1-\tfrac{1}{6}\left(\frac{4p_n^2}{n^2}-1\right)\sin^2\tfrac{1}{2}n\Delta\right],$$

$$\cos p_n\Delta = \cos\tfrac{1}{2}n\Delta\left[1-\tfrac{1}{2}\left(\frac{4p_n^2}{n^2}-1\right)\sin^2\tfrac{1}{2}n\Delta\right],$$

where n is 2, 1, or 4.

The p's are so nearly equal to 1, $\frac{1}{2}$, 2 respectively, that the second terms are small. In the case of the second term of $\sin p_1\Delta$, $\cos p_1\Delta$ we have the factor $\sin^2\Delta$, which is equal to $\frac{1}{2}$ even when Δ is 90°; also the height of the quater-diurnal tide is small. Thus, both for the diurnal and quater-diurnal tides, the second term may be neglected.

In the case of the semi-diurnal terms, indeed, the correction is so small that it may certainly be neglected if Δ be less than 45° or 50°, and without much loss of accuracy even for larger values of Δ. We shall in the first place neglect these second terms entirely.

Now let

$$\left.\begin{aligned} F &= \Sigma\, Hp\sin\vartheta \\ G &= \Sigma\, Hp^2\cos\vartheta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

with suffixes 2, 1, 4 for semi-diurnal, diurnal, quater-diurnal terms.

The constituent terms of F, G will be written separately below, for example $F_s = H_s p_s \sin\vartheta_s$.

Then the equation $dh/d\Delta = 0$, leads to

$$F_2\cos\Delta + G_2\sin\Delta \pm (F_1\cos\tfrac{1}{2}\Delta + 2G_1\sin\tfrac{1}{2}\Delta) + F_4\cos 2\Delta + \tfrac{1}{2}G_4\sin 2\Delta = 0. \quad . \quad (33).$$

With regard to the additional terms referred to above, the values of the semi-diurnal p's are, $p_m = 1$, $p_{''} = 1{\cdot}038$, $p_s = 1{\cdot}035$, and hence $p_{''}^2 - 1$ and $p_s^2 - 1$ are both nearly equal to ·072. Therefore if (33) be regarded as the fundamental equation, the additional terms may be taken into account by supposing that there are corrections to F_2 and G_2 given by

$$\left.\begin{aligned} \delta F_2 &= -{\cdot}012\sin^2\Delta\,.\,3F_2 \\ \delta G_2 &= -{\cdot}012\sin^2\Delta\,(G_{''} + G_s) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

The equation (33) may be solved thus:—
take

$$\tan\Delta_0 = -\frac{F_2 \pm F_1}{G_2 \pm G_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Then if we put

$$D_o = F_2 \sin \Delta_o - G_2 \cos \Delta_o \pm \tfrac{1}{2}(F_1 \sin \tfrac{1}{2}\Delta_o - 2G_1 \cos \tfrac{1}{2}\Delta_o) + 2(F_4 \sin 2\Delta_o - \tfrac{1}{2} G_4 \cos 2\Delta_o)$$

$$N_o = F_2 \cos \Delta_o + G_2 \sin \Delta_o \pm (F_1 \cos \tfrac{1}{2}\Delta_o + 2G_1 \sin \tfrac{1}{2}\Delta_o) + (F_4 \cos 2\Delta_o + \tfrac{1}{2} G_4 \cos 2\Delta_o) \quad . \quad . \quad . \quad (36),$$

and

$$\delta\Delta = \frac{N_o}{D_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

the solution of the equation (33) is $\Delta = \Delta_o + \delta\Delta$; and if D is the value of D_o when Δ replaces Δ_o, it is clear that

$$\delta D = D - D_o = \delta\Delta\{(F_2 \cos \Delta_o + G_2 \sin \Delta_o) \pm \tfrac{1}{4}(F_1 \cos \tfrac{1}{2}\Delta_o + 2G_1 \sin \tfrac{1}{2}\Delta_o) + 4(F_4 \cos 2\Delta_o + \tfrac{1}{2} G_4 \sin 2\Delta_o)\} \quad . \quad . \quad (38).$$

The angle Δ has to be reduced to time by division by $2(\gamma - \dot{l})$. As in the case of the reduction of the mean interval

$$\frac{\Delta}{2(\gamma - \dot{l})} = \frac{\Delta}{2(\gamma - \sigma)}\left(1 + \tfrac{2}{3}\frac{P\sigma}{\gamma - \sigma}\right).$$

Now the greatest value of P is about $\frac{1}{5}$, and $\sigma/(\gamma - \sigma)$ is ·038, hence $\frac{2}{3} P\sigma/(\gamma - \sigma)$ is at greatest ·005; but the inequality in the interval is rarely more than 2^h, and even if $\Delta/2(\gamma - \sigma)$ amounted to 3^h, the second term would only be about 1^m. Therefore I neglect the parallactic inequality in the reduction of Δ to time, and simply divide Δ by $2(\gamma - \sigma)$.

The value of $2(\gamma - \sigma)$ is 28°·98 per hour, and the reciprocal of this is ·0345, or $\frac{1}{30} + \frac{21}{20} \cdot \frac{1}{30^2}$.

Hence, if Δ be given in degrees, we have to multiply by ·0345, or its equivalent, to find its value in hours.

If any correction be given in circular measure the reduction to time is also very simple, for 57°·296 × ·0345 = 1^h·977 = 119^m, or

$$\frac{1}{2(\gamma - \sigma)}(\text{circ. meas.}) = 119^m \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

Thus, we have for H. W.,

$$\left.\begin{aligned} \Delta &= \Delta_o + \delta\Delta \\ D &= D_o + \delta D \\ I &= \cdot 0345\,\Delta = (\tfrac{1}{30} + \tfrac{21}{20} \cdot \tfrac{1}{30^2})\,\Delta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

Turning now to the expression for the height: it is expressible as the sum of three terms of the form

$$\Sigma(H \cos \vartheta \cos p\Delta - H \sin \vartheta \sin p\Delta).$$

Let

$$\left.\begin{aligned} &A = \Sigma H \cos \vartheta, \\ \text{then}\quad &h = A_2 \cos\Delta - F_2 \sin\Delta \pm (A_1 \cos \tfrac{1}{2}\Delta - 2F_1 \sin \tfrac{1}{2}\Delta) + A_4 \cos 2\Delta - \tfrac{1}{2}F_4 \sin 2\Delta \end{aligned}\right\} \quad (41),$$

with a correction to the height corresponding with

$$\left.\begin{aligned} \delta A_2 &= -\cdot 012 \sin^2 \Delta \,.\, 3\,(A_{''} + A_s) \\ \delta F_2 &= -\cdot 012 \sin^2 \Delta \,.\, F_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42).$$

In §§ 8, 9 we shall have to consider the variations of the interval and height due to variations of semi-diurnal and diurnal A, F, G. It is clear from the preceding formulæ and from (39) that

$$\left.\begin{aligned} \delta I &= \frac{119^{\mathrm{m}}}{D} \{\delta F_2 \cos\Delta + \delta G_2 \sin\Delta \pm (\delta F_1 \cos \tfrac{1}{2}\Delta + 2\,\delta G_1 \sin \tfrac{1}{2}\Delta)\} \\ \delta h &= \delta A_2 \cos\Delta - \delta F_2 \sin\Delta \pm (\delta A_1 \cos \tfrac{1}{2}\Delta - 2\,\delta F_1 \sin \tfrac{1}{2}\Delta) \end{aligned}\right\} \quad (43).$$

A particular case of the application of (43) is to the computation of the corrections referred to in (34) and (42).

§ 7. *On Evanescent Tides.*

At certain parts of the lunation, the diurnal tides sometimes suffice to annul one H. W. and one L. W., so that there is only one tide a day, perhaps for several days running.

If the inferior H. W. be watched as the condition of evanescence approaches, it will be seen to become smaller and smaller, and to occur later and later or earlier and earlier, and the adjacent L. W. undergoes similar changes. In the limit H. W. and L. W. coalesce, and in a tide diagram the coalescence appears as a point of contrary reflexure with horizontal tangent. Beyond this point, the reflexure is still maintained, although the tangent is not horizontal; finally, the tangent again becomes horizontal, and the double H. W. and L. W. again reappear.

Now in the use of the method of this paper, the loss of the double tide is very inconvenient, and I therefore propose to take the point of reflexure as representing both H. W. and L. W. during evanescence.

If N cannot vanish there is evanescence, and the point of reflexure is given by $D = 0$. The limit of the H. W. is given by $N = 0$, $D = 0$ simultaneously, and beyond this only D can vanish. The vanishing of D is taken to represent H. W.

Accordingly, when N cannot vanish, we proceed to make D vanish thus:—

$$0 = F_2 \sin \Delta - G_2 \cos \Delta \pm \tfrac{1}{2}(F_1 \sin \tfrac{1}{2}\Delta - 2\,G_1 \cos \tfrac{1}{2}\Delta) + 2\,(F_4 \sin 2\Delta - \tfrac{1}{2}\,G_4 \cos 2\Delta) \quad . \quad . \quad (44).$$

As a first approximation put

$$\tan \Delta_0 = \frac{G_2 \pm G_1}{F_2 \pm \frac{1}{4} F_1}$$

Then writing

$$E_o = -\{F_2 \cos \Delta_o + G_2 \sin \Delta_o \pm \tfrac{1}{4}(F_1 \cos \tfrac{1}{2}\Delta_o + 2G_1 \sin \tfrac{1}{2}\Delta_o) + 4\,(F_4 \cos 2\Delta_o + \tfrac{1}{2}\,G_4 \sin 2\Delta_o)\} \quad . \quad . \quad . \quad . \quad (45),$$

we have

$$\Delta = \Delta_o + \delta\Delta, \text{ where } \delta\Delta = \frac{D_o}{E_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

In the rest of the calculation this value of Δ is to be treated exactly as though it had been determined by the former method.

The corrections M, N, P, Q, R, S considered in succeeding sections, however, present a difficulty. In this case, Δ will always be very nearly $\pm 90°$, and I propose to compute P, Q, S (see §§ 8, 9), as though that were the true value of Δ.

But the correctional terms M′, N′, R, defined in §§ 8, 9, become theoretically infinite, and we are therefore compelled not to compute them, and to fill up the hiatus in the manner shown in the example below.

The process here suggested is a makeshift, but it is sufficient for the construction of a trustworthy tide-table, since the real occurrence at these times is a long period of nearly slack water, with or without a small maximum and minimum.

§ 8. *Parallactic Corrections to Time and Height.*

We will first consider the parallactic correction to the mean interval; we saw at the beginning of § 6 that there is a correction to the interval due to the moon's unequal motion in longitude of

$$\frac{\kappa_m}{2\,(\gamma - \sigma)} \cdot \tfrac{2}{3} \frac{P\sigma}{\gamma - \sigma}.$$

Now according to (31) $\kappa_m/2\,(\gamma - \sigma)$ is i, the mean interval, which we will suppose expressed in hours; also P is $\frac{1}{19}\,\Pi$ and $\sigma/(\gamma - \sigma)$ is ·03788, and $\frac{2}{3}$ of 60^m is 40^m; hence the correction in minutes is

$$i \,.\, 40^m \,.\, {\cdot}03788 \,.\, \tfrac{1}{19} \,.\, \Pi = i \times 0^m{\cdot}0797\,\Pi \quad . \quad . \quad . \quad . \quad . \quad (47).$$

Now turning to the other parallactic terms :—

Let us introduce the following notation, which follows the same plan as that used before, viz. :—

$$\left.\begin{matrix} {}_m\mathrm{A} \\ {}_m\mathrm{F} \\ {}_m\mathrm{G} \end{matrix}\right\} = \alpha' \mathrm{H}_m \left.\begin{matrix} \cos \\ p_m \sin \\ p_m^2 \cos \end{matrix}\right\} {}_m\vartheta, \qquad \left.\begin{matrix} {}_{''}\mathrm{A} \\ {}_{''}\mathrm{F} \\ {}_{''}\mathrm{G} \end{matrix}\right\} = \cdot 036\ \mathrm{H}_{''} \left.\begin{matrix} \cos \\ {}_{''}p \sin \\ {}_{''}p^2 \cos \end{matrix}\right\} {}_{''}\vartheta,$$

$$\left.\begin{matrix} {}_o\mathrm{A} \\ {}_o\mathrm{F} \\ {}_o\mathrm{G} \end{matrix}\right\} = \beta' \mathrm{H}_o \left.\begin{matrix} \cos \\ {}_o p \sin \\ {}_o p^2 \cos \end{matrix}\right\} {}_o\vartheta, \qquad \left.\begin{matrix} {}_{'}\mathrm{A} \\ {}_{'}\mathrm{F} \\ {}_{'}\mathrm{G} \end{matrix}\right\} = \cdot 036\ \mathrm{H}_{'} \left.\begin{matrix} \cos \\ {}_{'}p \sin \\ {}_{'}p^2 \cos \end{matrix}\right\} {}_{'}\vartheta.$$

Then a comparison of the method of § 6 with (28) § 5 shows that the corrections to the time and height are to be found by applying corrections to the A's, F's, G's, as follows :

$$\begin{aligned} \delta\mathrm{F}_2 &= \Pi\,({}_m\mathrm{F} + {}_{''}\mathrm{F}) = \Pi l_2, \quad \delta\mathrm{G}_2 = \Pi\,({}_m\mathrm{G} + {}_{''}\mathrm{G}) = \Pi m_2, \\ \delta\mathrm{F}_1 &= \Pi\,({}_o\mathrm{F} + {}_{'}\mathrm{F}) = \Pi l_1, \quad \delta\mathrm{G}_1 = \Pi\,({}_o\mathrm{G} + {}_{'}\mathrm{G}) = \Pi m_1, \\ \delta\mathrm{A}_2 &= \Pi\,({}_m\mathrm{A} + {}_{''}\mathrm{A}) = \Pi z_2, \\ \delta\mathrm{A}_1 &= \Pi\,({}_o\mathrm{A} + {}_{'}\mathrm{A}) = \Pi z_1 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (48). \end{aligned}$$

Then comparing (47) (48) with (39) and (43) we see that it

$$\begin{aligned} \mathrm{R} &= 119^{\mathrm{m}} \frac{1}{\mathrm{D}} \{l_2 \cos\Delta + m_2 \sin\Delta \pm (l_1 \cos\tfrac{1}{2}\Delta + 2m_1 \sin\tfrac{1}{2}\Delta)\} + 0^{\mathrm{m}}\cdot 08\,.\,i \\ \mathrm{S} &= \{z_2 \cos\Delta - l_2 \sin\Delta \pm (z_1 \cos\tfrac{1}{2}\Delta - 2l_1 \sin\tfrac{1}{2}\Delta)\} \quad . \ . \ . \ . \quad (49), \end{aligned}$$

the corrections to the interval and height are

$$\left.\begin{aligned} \delta I_{\mathrm{P}} &= \Pi\mathrm{R}, \\ \delta h_{\mathrm{P}} &= \Pi\mathrm{S}, \end{aligned}\right\} \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (50).$$

§ 9. *Corrections to Time and Height for Longitude of Moon's Node.*

Here again, we treat the terms as corrections to A, F, G.
Let

$$
\begin{aligned}
c_2 &= \qquad\qquad\qquad\quad \cdot 283\, H_{//} p_{//} \sin \vartheta_{//} = \qquad\quad \cdot 283\, F_{//},\\
d_2 &= -\cdot 037\, H_m p_m \cos \vartheta_m - \cdot 308\, H_{//} p_{//} \cos \vartheta_{//} = -\cdot 037\, G_m - \cdot 296\, G_{//},\\
e_2 &= -\cdot 037\, H_m p_m^2 \cos \vartheta_m + \cdot 283\, H_{//} p_{//}^2 \cos \vartheta_{//} = -\cdot 037\, G_m + \cdot 283\, G_{//},\\
f_2 &= \qquad\qquad\qquad\quad \cdot 308\, H_{//} p_{//}^2 \sin \vartheta_{//} = \qquad\quad \cdot 319\, F_{//},\\
a_2 &= -\cdot 037\, H_m \cos \vartheta_m + \cdot 283\, H_{//} \cos \vartheta_{//} = -\cdot 037\, A_m + \cdot 283\, A_{//},\\
b_2 &= \qquad\qquad\qquad\quad \cdot 308\, H_{//} \sin \vartheta_{//} = \qquad\quad \cdot 296\, F_{//},\\
c_1 &= \cdot 188\, H_o p_o \sin \vartheta_o + \cdot 115\, H_{/} p_{/} \sin \vartheta_{/} = \cdot 188\, F_o + \cdot 115\, F_{/},\\
d_1 &= \cdot 188\, H_o p_o \cos \vartheta_o - \cdot 154\, H_{/} p_{/} \cos \vartheta_{/} = \cdot 391\, G_o - \cdot 297\, G_{/},\\
e_1 &= \cdot 188\, H_o p_o^2 \cos \vartheta_o + \cdot 115\, H_{/} p_{/}^2 \cos \vartheta_{/} = \cdot 188\, G_o + \cdot 115\, G_{/},\\
f_1 &= -\cdot 188\, H_o p_o^2 \sin \vartheta_o + \cdot 154\, H_{/} p_{/}^2 \sin \vartheta_{/} = -\cdot 0905\, F_o + \cdot 080\, F_{/},\\
a_1 &= \cdot 188\, H_o \cos \vartheta_o + \cdot 115\, H_{/} \cos \vartheta_{/} = \cdot 188\, A_o + \cdot 115\, A_{/},\\
b_1 &= -\cdot 188\, H_o \sin \vartheta_o + \cdot 154\, H_{/} \sin \vartheta_{/} = -\cdot 391\, F_o + \cdot 297\, F_{/}. \qquad (51).
\end{aligned}
$$

A comparison with (29) then, shows that

$$
\begin{aligned}
\delta F_2 &= c_2 \cos 8 + d_2 \sin 8, \qquad \delta F_1 = c_1 \cos 8 + d_1 \sin 8,\\
\delta G_2 &= e_2 \cos 8 + f_2 \sin 8, \qquad \delta G_1 = e_1 \cos 8 + f_1 \sin 8,\\
\delta A_2 &= a_2 \cos 8 + b_2 \sin 8, \qquad \delta A_1 = a_1 \cos 8 + b_1 \sin 8 \quad . \quad . \quad . \quad (52).
\end{aligned}
$$

Then putting

$$
\begin{aligned}
M' - \delta M &= 119^m \frac{1}{D} \{c_2 \cos \Delta + e_2 \sin \Delta \pm (c_1 \cos \tfrac{1}{2}\Delta + 2e_1 \sin \tfrac{1}{2}\Delta)\},\\
N' - \delta N &= 119^m \frac{1}{D} \{d_2 \cos \Delta + f_2 \sin \Delta \pm (d_1 \cos \tfrac{1}{2}\Delta + 2f_1 \sin \tfrac{1}{2}\Delta)\},\\
P' &= a_2 \cos \Delta - c_2 \sin \Delta \pm (a_1 \cos \tfrac{1}{2}\Delta - 2c_1 \sin \tfrac{1}{2}\Delta),\\
Q' &= b_2 \cos \Delta - d_2 \sin \Delta \pm (b_1 \cos \tfrac{1}{2}\Delta - 2d_1 \sin \tfrac{1}{2}\Delta) \quad . \quad (53).
\end{aligned}
$$

The corrections to interval and height, as far as concerns the investigation up to the present point, are

$$
\begin{aligned}
\delta I_8 &= (M' - \delta M) \cos 8 + (N' - \delta N) \sin 8,\\
\delta h_8 &= P' \cos 8 + Q' \sin 8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).
\end{aligned}
$$

§ 10. *Reference to the Moon's True Transit.*

The intervals have been referred to the transits of a fictitious satellite whose R. A. is equal to l, and we now require corrections so as to refer to the moon's true transit.

Let T, T′ be the times of true and fictitious transits, and let α be the moon's R. A Then if t denotes the mean solar hour angle at time T,

$$t + h - \alpha = q\pi,$$

and, dropping the suffix o to l, at time T′

$$t + h - l = q\pi,$$

where q is an even integer at upper, and an odd one at lower transit.

If α be the value of the moon's R. A. at time T′, then its value at T is $\alpha + \dot{\alpha}\,(\mathrm{T} - \mathrm{T}')$; also the $t + h$ of the first equation is equal to the $t + h$ of the second corrected by $\gamma\,(\mathrm{T} - \mathrm{T}')$.

Hence the two equations may be written

$$t + h + \gamma\,(\mathrm{T} - \mathrm{T}') - \alpha - \dot{\alpha}\,(\mathrm{T} - \mathrm{T}') = q\pi,$$
$$t + h - l = q\pi.$$

Hence

$$\mathrm{T} - \mathrm{T}' = \frac{\alpha - l}{\gamma - \dot{\alpha}}.$$

It will afford a sufficiently close approximation if we replace $\dot{\alpha}$ by the moon's mean motion σ, so that

$$\mathrm{T}' = \mathrm{T} + \frac{l - \alpha}{\gamma - \sigma}.$$

We have, therefore, to find the excess of the moon's longitude above her R. A. at the time of fictitious transit.

It will be seen from fig. 2 that we have to determine the relationship between the R. A. and longitude, both measured from the intersection of the orbit and equator. The formula is exactly analogous with that which gives the reduction of longitudes to the ecliptic, so that

$$l - \xi = \alpha - \nu + \tan^2 \tfrac{1}{2} I \sin 2\,(l - \xi) - \tfrac{1}{2} \tan^4 \tfrac{1}{2} I \sin 4\,(l - \xi) + \ldots .$$

The last of these terms is very small and may be neglected, so that

$$l - \alpha = -\,(\nu - \xi) + \tan^2 \tfrac{1}{2} I \cos 2\xi \sin 2l - \tan^2 \tfrac{1}{2} I \sin 2\xi \cos 2l.$$

By the formula (7)

$$\tan^2 \tfrac{1}{2} I = \tan^2 \tfrac{1}{2} \omega\,[1 + 2i \operatorname{cosec} \omega \cos \mathfrak{S}\,],$$
$$\sin 2\xi = 2i \cot \omega \sin \mathfrak{S},$$
$$\cos 2\xi = 1,$$
$$\nu - \xi = i \tan \tfrac{1}{2} \omega \sin \mathfrak{S}.$$

Hence

$$l - \alpha = \tan^2 \tfrac{1}{2}\omega \sin 2l + \cos \text{Ø} \left[2i \frac{\tan^2 \frac{1}{2}\omega}{\sin \omega} \sin 2l \right] + \sin \text{Ø} \left[- 2i \tan^2 \tfrac{1}{2}\omega \cot \omega \cos 2l - i \tan \tfrac{1}{2}\omega \right].$$

But $2l = 2\,\text{☽} + 2\odot - 10^\circ = \Theta$, suppose; and

$$\frac{\tan^2 \frac{1}{2}\omega}{\gamma - \sigma} = 0^{\text{h}}\cdot 172 = 10^{\text{m}}\cdot 32, \quad \frac{2i \tan^2 \frac{1}{2}\omega}{(\gamma - \sigma) \sin \omega} = 4^{\text{m}}\cdot 60, \quad \frac{2i \tan^2 \frac{1}{2}\omega \cot \omega}{\gamma - \sigma} = 4^{\text{m}}\cdot 22,$$

and $\dfrac{i \tan \frac{1}{2}\omega}{\gamma - \sigma} = 4^{\text{m}}\cdot 41.$

Let us put then

$$\left.\begin{aligned} \Theta &= 2\,\text{☽} + 2\odot - 10^\circ, \\ \delta\text{T} &= 0^{\text{h}}\cdot 172 \sin \Theta, \\ \delta\text{M} &= 4^{\text{m}}\cdot 60 \sin \Theta, \\ \delta\text{N} &= - [4^{\text{m}}\cdot 22 \cos \Theta + 4^{\text{m}}\cdot 41], \\ \text{T}' &= \text{T} + \delta\text{T} + \delta\text{M} \cos \text{Ø} + \delta\text{N} \sin \text{Ø}, \end{aligned}\right\} \quad \ldots\ldots \quad (55).$$

and we have

It is now clear that, when the tide is referred to the moon's true transit, δT must be added to I, and δM, δN must be added to M′ − δM, N′ − δN as given in (53) to find M′, N′.

Let $\mathfrak{A}$ be the height of mean sea level above the datum adopted for the tide-table, and let

$$\text{B}_o = \mathfrak{A} + \text{H}_{ssa} \cos \vartheta_{ssa} \quad \ldots\ldots\ldots \quad (56).$$

Then i denotes the mean interval from moon's transit to H. W., and the interval is

$$i + I + \delta\text{T} + \text{M}' \cos \text{Ø} + \text{N}' \sin \text{Ø} + \delta I_{\text{P}};$$

and the height is

$$\text{B}_o + h + \text{P}' \cos \text{Ø} + \text{Q}' \sin \text{Ø} + \delta h_{\text{P}} + \text{H}_{sa} \cos \vartheta_{sa}.$$

In these formulæ all the quantities are functions of ☽; now although ☽ can easily be found from the 'Nautical Almanac,' it is not tabulated, and the difficulty of using the tide-table would be considerably augmented if it were necessary to compute ☽ for each tide. We shall proceed, then, to convert our formulæ into others, in which there is direct reference to moon's true transit, although there is some loss of accuracy in the process, and the amount of computation required to form the table is increased.

We have seen that the time T′ of fictitious transit is given by

$$\text{T}' = \text{T} + \delta\text{T} + \delta\text{M} \cos \text{Ø} + \delta\text{N} \sin \text{Ø},$$

and that $(\gamma - \eta)\,\mathrm{T}' + h - l = q\pi$ where h and l correspond to the time of fictitious transit.

Now

$$l_{,} = h + 2e_{,} \sin (l_{,} - 281°),$$

where $e_{,}$ is the eccentricity of the sun's orbit, and 281° the longitude of perigee.

Also

$$l = ☽ + l_{,}.$$

Hence

$$l - h = ☽ + 2e_{,} \sin (l_{,} - 281°).$$

But

$$\frac{2e_{,}}{\gamma - \eta} = 7^{\mathrm{m}}{\cdot}69 = 0^{\mathrm{h}}{\cdot}128;\ \text{also}\ l_{,} = ☉ - 5°,\ \text{and}\ -286° = +74°.$$

Therefore

$$\mathrm{T}' = \frac{q\pi + ☽}{\gamma - \eta} + 7^{\mathrm{m}}{\cdot}69 \sin (☉ + 74°),\ \text{and}$$

$$\frac{q\pi + ☽}{\gamma - \eta} = \mathrm{T} + \delta\mathrm{T} + \delta\mathrm{M} \cos ☒ + \delta\mathrm{N} \sin ☒ - 7^{\mathrm{m}}{\cdot}69 \sin (☉ + 74°).$$

This formula connects ☽ with T, the time of moon's transit.

Now I is a function of ☽ or of $(q\pi + ☽)/(\gamma - \eta)$; hence if I (T) be the value of I for which $☽ = (\gamma - \eta)\,\mathrm{T} - q\pi$, and if I (☽) be the true value of I,

$$I(☽) = I(\mathrm{T}) + \frac{dI}{d\mathrm{T}} \{\delta\mathrm{T} + \delta\mathrm{M} \cos ☒ + \delta\mathrm{N} \sin ☒ - 7^{\mathrm{m}}{\cdot}69 \sin (☉ + 74°)\};$$

and similarly

$$h(☽) = h(\mathrm{T}) + \frac{dh}{d\mathrm{T}} \{\delta\mathrm{T} + \delta\mathrm{M} \cos ☒ + \delta\mathrm{N} \sin ☒ - 0^{\mathrm{h}}{\cdot}128 \sin (☉ + 74°)\}.$$

These expressions have now to be substituted in those for the height and interval; but in the small terms δT, M′, N′, &c., we may regard ☽ as denoting $(\gamma - \eta)\,\mathrm{T} - q\pi$.

In carrying out the substitutions preparations will be made for computation.

First put

$$\delta I = \frac{dI}{d\mathrm{T}} \delta\mathrm{T} = \frac{dI}{d\mathrm{T}} \times 10^{\mathrm{m}}{\cdot}32 \sin \Theta,$$

$$= \left(2 \frac{dI}{d\mathrm{T}}\right) \cdot (2{\cdot}34)\,(2^{\mathrm{m}}{\cdot}2 \sin \Theta) \quad \ldots\ldots\ldots \quad (57).$$

Similarly, since $2^{\mathrm{m}}{\cdot}2$ is $\frac{11}{300}$ hrs., we put

$$\delta h = \left(2 \frac{dh}{d\mathrm{T}}\right) (2{\cdot}34)\,(\tfrac{11}{300} \sin \Theta) \quad \ldots\ldots\ldots \quad (58).$$

Next put

$$M'' = \frac{dI}{dT}\delta M = \frac{dI}{dT} \times 4^{m}\cdot 60 \sin\Theta,$$

$$= \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2 \sin\Theta)\left(1 + \tfrac{1}{20}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

and

$$-N'' = -\frac{dI}{dT}\delta N,$$

$$= \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2) + \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2 \cos\Theta)\left(1 - \tfrac{1}{20}\right). \quad . \quad . \quad . \quad . \quad (60).$$

Similarly put

$$P'' = \left(2\frac{dh}{dT}\right)\left(\tfrac{11}{300}\sin\Theta\right)\left(1 + \tfrac{1}{20}\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$-Q'' = \left(2\frac{dh}{dT}\right)\left(\tfrac{11}{300}\right) + \left(2\frac{dh}{dT}\right)\left(\tfrac{11}{300}\cos\Theta\right)\left(1 - \tfrac{1}{20}\right) . \quad . \quad . \quad . \quad (62).$$

Lastly put

$$\mathfrak{i} = -\left(2\frac{dI}{dT}\right)\{3^{m}\cdot 85 \sin(\odot + 74^{\circ})\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\mathfrak{h}' = -\left(2\frac{dh}{dT}\right)\{0^{h}\cdot 064 \sin(\odot + 74^{\circ})\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

With this notation

$$I(\mathfrak{D}) = I(T) + \delta I + M'' \cos 2\mathfrak{D} + N'' \sin 2\mathfrak{D} + \mathfrak{i},$$
$$h(\mathfrak{D}) = h(T) + \delta h + P'' \cos 2\mathfrak{D} + Q'' \sin 2\mathfrak{D} + \mathfrak{h}.$$

These are now to be substituted in the expressions for interval and height, and in doing so we may drop the (T) after the I and h.

I write

$$\mathfrak{I} = I + \delta T + i + \delta I,$$
$$\mathfrak{H} = B_{o} + h + \delta h,$$
$$\mathfrak{h} = \mathfrak{h}' + H_{sa}\cos\vartheta_{sa},$$
$$M = M' + M'', \quad N = N' + N'',$$
$$P^{*} = P' + P'', \quad Q = Q' + Q'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Our formulæ are designed to serve for any time of year and its opposite; now since $\mathfrak{i}$ and $\mathfrak{h}$ involve the sun's longitude they change their signs in six months, and we write $\pm\mathfrak{i}$, $\pm\mathfrak{h}$, and it is to be understood that the upper sign is to be used for the time of year under computation and the lower for its opposite.

The interval is then

* This P will not be confused with the P defined in § 1, which has been replaced by Π.

$$\mathfrak{I} \pm \mathfrak{i} + \mathrm{M}\cos\mathfrak{D} + \mathrm{N}\sin\mathfrak{D} + \mathrm{R}\,\Pi \quad . \quad . \quad . \quad . \quad (66),$$

and the height is

$$\mathfrak{H} \pm \mathfrak{h} + \mathrm{P}\cos\mathfrak{D} + \mathrm{Q}\sin\mathfrak{D} + \mathrm{S}\,\Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (67).$$

One part of M, N, P, Q arises from a true change in the tide when the longitude of the moon's node changes, and the remainder (nearly equally large) merely depends on the reference to true instead of fictitious transit. The quantities $\mathfrak{i}$ and $\mathfrak{h}$ depend partly on a portion of the equation of time and partly on the annual tide.

We must now explain the computation of $2\,dI/d\mathrm{T}$ and of $2\,dh/d\mathrm{T}$.

If u_o, u_1, . . . u_{23} are cyclical values of a function, the symmetrical interpolation formula in the neighbourhood of u_m is

$$u_{x+m} = u_m + \tfrac{1}{2}x(\Delta\mathrm{E}^{-1} + \Delta)\,u_m + \frac{x^2}{2!}\Delta^2\mathrm{E}^{-1}\,u_m + \tfrac{1}{2}\frac{x(x^2-1)}{3!}(\Delta^3\mathrm{E}^{-2} + \Delta^3\mathrm{E}^{-1})\,u_m + \ldots,$$

where

$$\mathrm{E}u_m = u_{m+1} \quad \text{and} \quad \Delta u_m = u_{m+1} - u_m.$$

Then when $x = 0$,

$$2\frac{du_m}{dx} = (\Delta\mathrm{E}^{-1} + \Delta)\,u_m - \tfrac{1}{6}(\Delta^3\mathrm{E}^{-2} + \Delta^3\mathrm{E}^{-1})\,u_m \ldots,$$

or

$$2\frac{du_m}{dx} = (u_{m+1} - u_{m-1}) - \tfrac{1}{6}\Delta^2\mathrm{E}^{-1}(u_{m+1} - u_{m-1}) + \ldots$$

In the present case the first term will usually suffice, but the second term may be easily computed.

In order then to compute the required differential coefficients we arrange the I's and h's in two columns, the even entries in one column and the odd in another, and take the differences of the two columns independently of one another.

§ 11. *The Correction for Solar Parallax.*

The terms depending on solar parallax arise from the potential $\mathrm{V}_{P_,}$ in (15), but the correction is so small that I shall omit it from the example below. It is well, however, to show how it may be computed. The only terms of importance are those in H_s, $\mathrm{H}_,$ corresponding to the tides S_2, K_1.

The variability of $P_,$ enters into the calculation in the form of corrections to A_2, G_2, F_2, A_1, G_1, F_1.

The sun's parallax is approximately

$$1 + e_, \cos(l_, - 281^\circ), \text{ and hence } P_, = 3e_, \cos(\odot + 74^\circ).$$

Now

$$3e_{,} \text{ is } \cdot 0504, \text{ and } 119^{m} \times 0504 = 6^{m}\cdot 0.$$

Then, since

$$\left.\begin{matrix}\delta F_2\\ \delta G_2\\ \delta A_2\end{matrix}\right\} = P_{,}\left.\begin{matrix}F_s\\ G_s\\ A_s\end{matrix}\right\}, \quad \left.\begin{matrix}\delta F_1\\ \delta G_1\\ \delta A_1\end{matrix}\right\} = \cdot 317\, P_{,}\left.\begin{matrix}F_{,}\\ G_{,}\\ A_{,}\end{matrix}\right\},$$

it follows that

$$\delta I_{P_{,}} = \frac{6^{m}\cdot 0}{D}\cos(\odot + 74^{\circ})\{F_s\cos\Delta + G_s\sin\Delta \pm \cdot 317\,(F_{,}\cos\tfrac{1}{2}\Delta + 2G_{,}\sin\tfrac{1}{2}\Delta)\}$$

$$\delta h_{P_{,}} = \cdot 050\cos(\odot + 74^{\circ})\{A_s\cos\Delta - F_s\sin\Delta \pm \cdot 317\,(A_{,}\cos\tfrac{1}{2}\Delta - 2F_{,}\sin\tfrac{1}{2}\Delta)\}.$$

These must be deemed to be corrections to $\mathfrak{i}$ and $\mathfrak{y}$, since they change signs in six months.

§ 12. *The Formulæ for a Low Water Table*

The formulæ (27), (28), (29), (30), for the height of water would (except in one detail) be applicable to L. W., but they would not be convenient, because Δ oscillates about $-\pi$ at the L. W. which precedes the H. W. for which mean Δ is zero.

Hence put $\Delta = \delta - \pi$.

The L. W. formulæ may be made exactly similar to those for H. W. by making the heights negative, and this condition is satisfied by adding π to all the arguments. Thus the formulæ for the height may be written

$$-\Sigma\, H\cos(p\,\delta + \theta),$$

where

$$\theta = \vartheta + (1 - p)\,\pi.$$

A similar change may be made in the nodal terms, but the parallactic terms require further consideration. The term $\cos(p_m\Delta + {}_m\vartheta)$ changes, not only because its sign is to be changed and Δ is to be replaced by $\delta - \pi$, but also because, as appears in § 3, ${}_m\vartheta$ changes.

For H. W.

$${}_m\vartheta = p_m\kappa_m - {}_m\kappa = p_n\kappa_m - \kappa_n + \frac{\kappa_m - \kappa_n}{9\cdot 11 H_n/H_m - 1},$$

but for L. W. κ_m must be replaced by $\kappa_m - \pi$ wherever it is multiplied by p; hence, for L. W., $p_m\kappa_m - {}_m\kappa$ must have $(p_m - p_n)\,\pi$ or $(1 - p_n)\,\pi$ added to its previous value.

Hence the term in $\cos(p_m\Delta + {}_m\vartheta)$ for H. W. corresponds in the expression for $-h_P$ for L. W., to a term involving

$$\cos[p_m\delta + (1 - p_m)\,\pi + (1 - p_n)\,\pi + {}_m\vartheta]$$

Since $p_m = 1$, the required term is $\cos(p_m\delta + {}_m\theta)$, where

$${}_m\theta = {}_m\vartheta + (1 - p_n)\,\pi.$$

Again the parallactic term for O involves $\cos({}_op\Delta + {}_o\vartheta)$ for H. W., and treating it in the same way, the corresponding term in $-h_P$ for L. W. is found to be

$$\cos({}_op\delta + {}_o\theta),$$

where

$${}_o\theta = (1 - {}_op)\,\pi + (p_o - p_q)\,\pi + {}_o\vartheta.$$

But

$${}_op = p_o - \frac{\epsilon}{\beta}.$$

Hence

$${}_o\theta = {}_o\vartheta + (1 + \frac{\epsilon}{\beta} - p_o)\,\pi.$$

We have also

$${}_{''}\theta = {}_{''}\vartheta + (1 - {}_{''}p)\,\pi, \qquad {}_{'}\theta = {}_{'}\vartheta + (1 - {}_{'}p)\,\pi.$$

Then the L. W. formulæ for depths below mean sea level are similar in form to those for H. W. for elevation above mean sea level, with δ in place of Δ, and with θ's in the place of ϑ's.

The connection of θ's with ϑ's is given in the following table.

Initial.	Principal and nodal terms.	Parallactic terms.
M_2	$\theta_m = \vartheta_m = 0$	${}_m\theta = {}_m\vartheta + (1 - p_n)\,\pi = {}_m\vartheta + 3^\circ{\cdot}4$
K_2	$\theta_{''} = \vartheta_{''} + (1 - p_n)\,\pi$	${}_{''}\theta = \vartheta_{''} + (1 - {}_{''}p)\,\pi$
	$= \vartheta_{''} - 7^\circ{\cdot}0$	$= {}_{''}\vartheta - 13^\circ{\cdot}7$
S_2	$\theta_s = \vartheta_s + (1 - p_s)\,\pi$	
	$= \vartheta_s - 6^\circ{\cdot}4$	(assume $\beta = \frac{4}{3}$)
O	$\theta_o = \vartheta_o + (1 - p_o)\,\pi$	${}_o\theta = {}_o\vartheta + (1 + \frac{\epsilon}{\beta} - p_q)\,\pi$
	$= \vartheta_o + \frac{1}{2}\pi + 3^\circ{\cdot}4$	$= {}_o\vartheta + \frac{1}{2}\pi + 8^\circ{\cdot}5$
K_1	$\theta_{'} = \vartheta_{'} + (1 - p_{'})\,\pi$	${}_{'}\theta = {}_{'}\vartheta + (1 - {}_{'}p)\,\pi$
	$= \vartheta_{'} + \frac{1}{2}\pi - 3^\circ{\cdot}4$	$= {}_{'}\vartheta + \frac{1}{2}\pi - 6^\circ{\cdot}8$
P	$\theta_p = \vartheta_p + (1 - p_p)\,\pi$	
	$= \vartheta_p + \frac{1}{2}\pi - 2^\circ{\cdot}9$	
M_4	$\theta_{2m} = \vartheta_{2m} + (1 - 2p_m)\,\pi$	
	$= \vartheta_{2m} - \pi$	
S_4	$\theta_{2s} = \vartheta_{2s} + (1 - 2p_s)\,\pi$	
	$= \vartheta_{2s} - \pi - 12^\circ{\cdot}8$	

All the θ's differ from the ϑ's by a small angle, or by an angle nearly equal to 90° or 180°. Where the difference is small the A, G, F for L. W. will be nearly equal to

those for H. W.; where the difference is nearly 90° the A, G, F for L. W. will be nearly the same as those for H. W., a quarter year earlier or later; and where the difference is nearly 180° the A, G, F for L. W. will be nearly equal and opposite to those for H. W. Hence we may set aside the change of 90° and 180° to be satisfied by a shift of a quarter year, or by change of sign. Suppose then, that $\theta = \vartheta + \alpha$, where α is small, and let [A], [G], [F] denote the values of L. W. A, G, F; then, remembering that

$$A = H\cos\vartheta, \qquad G = Hp^2\cos\vartheta, \qquad F = Hp\sin\vartheta,$$

and that [A], [G], [F] are represented by similar formulæ with θ in place of ϑ, we have

$$[A] = A - \frac{\sin\alpha}{p}F,$$
$$[G] = G - p\sin\alpha\, F,$$
$$[F] = F + p\sin\alpha\, A.$$

The values of α are given above, and those of the p's are known; hence it is easy to compute formulæ of transition from L. W. to H. W.

The rules given below in the example are derived from these formulæ, but the coefficients $\sin\alpha/p$ and $p\sin\alpha$ are given in round numbers appropriate for computation, and are sometimes treated as zero.

Part II.—Computation.

Remarks on the Computations.

The multiplications are supposed to be done with Crelle's Bremiker's multiplication table.*

The other tables required are tables of squares, natural tangents, circular measure, and a traverse table. Bottomley's tables† are convenient for the purpose, because they give no more than is required. The nautical traverse table, such as that in Inman's tables, or in Chambers' logarithms, is used for finding such quantities as $H\cos\vartheta$ and $H\sin\vartheta$, for if H is "Distance," $H\cos\vartheta$ is "Lat." and $H\sin\vartheta$ is "Dep.," and the position of the decimal point is determined by inspection. For the use of this table it is advantageous to have only angles with a whole number of degrees, so as to avoid cross interpolations‡; the whole calculation is therefore conducted so as to avoid broken degrees. A traverse table is commonly given for

* 'Rechentafeln,' Berlin, Georg Reimer.

† 'Four-figure Mathematical Tables,' by J. T. Bottomley, F.R.S. Macmillan.

‡ Inman's Traverse Table is arranged so that the interpolation for a fraction of a degree is not very awkward.

"Distances" from 0 to 300, hence, if the "Distance" involves three digits and lies between 300 and 999, an interpolation is required, so as to use the entries between 30 and 99; this interpolation can easily be made by inspection.

All the angles are entered so that the significant part is less than 90°, by treating them as + or −, with π or 180° added where necessary. This facilitates the use of the traverse table.

It is best to determine the signs of the cosines and sines independently from their numerical values, and, accordingly, in the example where ·000 is entered as the value of a sine or cosine, it has a sign attached to it.

I suppose the computer to be able to add up a short column of figures, where some of the entries are + and others −. This is an arithmetical process not much practised, but easily acquired.

The sequences of angles and of cosines and sines, which occur frequently below, appertain (except in the cases of Sa and Ssa) to values of the excess of moon's longitude over sun's (for which the symbol used is ☽) at intervals of 15°, beginning with ☽ = 0°, and ending with ☽ = 345°, 24 values in all. But in the earlier part of the computation, the beginning of the sequence occurs at a different part of the column at different times of the year. Thus, a list of months is written in the margin, to show where we are to begin at any specified time of year. Strictly speaking these months are the times when the sun's longitude + 5° (for which the symbol used is ☉) is equal to a multiple of 30°; thus, when ☉ is 0°, we have March 15th, when ☉ is 30°, April 14th, and so on, as shown in Table VI. If the number of degrees in ☽ be reduced to time, at the rate of 15° per hour, we have, approximately, the time of the moon's transit, and in the later stages of the computation the time of Moon's transit is made to replace ☽. The sequences of angles are found by adding multiples of 30°, adding or subtracting multiples of 15°, or adding multiples of 60° (see Table II.) to certain initial angles (see Table I.). When the sequence has been carried so far that the next addition would reproduce the first angle with π added to it, it is unnecessary to proceed further. In the sequences of cosines and sines of such angles, when we have got to this same point, it is unnecessary to proceed further, since the remainder is the same as the beginning, with the sign changed.

In subsequent stages where a constant has to be added to a sequence, the new sequence will have double as many entries as the old, the first half being formed by addition, and the second half by subtraction; but in repeating the new sequence the signs are *not* to be changed. This follows immediately from what has been said of the signs of sequences.

Before proceeding with the computations I give some tables and rules of general applicability to all ports. It will be best for the computer who is learning the process to pass straight to the example, and to refer back to these tables as they are required; but I give them in the first place, because they will be wanted in the case of any other port.

Tables and Rules applicable to all Ports

TABLE I.—For finding the K's, the initial entries of the several sequences, for H. W. (See (26), § 5.)

Initials.	Principal and nodal terms.	Parallactic terms.
M_2	$K_m = p_m \kappa_m - \kappa_m = 0$	${}_mK = p_n \kappa_m - \kappa_n + \frac{\kappa_m - \kappa_n}{9{\cdot}11\ H_n/H_m - 1}$
K_2	${}_{\prime\prime}K = p_{\prime\prime} \kappa_m - \kappa_{\prime\prime} - 10°$	${}_{\prime\prime}K = K_{\prime\prime} + ({}_{\prime\prime}p - p_{\prime\prime})\ \kappa_m = K_{\prime\prime} + {\cdot}038\ \kappa_m$
S_2	$K_s = p_s \kappa_m - \kappa_s$	
O	$K_o = p_o \kappa_m - \kappa_o + 95°$	${}_oK = p_q \kappa_m - \kappa_q - \frac{{\cdot}013}{\beta} \kappa_m + \frac{\kappa_o - \kappa_q}{9{\cdot}11\ H_q/H_o - 1} + 95°$
K_1	${}_{\prime}K = p_{\prime} \kappa_m - \kappa_{\prime} - 95°$	${}_{\prime}K = K_{\prime} + ({}_{\prime}p - p_{\prime})\ \kappa_m = K_{\prime} + {\cdot}0192\ \kappa_m$
P	$K_p = p_p \kappa_m - \kappa_p + 95°$	
M_4	$K_{2m} = 2p_m \kappa_m - \kappa_{2m}$	
S_4	$K_{2s} = 2p_s \kappa_m - \kappa_{2s}$	
Sa	$K_{sa} = -\kappa_{sa} - 5°$	
Ssa	$K_{ssa} = -\kappa_{ssa} - 10°$	

TABLE II.—For finding the sequences of the ϑ's by putting n successively equal to 0, 1, 2, 3, &c. (See (26), § 5.)

Initials.	Principal and nodal terms. ϑ	Parallactic terms. ϑ
M_2	K_m	${}_mK$
K_2	$K_{\prime\prime} + 30°n$	${}_{\prime\prime}K + 30°n$
S_2	$K_s + 30°n$	
O	$K_o - 15°n$	${}_oK - 15°n$
K_1	$K_{\prime} + 15°n$	${}_{\prime}K + 15°n$
P	${}_{\frac{1}{2}}K_p + 15°n$	
M_4	K_{2m}	
S_4	$K_{2s} + 60°n$	
Ssa	$K_{ssa} + 60°n$	
Sa	${}_{\frac{1}{2}}K_{sa} + 10°n$	

N.B.—The sequence for Sa is required under conditions which differ from those of the other ϑ's.

TABLE III.—The numerical values of the p's. (See § 5.)

Initials.	Principal and nodal terms.	Parallactic terms.
M_2	$p_m = 1$, $p_m^2 = 1$	
K_2	$p_{''} = 1{\cdot}038$, $p_{''}^2 = 1{\cdot}078$	${}_{''}p = 1{\cdot}076$, ${}_{''}p^2 = 1{\cdot}158$
S_2	$p_s = 1{\cdot}035$, $p_s^2 = 1{\cdot}071$	
N	$p_n = {\cdot}981$	
O	$p_o = {\cdot}481$, $p_o^2 = {\cdot}231$	${}_{o}p = {\cdot}481 - {\cdot}013\,\beta^{-1}$, ${}_{o}p^2 = {\cdot}231 - {\cdot}012\,\beta^{-1}$
K_1	$p_{'} = {\cdot}519$, $p_{'}^2 = {\cdot}269$	${}_{'}p = {\cdot}538$, ${}_{'}p^2 = {\cdot}289$
P	$p_p = {\cdot}516$, $p_p^2 = {\cdot}266$	
Q	$p_q = {\cdot}462$	

TABLE IV.—For computing corrections for reference to moon's transit, viz., δT, δM, δN, and the sequence Θ. (See (55) § 10.)

Sequence. $(-10^\circ + 30^\circ n)$. Θ	
March -10°	June $\pi - 10^\circ$
$+20$	$\pi + 20$
April $+50$	July $\pi + 50$
$+80$	$\pi + 80$
May $\pi - 70$	August -70
$\pi - 40$	-40

Repeat the sequence.

δT $= 0^h{\cdot}172 \sin \Theta$.	δM $= 4^m{\cdot}60 \sin \Theta$.	$-4^m{\cdot}22 \cos \Theta$.	δN $= -4^m{\cdot}22 \cos \Theta - 4^m{\cdot}41$.
h.	m.	m.	m.
March $-{\cdot}030$	March $-0{\cdot}80$	March $-4{\cdot}16$	March $-8{\cdot}57$
$+{\cdot}059$	$+1{\cdot}57$	$-3{\cdot}97$	$-8{\cdot}38$
April $+{\cdot}132$	April $+3{\cdot}52$	April $-2{\cdot}71$	April $-7{\cdot}12$
$+{\cdot}169$	$+4{\cdot}53$	$-0{\cdot}73$	$-5{\cdot}14$
May $+{\cdot}162$	May $+4{\cdot}32$	May $+1{\cdot}45$	May $-2{\cdot}96$
$+{\cdot}111$	$+2{\cdot}96$	$+3{\cdot}23$	$-1{\cdot}18$
June $+{\cdot}030$	June $+0{\cdot}80$	Rep. and ch.	June $-0{\cdot}25$
$-{\cdot}059$	$-1{\cdot}57$		$-0{\cdot}44$
July $-{\cdot}132$	July $-3{\cdot}52$		July $-1{\cdot}70$
$-{\cdot}169$	$-4{\cdot}53$		$-3{\cdot}68$
August $-{\cdot}162$	August $-4{\cdot}32$		August $-5{\cdot}86$
$-{\cdot}111$	$-2{\cdot}96$		$-7{\cdot}64$

The sequences for δT, δM, δN are to be repeated without change of sign. To find the succession of values for any month we begin with the entry opposite to that

month, read on down to the bottom, and then begin again at the top. For example, δT for July begins with $-$ ·132, and then, after going on down to $-$ ·111, it begins again at the top with $-$ ·030.

TABLE V.—For corrections due to part of the equation of time. (See (63) (64) § 10.)

The following is a table of $-3^m{\cdot}85 \sin(\odot + 74°)$ (which I call c), and of the same when the hour is unit of time (which I call d).

N.B.—286° is the longitude of sun's perigee + 5°, and 74° is its supplement to 360°.

	c. $-3^m{\cdot}85 \sin(\odot + 74°)$.	d. $-0^h{\cdot}064 \sin(\odot + 74°)$.
	m.	h.
March . . .	$-3{\cdot}7$	$-{\cdot}062$
April	$-3{\cdot}7$	$-{\cdot}062$
May	$-2{\cdot}8$	$-{\cdot}046$
June	$-1{\cdot}1$	$-{\cdot}018$
July	$+0{\cdot}9$	$+{\cdot}016$
August . . .	$+2{\cdot}7$	$+{\cdot}045$

TABLE VI.—Dates and Limits of Applicability of the Tide-Tables.

Heading for tide-table.	⊙	Applicability by reference to Sun's long. at Moon's transit. Sun's longitude.	Heading for tide-table.	⊙	Applicability by reference to Sun's long. at Moon's transit. Sun's longitude.
	°	° °		°	° °
March 15	0	from 350 to 0	Sept. 17	180	from 170 to 180
,, 25	10	,, 0 ,, 10	,, 28	190	,, 180 ,, 190
April 4	20	,, 10 ,, 20	Oct. 8	200	,, 190 ,, 200
,, 14	30	,, 20 ,, 30	,, 18	210	,, 200 ,, 210
,, 25	40	,, 30 ,, 40	,, 28	220	,, 210 ,, 220
May 5	50	,, 40 ,, 50	Nov. 7	230	,, 220 ,, 230
,, 15	60	,, 50 ,, 60	,, 17	240	,, 230 ,, 240
,, 26	70	,, 60 ,, 70	,, 27	250	,, 240 ,, 250
June 5	80	,, 70 ,, 80	Dec. 7	260	,, 250 ,, 260
,, 16	90	,, 80 ,, 90	,, 16	270	,, 260 ,, 270
,, 26	100	,, 90 ,, 100	,, 26	280	,, 270 ,, 280
July 7	110	,, 100 ,, 110	Jan. 5	290	,, 280 ,, 290
,, 17	120	,, 110 ,, 120	,, 15	300	,, 290 ,, 300
,, 28	130	,, 120 ,, 130	,, 25	310	,, 300 ,, 310
Aug. 7	140	,, 130 ,, 140	Feb. 4	320	,, 310 ,, 320
,, 17	150	,, 140 ,, 150	,, 13	330	,, 320 ,, 330
,, 28	160	,, 150 ,, 160	,, 23	340	,, 330 ,, 340
Sept. 7	170	,, 160 ,, 170	March 5	350	,, 340 ,, 350

This table gives the days of the year on which ⊙, or Sun's longitude + 5°, is nearly equal to a multiple of 10°. These days are used as headings to the several tide-tables. It is intended that the tables shall be used without an interpolation for

the time of year, which ought strictly to be made. When the time of a particular moon's transit, with reference to which a tide is to be calculated, falls nearly halfway between any two of the specified days, it becomes uncertain which of the two adjoining tables should be used, and the question can only be decided by reference to the Sun's longitude. A column is therefore given of the limits of applicability of the table.

It would be easy, by means of a table of four columns referring to leap year, to give the Greenwich times at which the Sun's longitude is 0°, 10°, 20°, &c., which would be accurate enough for the present purpose during some twenty-five years.

VII.—*The Choice of a Unit of Length.*

In a calculation of this kind it is advantageous to reduce the number of digits as far as possible, consistently with due accuracy, and it is convenient to omit the decimal point when we deal with heights.

The diurnal tides are so various at different places that no general rule can be made to depend on them.

It is required to express as many of the heights as possible by two digits, and it will be best to take such a unit that much of the work shall be conducted with 70's and 80's, but to allow a margin and not to try to bring them into the 90's.

After consideration I think it is best to take such a unit that $\alpha' H_m$ (see below) shall be expressed by 70 or 80. Since α' is usually about $\frac{4}{57}$, or say $\frac{1}{14}$, then when H_m is given in feet and decimals (or any other unit), we are to multiply the heights by a simple factor lying between $14 \times 70 \div H_m$ and $14 \times 80 \div H_m$.

The rule therefore is:—

Multiply the heights by a factor lying between $1000 \div H_m$ *and* $1200 \div H_m$, *and omit decimals.*

In the example below it would have been best to multiply all the heights by 700. We should then have $H_m = 1098$, $H_s = 488$, &c., and this would have made $\alpha' H_m$ equal to 77. The final step in the calculation would then have been to divide all the heights by 700.

In my example I have not followed this plan, and accordingly the decimal point is retained, and an unnecessary number of digits has been written.

VIII.—*Rules for the Calculation of a L. W. Table.*

It will be more convenient to state these rules as part of the example for the Port of Aden, although they are of course generally applicable to all ports.

Example of Formation of a Tide-Table.

Table of Constants for the Port of Aden.

	ft.		ft.
M_2 H_m	1·568	O H_o	·653
κ_m	229°	κ_o	38°
S_2 H_s	·697	K_1 $H_{\prime}$	1·299
κ_s	248°	$\kappa_{\prime}$	36°
K_2 $H_{\prime\prime}$	·201	P H_p	·388
$\kappa_{\prime\prime}$	244°	κ_p	33°
N H_n	·427	Q H_q	·151
κ_n	225°	κ_q	42°
L H_l	·046	Sa H_{sa}	·390
κ_l	230°	κ_{sa}	357°
M_4 H_{2m}	·007	Ssa H_{ssa}	·095
κ_{2m}	314°	κ_{ssa}	126°
S_4 H_{2s}	·006	$\mathfrak{A}$	3·859
κ_{2s}	271°		

These are the results of four years of observation, and are the constants from which the tide-table is to be computed.

Formation of Sequences (see Tables I., II., III., and § 6).

K_2.	S_2.	O.	K_1.
$p_{\prime\prime}\kappa_m$ = 238°	$p_s\kappa_m$ = 237°	$p_o\kappa_m$ = 110°	$p_{\prime}\kappa_m$ = 119°
$-\kappa_{\prime\prime}$ = − 244	$-\kappa_s$ = − 248	$-\kappa_o$ = − 38	$-\kappa_{\prime}$ = − 36
− 10° = − 10		+ 95° = 95	− 95° = − 95
		167	
$K_{\prime\prime}$ = − 16	K_s = − 11	$K_o = \pi$ − 13	$K_{\prime}$ = − 12

P.	M_4.	S_4.
$p_p\kappa_m$ = 118°	$2p_m\kappa_m$ = 458°	$2p_s\kappa_m$ = 474°
$-\kappa_p$ = − 33	$-\kappa_{2m}$ = − 314	$-\kappa_{2s}$ = − 271
+ 95° 95		
180	144	203
$K_p = \pi$ + 0	$K_{2m} = \pi$ − 36	$K_{2s} = \pi$ + 23

Sequences of Angles.

M_2. ϑ_m.	K_2. $(K_{\prime\prime}+30°n)$ $\vartheta_{\prime\prime}$.	S_2. $(K_s+30°n)$ ϑ_s.	O. $(K_o-15°n)$ ϑ_o.	K_1. $(K_\prime+15°n)$ $\vartheta_\prime$.	P. $(K_p+15°n)$ ϑ_p.	M_4. (K_{2m}) ϑ_{2m}.	S_4. $(K_{2s}+60°n)$ ϑ_{2s}.
°	°	°	°	°	°	°	°
0°.	Mar. − 16	− 11	Mar. π − 13	Mar. − 12	Mar. π + 0	π − 36	π + 23
(a const.)	+ 14	+ 19	π − 28	+ 3	π + 15	(const.)	π + 83
	April + 44	+ 49	April π − 43	April + 18	Feb. π + 30		− 37
	+ 74	+ 79	π − 58	+ 33	π + 45		
	May π − 76	π − 71	May π − 73	May + 48	Jan. π + 60		
	π − 46	π − 41	π − 88	+ 63	π + 75		
			June + 77	June + 78	Dec. π + 90		
			+ 62	− 87	− 75		
			July + 47	July − 72	Nov. − 60		
			+ 32	− 57	− 45		
			Aug. + 17	Aug. − 42	Oct. − 30		
			+ 2	− 27	− 15		

All months.

Semi-diurnal: $A = H\cos\vartheta$, $G = Hp^2\cos\vartheta$, $F = Hp\sin\vartheta$.

M_2.	S_2.			K_2.		
	A_s. $(H_s=·697.)$	G_s. $(H_s p_s^2=·746.)$	F_s. $(H_s p_s=·721.)$	$A_{\prime\prime}$. $(H_{\prime\prime}=·201.)$	$G_{\prime\prime}$. $(H_{\prime\prime}p_{\prime\prime}^2=·217)$	$F_{\prime\prime}$. $(H_{\prime\prime}p_{\prime\prime}=·209.)$
$H_m=H_m p_m^2=1·568$	+ ·684	+ ·732	− ·137	Mar. + ·193	+ ·209	− ·058
$A_m=G_m=1·568$	+ ·659	+ ·705	+ ·234	+ ·195	+ ·211	+ ·051
$F_m=0$	+ ·457	+ ·489	+ ·544	April + ·145	+ ·156	+ ·145
(constants)	+ ·133	+ ·142	+ ·708	+ ·055	+ ·060	+ ·201
	− ·227	− ·243	+ ·682	May − ·049	− ·053	+ ·203
	− ·526	− ·563	+ ·473	− ·140	− ·151	+ ·150

All months.

Repeat the sequences for K_2 and S_2 changing the signs.

Add M_2 to S_2.

$A_m + A_s$.	$G_m + G_s$.	$F_m + F_s$.
2·252	2·300	
2·227	2·273	
2·025	2·057	
1·701	1·710	
1·341	1·325	
1·042	·005	
·884	·836	
·909	·863	
1·111	1·079	
1·435	1·426	
1·795	1·811	
2·094	2·131	

All months. Same as F_s.

Repeat without change.

Then write out the three K_2 sequences, with the months, *in extenso*, each on a separate strip of paper; place the $A_{//}$ strip opposite to the $A_m + A_s$ table, so that the month for which the sequence is required falls in the first place; *e.g.*, for March put + ·193, for April put + ·145, for May put − ·049, for June put − ·193, &c., in the first place opposite 2·252 of $A_m + A_s$. Then add the $A_m + A_s$ and $A_{//}$ tables together in a different way for each of the six months from March to August.

Proceed with the $G_{//}$, $F_{//}$ strips in the same way. The next following table is formed in this way.

Semi-diurnal: $A_2 = A_m + A_{//} + A_s$, $G_2 = G_m + G_{//} + G_s$, $F_2 = F_m + F_{//} + F_s$.

March.			April.			&c.
A_2.	G_2.	F_2.	A_2.	G_2.	F_2.	
2·445	2·509	−·195	2·397	2·456	+·008	Continue as far as August inclusive.
2·422	2·484	+·285	2·282	2·333	+·435	
2·170	2·213	+·689	1·976	2·004	+·747	
1·756	1·770	+·909	1·561	1·559	+·858	
1·292	1·272	+·885	1·148	1·116	+·740	
·902	·854	+·623	·847	·794	+·422	
·691	·627	+·195	·739	·680	−·008	
·714	·652	−·285	·854	·803	−·435	
·966	·923	−·689	1·160	1·132	− 747	
1·380	1·366	−·909	1·575	1·577	−·858	
1·844	1·864	−·885	1·988	2·020	−·740	
2·234	2·282	−·623	2·289	2·342	−·422	

Repeat without change.

Diurnal: $A = H \cos \vartheta$, $G = Hp^2 \cos \vartheta$, $F = Hp \sin \vartheta$.

O.				K_1.				P.			
	A_o. (H_o = ·653.)	G_o. ($H_o p_o^2$ = ·151.)	F_o. ($H_o p_o$ = ·314.)		$A_{,}$. ($H_{,}$ = 1·299.)	$G_{,}$. ($H_{,} p_{,}^2$ = ·350.)	$F_{,}$. ($H_{,} p_{,}$ = ·674.)		A_p. (H_p = ·388.)	G_p. ($H_p p_p^2$ = ·103.)	F_p. ($H_p p_p$ = ·200.)
Mar.	−·636	−·147	+·071	Mar.	+1·271	+·342	−·140	Mar.	−·388	−·103	·000
	−·577	−·133	+·148		+1·297	+·350	+·035		−·375	−·100	−·052
April	−·477	−·110	+·214	April	+1·235	+·333	+·208	Feb.	−·336	−·089	−·100
	−·346	−·080	+·266		+1·089	+·294	+·367		−·275	−·073	−·141
May	−·191	−·044	+·300	May	+ ·869	+·234	+·501	Jan.	−·194	−·052	−·173
	−·023	−·005	+·314		+ ·590	+·159	+·601		−·100	−·027	−·193
June	+·147	+·034	+·306	June	+ ·270	+·073	+·659	Dec.	·000	·000	−·200
	+·307	+·071	+·278		− ·068	−·018	+·673		+·100	+·027	−·193
July	+·445	+·103	+·230	July	− ·402	−·108	+·641	Nov.	+·194	+·052	−·173
	+·554	+·128	+·166		− ·708	−·191	+·565		+·275	+·073	−·141
Aug.	+·625	+·144	+·092	Aug.	− ·965	−·260	+·451	Oct.	+·336	+·089	−·100
	+·653	+·151	+·011		−1·157	−·312	+·306		+·375	+·100	−·052

Repeat changing signs.

Add together the O and K_1 sequences as they stand above :—

$O + K_1$.

	$A_o + A_,$.	$G_o + G_,$.	$F_o + F_,$.
March	+ ·635	+ ·195	− ·069
	+ ·720	+ ·217	+ ·183
April	+ ·758	+ ·223	+ ·422
	+ ·743	+ ·214	+ ·633
May	+ ·678	+ ·190	+ ·801
	+ ·567	+ ·154	+ ·915
June	+ ·417	+ ·107	+ ·965
	+ ·239	+ ·053	+ ·951
July	+ ·043	− ·005	+ ·871
	− ·154	− ·063	+ ·731
August	− ·340	− ·116	+ ·543
	− ·504	− ·161	+ ·317

Repeat changing signs.

Write out the P sequences *in extenso* with the months on the margin (24 entries), each on a separate strip of paper ; place the A_p strip opposite the $A_o + A_,$ table so that any chosen month in one agrees with that month on the other ; add the two tables together, making the first entry in the new sequence that opposite which the chosen month is written. For example the April entry in the A_p sequence (completed) is − ·336, and this added to + ·758, the April entry of $A_o + A_,$, gives + ·422, which is the initial entry in the diurnal sequence for A_1 corresponding to April in the following table. G_p and F_p are operated on in the same way.

The first time the computer does this sort of work he may find it convenient to write out the $O + K_1$ sequences *in extenso*, so as to see exactly how the computation runs, but it will be found with a little practice that this is unnecessary.

Diurnal: $A_1 = A_o + A_, + A_p$, $G_1 = G_o + G_, + G_p$, $F_1 = F_o + F_, + F_p$.

March.			April.			&c.
A_1.	G_1.	F_1.	A_1.	G_1.	F_1.	
+ ·247	+ ·092	− ·069	+ ·422	+ ·134	+ ·522	Continue up to August, inclusive.
+ ·345	+ ·117	+ ·131	+ ·368	+ ·114	+ ·685	
+ ·422	+ ·134	+ ·322	+ ·290	+ ·087	+ ·801	
+ ·468	+ ·141	+ ·492	+ ·192	+ ·054	+ ·863	
+ ·484	+ ·138	+ ·628	+ ·081	+ ·018	+ ·865	
+ ·467	+ ·127	+ ·722	− ·036	− ·020	+ ·810	
+ ·417	+ ·107	+ ·765	− ·151	− ·057	+ ·698	
+ ·339	+ ·080	+ ·758	− ·254	− ·090	+ ·538	
+ ·237	+ ·047	+ ·698	− ·340	− ·116	+ ·343	
+ ·121	+ ·010	+ ·590	− ·404	− ·134	+ ·124	
− ·004	− ·027	+ ·443	− ·441	− ·143	− ·104	
− ·129	− ·061	+ ·265	− ·445	− ·144	− ·324	

Quater-diurnal: $A = H \cos \vartheta$, $G = H (2p)^2 \cos \vartheta$, $F = H (2p) \cos \vartheta$.

M_4.			S_4.		
A_{2m}. ($H_{2m} = \cdot 007$.)	G_{2m}. ($4H_m = \cdot 028$.)	F_{2m}. ($2H_m = \cdot 014$.)	A_{2s}. ($H_{2s} = \cdot 006$.)	G_{2s}. ($4H_{2s}p_s^2 = \cdot 026$.)	F_{2s}. ($2H_{2s}p_s = \cdot 012$.)
$-\cdot 006$ const.	$-\cdot 023$ const.	$+\cdot 008$ const.	All months. $-\cdot 006$	$-\cdot 024$	$-\cdot 005$
			$-\cdot 001$	$-\cdot 003$	$-\cdot 012$
			$+\cdot 005$	$+\cdot 021$	$-\cdot 007$
			Repeat and change.	Repeat and change.	Repeat and change.

Quater-diurnal: $A_4 = A_{2m} + A_{2s}$, $G_4 = G_{2m} + G_{2s}$, $F_4 = F_{2m} + F_{2s}$.

A_4.	G_4.	F_4.
All months. $-\cdot 012$	$-\cdot 047$	$+\cdot 003$
$-\cdot 007$	$-\cdot 026$	$-\cdot 004$
$-\cdot 001$	$-\cdot 002$	$+\cdot 001$
$\cdot 000$	$+\cdot 001$	$+\cdot 013$
$-\cdot 005$	$-\cdot 020$	$+\cdot 020$
$-\cdot 011$	$-\cdot 044$	$+\cdot 015$

Repeat without change of sign.

Semi-annual and Mean Water. (See (56), § 10.)

$$\begin{aligned} -\kappa_{ssa} &= \quad 234^\circ \\ -10^\circ &= - \quad 10 \\ &\quad\overline{\quad\quad 224} \\ K_{ssa} &= \pi + 44 \end{aligned}$$

Sequence.

$K_{ssa} + 60^\circ n$. ϑ_{ssa}.	$H_{ssa} \cos \vartheta_{ssa}$. ($H_{ssa} = \cdot 095$.)	$B_o = \mathfrak{A} + H_{ssa} \cos \vartheta_{ssa}$. ($\mathfrak{A} = 3 \cdot 859$.)
March $\pi + 44^\circ$	$-\cdot 068$	March 3·791
April $-\ 76$	$+\cdot 023$	April 3·882
May $-\ 17$	$+\cdot 091$	May 3·950.
	Repeat and change.	June 3·927
		July 3·836
		Aug. 3·768

Annual.

$$-\kappa_{sa} = -\ 357^\circ$$
$$-\ 5^\circ = -\ 5$$
$$\mathrm{K}_{sa} = -\ 2$$

Sequence.

$\mathrm{K}_{sa} + 10^\circ n_r$ ϑ_{sa}.			
March	$-\ 2^\circ$	June	$+\ 88^\circ$
	$+\ 8$		$\pi - 82$
	$+\ 18$		$\pi - 72$
April	$+\ 28$	July	$\pi - 62$
	$+\ 38$		$\pi - 52$
	$+\ 48$		$\pi - 42$
May	$+\ 58$	Aug.	$\pi - 32$
	$+\ 68$		$\pi - 22$
	$+\ 78$		$\pi - 12$

Annual Tide.

$\mathrm{H}_{sa} \cos \vartheta_{sa}$. ($\mathrm{H}_{sa} = \cdot 390$.)			
	ft.		ft.
March	$+\cdot 390$	June	$+\cdot 014$
	$+\cdot 386$		$-\cdot 054$
	$+\cdot 371$		$-\cdot 121$
April	$+\cdot 344$	July	$-\cdot 183$
	$+\cdot 307$		$-\cdot 240$
	$+\cdot 261$		$-\cdot 290$
May	$+\cdot 207$	Aug.	$-\cdot 331$
	$+\cdot 146$		$-\cdot 362$
	$+\cdot 081$		$-\cdot 381$

Mean Interval i, and Parallactic Correction to i. (See (31) § 6; and (47), § 8.)

$$\kappa_m = 229^\circ$$
$$\tfrac{1}{30}\kappa_m = 7{\cdot}633$$
$$\tfrac{1}{30^2} \cdot \tfrac{21}{20}\kappa_m = \cdot 267$$
$$i = 7^{\mathrm{h}}{\cdot}900$$

Retaining i in hours, parallactic correction to $i = +\ 0^{\mathrm{m}}{\cdot}08 \times i$
$= +\ 0^{\mathrm{m}}{\cdot}63.$

N.B.—$0^{\mathrm{m}}{\cdot}08$ is an absolute constant.

Parallactic Corrections. (See §§ 3, 8.)

36·4 H_n is greater than 8 H_m; therefore $\alpha = \frac{4}{3}$, $\alpha' H_m = \cdot 07\ H_m = \cdot 110$,
36·4 H_q is greater than 8 H_o; therefore $\beta = \frac{4}{3}$, $\beta' H_o = \cdot 07\ H_o = \cdot 046$.

[N.B.—When either of these inequalities is *less* instead of greater, put

$$\alpha = 12\cdot 14\ H_n/H_m - \tfrac{4}{3}, \quad \alpha' H_m = \cdot 639\ H_n - \cdot 07\ H_m.$$
$$\beta = 12\cdot 14\ H_q/H_o - \tfrac{4}{3}, \quad \beta' H_o = \cdot 639\ H_q - \cdot 07\ H_o.$$

If the N tide is unknown take $\alpha = 1$, $\alpha' = \frac{1}{19}$; if the Q tide is unknown take $\beta = 1$, $\beta' = \frac{1}{19}$.]

(Table III.) $${}_o p = \cdot 481 - \cdot 013\beta^{-1} = \cdot 481 - \cdot 010 = \cdot 471;$$
$${}_o p^2 = \cdot 231 - \cdot 012\beta^{-1} = \cdot 231 - \cdot 009 = \cdot 222.$$

(Table I.) Let

$$\gamma = \frac{\kappa_m - \kappa_n}{9\cdot 11\ H_n/H_m - 1};$$

the denominator is $9\cdot 11 \times \cdot 272 - 1 = 1\cdot 48$; the numerator is 4°.
Hence

$$\gamma = + 4^\circ \div 1\cdot 48 = + 3^\circ.$$

Let

$$\delta = \frac{\kappa_o - \kappa_q}{9\cdot 11\ H_q/H_o - 1} - \frac{\cdot 013}{\beta}\kappa_m;$$

the denominator of the first term is $9\cdot 11 \times \cdot 232 - 1 = 1\cdot 11$; the numerator is − 4°. Hence the first term is − 4°. The second term is $- \cdot 010 \times 229^\circ = - 2^\circ$. Hence

$$\delta = - 6^\circ.$$

(See Tables I., II., III.)

M_2.	K_2.	O.	K_o.
$p_n \kappa_m =$ 225	$\cdot 0384 \kappa_m =$ 9	$p_q \kappa_m =$ 106	$\cdot 0192 \kappa_m =$ 4
$- \kappa_n = -$ 225	$K_{II} = -$ 16	$- \kappa_q = -$ 42	$K_I = -$ 12
$+ \gamma$ 3		$+ \delta = -$ 6	
		$+ 95^\circ =$ 95	
		153	
${}_m K = +$ 3	${}_{II} K = -$ 7	${}_o K = \pi -$ 27	${}_I K = -$ 8

N.B.—My calculations were made on a principle, now abandoned, which led to slightly different values. I therefore now continue the calculation with

$$\alpha' H_m = \cdot 122, \quad \beta' H_o = \cdot 041,$$
$${}_m K = + 5^\circ, \quad {}_{II} K = - 8^\circ, \quad {}_o K = \pi - 19^\circ, \quad {}_I K = - 8^\circ.$$

Sequences of Angles.

M_2. $({}_m\mathrm{K}.)$ ${}_m\vartheta$.	K_2. $({}_{''}\mathrm{K}+30°\,n)$ ${}_{''}\vartheta$.	O. $({}_o\mathrm{K}-15°\,n)$ ${}_o\vartheta$.	K_1. $({}_{'}\mathrm{K}+15°\,n)$ ${}_{'}\vartheta$.
°	°	°	°
+ 5	March − 8	March π − 19	March − 8
const.	+ 22	π − 34	+ 7
	April + 52	April π − 49	April + 22
	+ 82	π − 64	+ 37
	May π − 68	May π − 79	May + 52
	π − 38	+ 86	+ 67
		June + 71	June + 82
		+ 56	π − 83
		July + 41	July π − 68
		+ 26	π − 53
		Aug. + 11	Aug. π − 38
		− 4	π − 23

Semi-diurnal.

${}_m\mathrm{A} = \alpha' \mathrm{H}_m \cos {}_m\vartheta$, ${}_m\mathrm{G} = \alpha' \mathrm{H}_m p_m^2 \cos {}_m\vartheta$, ${}_m\mathrm{F} = \alpha' \mathrm{H}_m p_m \sin {}_m\vartheta$;

${}_{''}\mathrm{A} = \cdot 036\ \mathrm{H}_{''} \cos {}_{''}\vartheta$, ${}_{''}\mathrm{G} = \cdot 036\ \mathrm{H}_{''}\, {}_{''}p^2 \cos {}_{''}\vartheta$, ${}_{''}\mathrm{F} = \cdot 036\ \mathrm{H}_{''}\, {}_{''}p \sin {}_{''}\vartheta$.

$z_2 = {}_m\mathrm{A} + {}_{''}\mathrm{A}$, $m_2 = {}_m\mathrm{G} + {}_{''}\mathrm{G}$, $l_2 = {}_m\mathrm{F} + {}_{''}\mathrm{F}$.

${}_m\mathrm{A}$. $(\alpha' \mathrm{H}_m = \cdot 122.)$	${}_{''}\mathrm{A}$. $(\cdot 036\ \mathrm{H}_{''} = \cdot 007.)$	${}_m\mathrm{G}$. $(\alpha' \mathrm{H}_m p_m^2 = \cdot 122.)$	${}_{''}\mathrm{G}$. $(\cdot 036\ \mathrm{H}_{''}\, {}_{''}p^2 = \cdot 008.)$	${}_m\mathrm{F}$. $(\alpha' \mathrm{H}_m p_m = \cdot 122.)$	${}_{''}\mathrm{F}$. $(\cdot 036\ \mathrm{H}_{''}\, {}_{''}p = \cdot 007.)$
+ ·122	March + ·007	+ ·122	March + ·008	+ ·011	March − ·001
const.	+ ·006	const.	+ ·007	const.	+ ·003
	April + ·004		April + ·005		April + ·006
	+ ·001		+ ·001		+ ·007
	May − ·003		May − ·003		May + ·006
	− ·006		− ·006		+ ·004
z_2		m_2		l_2	
	March + ·129		March + ·130		March + ·010
	+ ·128		+ ·129		+ ·014
	April + ·126		April + ·127		April + ·017
	+ ·123		+ ·123		+ ·018
	May + ·119		May + ·119		May + ·017
	+ ·116		+ ·116		+ ·015
	June + ·115		June + ·114		June + ·012
	+ ·116		+ ·115		+ ·008
	July + ·118		July + ·117		July + ·005
	+ ·121		+ ·121		+ ·004
	Aug. + ·125		Aug. + ·125		Aug. + ·005
	+ ·128		+ ·128		+ ·007

Repeat without change.

It might suffice if the parallactic correction to K_2 were neglected, in which case $z_2 = m_2 = {}_m\mathrm{A}$, $l_2 = {}_m\mathrm{F}$. The labour of making the correct table is, however, inconsiderable.

Diurnal: $_oA = \beta' H_o \cos {}_o\vartheta$, $_oG = \beta' H_o\, {}_op^2 \cos {}_o\vartheta$, $_oF = \beta' H_o\, {}_op \sin {}_o\vartheta$,

$_{'}A = \cdot 036 H_{'} \cos {}_{'}\vartheta$, $_{'}G = \cdot 036 H_{'}\, {}_{'}p^2 \cos {}_{'}\vartheta$, $_{'}F = \cdot 036 H_{'}\, {}_{'}p \sin {}_{'}\vartheta$,

$z_1 = {}_oA + {}_{'}A$, $m_1 = {}_oG + {}_{'}G$, $l_1 = {}_oF + {}_{'}F$.

	$_oA$. ($\beta' H_o = \cdot 041$.)	$_{'}A$. ($\cdot 036 H_{'} = \cdot 047$.)	$_oG$. ($\beta' H_o\, {}_op^2 = \cdot 009$.)	$_{'}G$. ($\cdot 036 H_{'}\, {}_{'}p^2 = \cdot 013$.)	$_oF$. ($\beta' H_o\, {}_op = \cdot 019$.)	$_{'}F$. ($\cdot 036 H_{'}\, {}_{'}p = \cdot 025$.)
March	−·039	+·047	−·009	+·013	+·006	−·003
	−·034	+·047	−·007	+·013	+·011	+·003
April	−·027	+·044	−·006	+·012	+·014	+·009
	−·018	+·038	−·004	+·010	+·017	+·015
May	−·008	+·029	−·002	+·008	+·019	+·020
	+·003	+·018	+·001	+·005	+·019	+·023
June	+·013	+·007	+·003	+·002	+·018	+·025
	+·023	−·006	+·005	−·002	+·016	+·025
July	+·031	−·018	+·007	−·005	+·012	+·023
	+·037	−·028	+·008	−·008	+·008	+·020
Aug.	+·040	−·037	+·009	−·010	+·004	+·015
	+·041	−·043	+·009	−·012	−·001	+·010
	z_1		m_1		l_1	
	March +·008		March +·004		March +·003	
	+·013		+·006		+·014	
	April +·017		April +·006		Apr. +·023	
	+·020		+·006		+·032	
	May +·021		May +·006		May +·039	
	+·021		+·006		+·042	
	June +·020		June +·005		June +·043	
	+·017		+·003		+·041	
	July +·013		July +·002		July +·035	
	+·009		·000		+·028	
	Aug. +·003		Aug. −·001		Aug. +·019	
	−·002		−·003		+·009	

Repeat these sequences, changing the signs.

Nodal Corrections. (See (51) (52) § 9.)

Find by reference to preceding sequences the following nine sequences:—

(i.) $-\cdot 0372\, A_m$, (ii.) $+\cdot 283\, A_{''}$, (iii.) $+\cdot 296\, F_{''}$, (iv.) $+\cdot 283\, F_{''}$, (v.) $-\cdot 0372\, G_m$,
(vi.) $-\cdot 296\, G_{''}$, (vii.) $-\cdot 0372\, G_m$, (viii.) $+\cdot 283\, G_{''}$, (ix.) $+\cdot 319\, F_{''}$.

Then the semi-diurnal sequences are as follows:—

a_2 is (i.) + (ii.); b_2 is (iii.); c_2 is (iv.); d_2 is (v.) + (vi.); e_2 is (vii.) + (viii.); f_2 is (ix.).

For example:—

	a_2.		a_2.
March . . .	− ·003	June . . .	− ·113
	− ·003		− ·113
April . . .	− ·017	July	− ·099
	− ·042		− ·074
May . . .	− ·072	August . . .	− ·044
	− ·098		− ·018

This, and the other semi-diurnal sequences are repeated without change of sign, and in all six of them the months run just as in this example, and denote the places at which to begin reading the sequence for the month in question.

The diurnal sequences are obtained thus :—

Find, from preceding sequences, the twelve following—

(i.) + ·188 A_o, (ii.) + ·115 $A_{,}$, (iii.) − ·391 F_o, (iv.) + ·297 $F_{,}$, (v.) + ·188 F_o, (vi.) + ·115 $F_{,}$, (vii.) + ·391 G_o, (viii.) − ·297 $G_{,}$, (ix.) + ·188 G_o, (x.) + ·115 $G_{,}$, (xi.) − ·0905 F_o, (xii.) + ·080 $F_{,}$.

Then

a_1 is (i.) + (ii.) ; b_1 is (iii.) + (iv.) ; c_1 is (v.) + (vi.) ;
d_1 is (vii.) + (viii.) ; e_1 is (ix.) + (x.) ; f_1 is (xi.) + (xii.).

For example :—

	a_1.		a_1.
March . . .	+ ·026	June . . .	+ ·059
	+ ·041		+ ·050
April . . .	+ ·052	July	+ ·038
	+ ·060		+ ·023
May	+ ·064	August . . .	+ ·007
	+ ·064		− ·010

This, and the other diurnal sequences, are repeated with change of sign, and the months in all six of them run just as in this example, and denote the places at which to begin reading the sequence for the month in question.

Calculation of Height, Interval, and Corrections for each Month. (See §§ 7, 8, 9, 10.)

Remarks.—Each column in the following computation is arranged exactly like the first, so that it is unnecessary to repeat the letters in the successive columns.

For the month of March, which serves as an example, we refer to the March sequences, and enter the twelve values of G_2 successively, in the top left-hand corners

of twelve columns; below these are entered the twelve values of G_1, and the twelve values of G_4, and on the right of the columns are put the twelve values of F_2, F_1, F_4. A similar statement is true of all the other symbols all the way down, and all the sequences are utilised up to twelve entries in each.

The divisions and multiplications may be done by CRELLE's table; Δ_o is found by a table of natural tangents, and $\delta\Delta$ is converted into degrees by a table of circular measure or radians. It is necessary to take as an approximate value of Δ_o the nearest even number of degrees.

From the places where the values of Δ_o are found, the left-hand side of each column corresponds to the time of moon's transit written at the head of the column, and the right-hand side to a time of moon's transit 12^h greater than the time specified. But the whole table for any month serves for its opposite (*e.g.*, September opposite to March), by transposing the words right and left in the preceding statement. Thus the whole computation has only to be made for six months (up to August inclusive), instead of for twelve. The diurnal terms with suffix 1, are written in the margin, with alternative signs, and the upper sign is to be used on the left, and the lower on the right of each column.

Thus, in finding, for example, $F_1 \cos \frac{1}{2}\Delta_o$ on the right we deem the F_1 written at the head to have its sign changed. Thus, in the column of 0^h we have $F_1 = -\cdot 069$ and on the right hand, $\Delta_o = +4^\circ$; then the required entry, on the right-hand, for $F_1 \cos \frac{1}{2}\Delta_o$ is $+\cdot 069 \cos(+2^\circ) = +\cdot 069$.

The values of δT, δM, δN are extracted from the sequences of those functions in Table IV., and they are the same on each side of the column. The value of B_o is taken from the sequence of the semi-annual tide and mean water, and changes only with the month.

The parallactic correction to the mean interval, i, is introduced in computing R. This is a constant of the port and is the same in all months.

In computing the height $\mathfrak{H}$, and its corrections, an approximate value of Δ is used, namely, the nearest even number of degrees; this approximate Δ will often be the same as Δ_o.

In this table it appears to me specially important that the signs of the sines and cosines should be determined independently of their numerical values.

Whereas in the right-hand of column 6^h we get, as a result of a second approximation, no value of Δ, the conjectural value $\Delta = +90^\circ$ is adopted for the computation of P, Q, S, and values of M, N, R are not computed.

The table has rows marked δI, δh, M″, N″, P″, Q″; all these are derived from a subsequent table of "*Corrections for reference to the Moon's transit.*" But it appears convenient to finish off the computation on this sheet, although we have to pause in the computation in order to calculate the said table of corrections.

March.

Interval.	0° or 0h.		15° or 1h.			90° or 6h.	
G_2	+2·509	F_2 — ·195	+2·484	+ ·285		+ ·627	+ ·195
G_1	+ ·092	F_1 — ·069	+ ·117	+ ·131		+ ·107	+ ·765
G_4	— ·047	F_4 + ·003	— ·026	— ·004	&c.	— ·047	+ ·003
$G_2 + G_1$	+2·601	$F_2 + F_1$ — ·264	+2·601	+ ·416		+ ·734	+ ·960
$G_2 - G_1$	+2·417	$F_2 - F_1$ — ·126	+2·367	+ ·154		+ ·520	— ·570
$\tan \Delta_o = -\frac{F_2 + F_1}{G_2 + G_1}$	+ ·102	$-\frac{F_2 - F_1}{G_2 - G_1}$ + ·052	— ·160	— ·065		−1·308	+1·096
Δ_o	+ 6°	+ 4°	−10°	−4°		−52°	+48°
α $F_2 \cos \Delta_o$	— ·194	— ·195	+ ·281	+ ·284		+ ·120	+ ·131
β $F_2 \sin \Delta_o$	— ·020	— ·014	— ·050	— ·020		— ·154	+ ·145
γ $G_2 \cos \Delta_o$	+2·495	+2·503	+2·446	+2·478		+ ·386	+ ·420
δ $G_2 \sin \Delta_o$	+ ·262	+ ·175	— ·431	— ·173		— ·493	+ ·466
ϵ $\pm F_1 \cos \frac{1}{2}\Delta_o$	— ·069	+ ·069	+ ·131	— ·131		+ ·688	— ·699
ζ $\pm F_1 \sin \frac{1}{2}\Delta_o$	— ·004	+ ·002	— ·011	+ ·005	&c.	— ·336	— ·311
η $\pm 2G_1 \cos \frac{1}{2}\Delta_o$	+ ·184	— ·184	+ ·233	— ·234		+ ·192	— ·196
θ $\pm 2G_1 \sin \frac{1}{2}\Delta_o$	+ ·010	— ·006	— ·020	+ ·008		— ·094	— ·087
λ $F_4 \cos 2\Delta_o$	+ ·003	+ ·003	— ·004	— ·004		— ·001	·000
μ $F_4 \sin 2\Delta_o$	+ ·001	+ ·000	+ ·001	+ ·001		— ·003	+ ·003
ν $\frac{1}{2}G_4 \cos 2\Delta_o$	— ·023	— ·023	— ·012	— ·013		+ ·006	+ ·002
ρ $\frac{1}{2}G_4 \sin 2\Delta_o$	— ·005	— ·003	+ ·004	+ ·002		+ ·023	— ·023
$\alpha + \delta$	+ ·068	— ·020	— ·150	+ ·111		— ·373	+ ·597
$\epsilon + \theta$	— ·059	+ ·063	+ ·111	— ·123		+ ·594	— ·786
$\lambda + \rho$	— ·002	·000	·000	— ·002		+ ·022	— ·023
Sum N_o	+ ·007	+ ·043	— ·039	— ·014		+ ·243	— ·212
$\zeta - \eta$	— ·188	+ ·186	— ·244	+ ·239		— ·528	— ·115
$\frac{1}{2}(\zeta - \eta)$	— ·094	+ ·093	— ·122	+ ·120		— ·264	— ·058
$\beta - \gamma$	−2·515	−2·517	−2·496	−2·498	&c.	— ·540	— ·275
$2(\mu - \nu)$	+ ·048	+ ·046	+ ·026	+ ·028		— ·018	+ ·002
Sum D_o	−2·561	−2·378	−2·592	−2·350		— ·822	— ·331
$\delta\Delta = N_o/D_o$	— ·003	— ·018	+ ·015	+ ·006		— ·296	+ ·640
(In degrees) $\delta\Delta$	— 0°·2	— 1°·0	+0°·9	+0°·3		−17 ·0	+36°·7
$\Delta = \Delta_o + \delta\Delta$	+5·8	+3·0	−9·1	−3·7		−68·6*	No H. W.*
$\frac{1}{30}\Delta$	·193	·100	·303	·123		2·287	
$\frac{1}{30^2}\frac{21}{20}\Delta$	7	4	11	4		80	
(In hours) I	+ ·200	+ ·104	— ·314	— ·127	&c.	−2·367	
δT	— ·030	— ·030	+ ·059	+ ·059		+ ·030	
	+ ·170	+ ·074	— ·255	— ·068		−2·337	
(Mean int.) i	7·900	7·900	7·900	7·900		7·900	
	8·070	7·974	7·645	7·832		5·563	
	8h 4m·2	7h 58m·4	7h 38m·7	7h 49m·9		5h 33m·8	
(See below) δI	+0·9	+ 0·4	— 1·8	— 0·8		+ 0·6	
☽	8h 5m	7h 59m	7h 37m	7h 49m		5h 34m	

Take upper sign of diurnal terms on left, lower on right

Continue up to 165° or 11h, twelve columns in all.

* Second approximation (see below) introduced here.

MARCH (continued).

Correction of D.	0° or 0h.		15° or 1h.			90° or 6h.		
$a + \delta$						− ·531*		
$\frac{1}{4}(\epsilon + \theta)$						+ ·126*		
$\frac{1}{4}(\lambda + \rho)$						+ ·044*		
Sum	This correction need only be made when D_o					− ·361*		
(In circ. meas.) $\delta\Delta$	is small and $\delta\Delta$ considerable				&c.	+ ·022*		&c.
Product δD						− ·008*		
D_o						− ·744*		
$D = D_o + \delta D$	−2·56		−2·38	−2·59	−2·35	− ·752*		
Height.								
A_2	2·445	$-F_2$	+ ·195	2·422	− ·285	·691	− ·195	
A_1	+ ·247	$-F_1$	+ ·069	+ ·345	− ·131	&c.	+ ·417	− ·765
A_4	− ·012	$-F_4$	− ·003	− ·007	+ ·004	− ·012	− ·003	
Approx. Δ	+ 6°		+ 4°	− 10°	− 4°	− 70*	None	
$A_2 \cos \Delta$	2·432		2·439	2·385	2·416	·236		
$-F_2 \sin \Delta$	+ ·020		+ ·014	+ ·050	+ ·020	+ ·183		
$\pm A_1 \cos \frac{1}{2}\Delta$	+ ·247		− ·247	+ ·344	− ·345	+ ·342		
$\mp 2F_1 \sin \frac{1}{2}\Delta$	+ ·007		− ·005	+ ·023	− ·009	+ ·878		
$A_4 \cos 2\Delta$	− ·012		− ·012	− ·007	− ·007	&c.	+ ·009	
$-\frac{1}{2}F_4 \sin 2\Delta$	− ·000		− ·000	− ·001	− ·000	+ ·001		&c.
Sum h	2·694	Upper sign on left; lower on right.	2·189	2·794	2·075	1·649	See below for evanescent tide.	
B_o	3·791		3·791	3·791	3·791	3·791		
	6·485		5·980	6·585	5·866	5·440		
(See below) δh	− ·004		+ ·001	+ ·004	− ·012	− ·014		
$\mathfrak{H}$	6·48		5·98	6·59	5·85	5·43		
Nodal correction.								
Δ	+ 6°		+ 4°	− 10°	− 4°	− 70°*	+ 90*	
$119^m \div D$	− 46m		− 50m	− 46m	− 51m	&c.	− 158m*	
c_2	− ·016	e_2	+ ·001	+ ·014	+ ·002	+ ·016	− ·117	
c_1	− ·003	e_1	+ ·011	+ ·032	+ ·015	+ ·134	+ ·014	
$c_2 \cos \Delta$	− ·016		− ·016	+ ·014	+ ·014	+ ·005		
$e_2 \sin \Delta$	+ 0		+ 0	− 0	− 0	+ ·110		
$\pm c_1 \cos \frac{1}{2}\Delta$	− 3		+ 3	+ 32	− 32	+ ·110		
$\pm 2e_1 \sin \frac{1}{2}\Delta$	+ 1		− 1	− 3	+ 1	− · 16		
Sum	− ·018	Upper sign on left; lower on right.	− ·014	+ ·043	− ·017	&c.	+ ·209	&c.
Mult. by 119/D	+ 0·8		+ 0·7	− 2·0	+ 0·9	− 33·0		
δM	− 0·8		− 0·8	+ 1·6	+ 1·6	+ 0·8		
Sum M′	0·0		− 0·1	− 0·4	+ 2·5	− 32·2		
(See below) M″	+ 0·4		+ 0·2	− 0·8	− 0·4	+ 0·3		
Sum M	+ 0·4		+ 0·1	− 1·2	+ 2·1	− 31·9	None	

March (continued).

Nodal correction—(continued).	0° or 0h.			15° or 1h.			90° or 6h.		
d_2	− ·120	f_2	− ·019	&c.	&c.	&c.	&c.	&c.	&c.
d_1	− ·159	f_1	− ·017						
$d_2 \cos \Delta$									
$f_2 \sin \Delta$	&c.								
$\pm d_1 \cos \frac{1}{2}\Delta$									
$\pm 2f_1 \sin \frac{1}{2}\Delta$		Compute like M		&c.	&c.	&c.	&c.	&c.	&c.
Sum									
Mult. by 119/D	&c.								
δN									
Sum N′	&c.								
(See below) N″									
Sum N								None	
a_2	− ·003	$-c_2$	+ ·016	− 003	− ·014		− ·113	− ·016	
a_1	+ ·026	$-c_1$	+ ·003	+ ·041	− ·032	&c.	+ ·059	− ·134	
$a_2 \cos \Delta$	− ·003		− ·003	− ·003	− ·003		− ·039	·000	
$-c_2 \sin \Delta$	+ 2		+ 1	+ 2	+ 1		+ 15	− 16	
$\pm a_1 \cos \frac{1}{2}\Delta$	+ 26	Upper on left; lower on right.	− 26	+ 41	− 41		+ 48	− 42	
$\mp 2c_1 \sin \frac{1}{2}\Delta$	+ 0		− 0	+ 6	− 2		+ ·154	+ ·190	&c.
Sum P′	+ ·025		− ·028	+ ·046	− ·045		+ ·178	+ ·132	
(See below) P″	− ·002		·000	+ ·002	− ·005		− ·006	+ ·005	
Sum P	+ ·023		− ·028	+ ·048	− ·050		+ ·172	+ ·137	
b_2	− ·017	$-d_2$	+ ·120	&c.	&c.		&c.	&c.	
b_1	− ·070	$-d_1$	+ ·159						
$b_2 \cos \Delta$									
$-d_2 \sin \Delta$	&c.								
$\pm b_1 \cos \frac{1}{2}\Delta$									
$\mp 2d_1 \sin \frac{1}{2}\Delta$		Compute like P		&c.	&c.	&c.	&c.	&c.	&c.
Sum Q′	&c.								
(See below) Q″									
Sum Q									
Parallactic corrections.									
l_2	+ ·010	m_2	+ ·130	+ ·014	+ ·129	&c.	+ ·012	+ ·114	&c.
l_1	+ ·003	m_1	+ ·004	+ ·014	+ ·006		+ ·043	+ ·005	
$l_2 \cos \Delta$	+ ·010		+ ·010	+ ·014	+ ·014		+ ·004		
$m_2 \sin \Delta$	+ 14		+ 9	− 22	− 9		− ·107		
$\pm l_1 \cos \frac{1}{2}\Delta$	+ 3	Upper sign on left; lower sign on right.	− 3	+ 14	− 14		+ 35		
$\pm 2m_1 \sin \frac{1}{2}\Delta$	+ 0		− 0	− 1	+ 0	&c.	− 6		&c.
Sum	+ ·027		+ ·016	+ ·005	− ·009		− ·074		
Mult. by 119/D	− 1·2		− 0·8	− 0·2	+ 0·5		+ 11·7		
Par. corr. to i	+ 0·6		+ 0·6	+ 0·6	+ 0·6		+ 0·6		
Sum R	− 0·6		− 0·2	+ 0·4	+ 1·1		+ 12·3	None	

MARCH (continued)

Parallactic corrections —(continued).	0° or 0h.			15° or 1h.			30° or 6h.		
z_2	+ ·129	$-l_2$	− ·010	&c.	&c.	&c.	&c.	&c.	&c.
z_1	+· 008	$-l_1$	− ·003						
$z_2\cos\Delta$									
$-l_2\sin\Delta$	&c.	Compute like		P and Q			&c.	&c.	&c.
$\pm z_1\cos\frac{1}{2}\Delta$									
$\mp 2l_1\sin\frac{1}{2}\Delta$	&c.								
Sum S	+ ·135		+ ·120	+ ·143	+ ·115		+ ·115	+ ·035	

SECOND approximation.

When the correction $\delta\Delta$ is large, as in the case of column 6^h, this is necessary.

Column of 90° or 6h.		
Assume Δ_o	−70°	+86°
α	+·067	+·014
β	−·183	+·195
γ	+·214	+·044
δ	−·598	+·625
ϵ	+·627	−·560
ζ	−·439	−·529
η	+·175	−·157
θ	−·123	−·146
λ	−·002	−·003
μ	−·002	+·000
ν	+·018	+·023
	+·015	−·003
$\alpha+\delta$	−·531	+·639
$\epsilon+\theta$	+·504	−·706
$\lambda+\rho$	+·011	−·006
Sum N_o	−·016	−·073
$\zeta-\eta$	−·614	−·377
$\frac{1}{2}(\zeta-\eta)$	−·307	−·189
$\beta-\gamma$	−·397	+·151
$2(\mu-\nu)$	−·040	−·046
Sum D_o	−·744	−·084
$\delta\Delta=N_o/D_o$	+·022	+·87
(In degrees) $\delta\Delta$	+1°·4	+50°
$\Delta=\Delta_o+\delta\Delta$	−68·6	No H. W.

It is concluded that as the correction in the second column is 50° there is no H. W. A conjectural value of $\Delta = +\ 90°$ is used above in computing P, Q, S.*

* There ought in strictness to be further corrections to ℐ and 𝔍, but they are of little importance Thus:—

Further correction to ℐ and 𝔍.

When Δ is greater than (say) 50° there are further corrections $[\delta I]$, $[\delta h]$ computed from

$$[\delta I] = -\ 2^h\,(1 - \tfrac{1}{8\,5})\,\frac{\cdot 012}{D}\,[3\,F_2 \sin^2\Delta \cos\Delta + (G_{,,} + G_s)\sin^3\Delta]$$

$$[\delta h] = -\ \cdot 012\,[3\,(A_{,,} + A_s)\sin^2\Delta\cos\Delta - F_2 \sin^3\Delta]$$

Thus in the column of 6^h on the left $\Delta = -\ 70°$; then compute thus:—

$F_2 = +\ \cdot 195$,	$G_2 = +\ \cdot 627$,	$A_2 = +\ \cdot 691$
$3F_2 = +\ \cdot 585$,	$-\ H_m = -\ 1\cdot 568$,	$-\ H_m = -\ 1\cdot 568$
	$G_2 - H_m = -\ \cdot 941$	$A_2 - H_m = -\ \cdot 877$
	$G_{,,} + G_s = -\ \cdot 941$	$3\,(A_2 - H_m) = -\ 2\cdot 631$
		$3\,(A_{,,} + A_s) = -\ 2\cdot 631$

By successive use of Traverse table,—

$$(G_{,,} + G_s)\sin\Delta = +\ \cdot 884, \qquad 3\,F_2 \sin\Delta = -\ \cdot 550$$
$$(G_{,,} + G_s)\sin^2\Delta = -\ \cdot 831, \qquad 3\,F_2 \sin^2\Delta = +\ \cdot 517$$
$$(G_{,,} + G_s)\sin^3\Delta = +\ \cdot 781$$
$$+\ 3\,F_2 \sin^2\Delta\cos\Delta = +\ \cdot 177$$
$$+\ \cdot 958$$
$$\times\ -\ \cdot 012$$
$$\text{Divide by D or } -\ \cdot 752 \quad -\ \cdot 0115$$
$$\times\ 2^h \quad +\ \cdot 0153$$
$$+\ \cdot 0306$$
$$-\ \tfrac{1}{8\,5} \quad -\ 4$$
$$[\delta I] = +\ \cdot 030$$
$$= +\ 1^m\cdot 8$$
$$\text{Previous } ℐ = 5^h\ 33\cdot 8$$
$$\text{Correct } ℐ = 5^h\ 36^m$$

Again, by successive use of Traverse table,—

$$3\,(A_{,,} + A_s)\sin\Delta = +\ 2\cdot 472, \qquad -\ F_2 \sin\Delta = +\ \cdot 183$$
$$3\,(A_{,,} + A_s)\sin^2\Delta = -\ 2\cdot 323, \qquad -\ F_2 \sin^2\Delta = -\ \cdot 172$$
$$3\,(A_{,,} + A_s)\sin^2\Delta\cos\Delta = -\ \cdot 795$$
$$-\ F_2 \sin^3\Delta = +\ \cdot 162$$
$$-\ \cdot 633$$
$$\times\ -\ \cdot 012$$
$$[\delta h] = +\ \cdot 008$$
$$\text{Previous } 𝔍 = 5\cdot 426$$
$$\text{Correct } 𝔍 = 5\cdot 43$$

Evanescent Tide. (See § 7.)

The right-hand column of 6^h leads to no H. W., and the tables of $\mathfrak{J}$ and $\mathfrak{H}$ must be completed by other formulæ.

The following calculation is very like the preceding one. The value Δ_o will always be nearly $\pm$ 90°, and in our example it is exactly + 90°.

The computation of M, N, R is to be omitted, and that of P, Q, S has been included in the general calculation with a conjectural $\Delta = +$ 90°.

MARCH.

Evanescent Tide.					Evanescent Tide (continued).			
			90° or 6^h.					90° or 6^h.
	G_2	+ ·627	F_2	+ ·195	$\delta\Delta = \frac{D_o}{E^o}$	,,		+ ·110
	G_1	+ ·107	F_1	+ ·765	(In degrees) $\delta\Delta$	,,		+ 6°·3
	G_4	− ·047	F_4	+ ·003				
			$\frac{1}{4}F_1$	+ ·191	$\Delta = \Delta_o + \delta\Delta$	,,		+ 96·3
	$G_2 + G_1$	,,	$F_2 + \frac{1}{4}F_1$	,,				
	$G_2 - G_1$	+ ·520	$F_2 - \frac{1}{4}F_1$	+ ·004	$\frac{1}{3}\Delta$	,,		3·210
	$\tan\Delta_o = \frac{G_2+G_1}{F_2+\frac{1}{4}F_1}$	,,	$\frac{G_2-G_1}{F_2-\frac{1}{4}F_1}$	+ 130·0	$\frac{1}{3}\sim\frac{2}{2}\frac{1}{0}\Delta$	,,		·112
	Δ_o	,,		+ 90°	I	,,		3·322
					δT	,,		+ ·030
a	$F_2\cos\Delta_o$	,,		·000				3·352
β	$F_2\sin\Delta_o$	,,		+ ·195	Mean int. i	,,		7·900
γ	$G_2\cos\Delta_o$	,,		·000				11·252
δ	$G_2\sin\Delta_o$	,,		+ ·627				$11^h\ 15^m{\cdot}1$
$\epsilon \pm$	$F_1\cos\frac{1}{2}\Delta_o$	,,		− ·541	(See below) δI	,,		+ 2·5
$\zeta \pm$	$F_1\sin\frac{1}{2}\Delta_o$	,,		− ·541				
$\eta \pm$	$2G_1\cos\frac{1}{2}\Delta_o$	,,	right.	− ·151	$\mathfrak{J}$	,,		$11^h\ 18^m$
$\theta \pm$	$2G_1\sin\frac{1}{2}\Delta_o$	,,		− ·151				
λ	$F_4\cos 2\Delta_o$	,,		− ·003	A_2	·691	F_2	− ·195
μ	$F_4\sin 2\Delta_o$	,,		·000	A_1	+ ·417	F_1	− ·765
ν	$\frac{1}{2}G_4\cos 2\Delta_o$	,,	lower to	+ ·024	A_4	− ·012	F_4	− ·003
ρ	$\frac{1}{2}G_4\sin 2\Delta_o$	,,		·000	Δ	,,		96° or π −84°
	$\zeta - \eta$	,,		− ·390				
					$A_2\cos\Delta$	,,		− ·072
	$\frac{1}{2}(\zeta-\eta)$	,,		− ·195	$-F_2\sin\Delta$	,,		− ·194
	$\beta - \gamma$	,,		+ ·195	$\pm A_1\cos\frac{1}{2}\Delta$	,,		− ·279
	$2(\mu - \nu)$	,,		− ·048	$\mp 2F_1\sin\frac{1}{2}\Delta$	,,		+ 1·137
					$A_4\cos 2\Delta$	,,		− ·012
	Sum D_o	,,	Upper sign to left;	− ·048	$-\frac{1}{2}F_4\sin 2\Delta$	,,		+ ·001
	$(\epsilon + \theta)$	,,		− ·692	Sum h	,,		+ ·581
					B_o	,,		3·791
	$\frac{1}{4}(\epsilon+\theta)$	,,		− ·173				4·372
	$a + \delta$	,,		+ ·627				
	$4(\lambda + \rho)$	,,		− ·012	(See below) δh	,,		+ ·011
	Sum $-E_o$	,,		+ ·442	$\mathfrak{H}$	,,		4·38
	E_o	,,		− ·442				

Corrections for Reference to Moon's Transit. (See (57)–(64), § 10.)

Of these corrections δI, δh, M″, N″, P″, Q″ have already been used in the preceding calculation, and we have to show how they are to be computed; we also have to compute $\mathfrak{i}$ and $\mathfrak{h}'$.

From March "intervals" and "heights" we extract I and h, and arrange them in double columns—the even entries in one column and the odd in another. The columns 0^h to 11^h afford the 12 values for 0^h to 11^h of I and h by means of their left hand entries, and they afford the 12 values for 12^h to 23^h by means of their right hand entries. The entry for 23^h is repeated at the top and that for 0^h at the bottom, so that each column has 13 entries, and thus each provides 12 first differences. After finding these differences the distinction of odd and even entries is unnecessary.

The numerical factors 2·2, $\frac{1}{20}$, 2·34, $\frac{11}{300}$, $\frac{1}{20}$, 2·34 in the legends at the top of the columns are absolute constants.

The Θ's are derived from the sequence in Table IV., beginning the sequence with the month treated.

The values of c and d are derived from Table V., for the month named; thus for March c is $-3^m\cdot7$ and d is $-0^h\cdot062$.

The values of M″, — N″, δI are found in columns vii., xi., xii. of the first table, and P″, — Q″, δh in vii., xi., xii. in the second table. The entries opposite 0^h to 11^h were used above on the left-hand side of columns 0^h to 11^h, and the entries opposite 12^h to 23^h were used above on the right-hand side of the columns 0^h to 12^h.

The quantity $\mathfrak{h}'$ is not a final result, but after *interpolated* values of $\mathfrak{h}'$ (see below on Interpolation) have been found, we shall add to it *computed* values of the annual tide, so as to form $\mathfrak{h}$.

The arithmetical processes involved in these tables are sufficiently explair ed by the instructions at the head of each column.

March.—Intervals.

	i.		$2\,dI/dT$. ii.	iii.	iv.	v.	vi.	M″. vii.	viii.	ix.	x.	− N″. xi.	δI. xii.	i. xiii.
	I.		Diff. i.	2·2 × ii.	Θ for Mar.	iii. sin iv.	$\frac{1}{20}$ v.	v. + vi.	iii. cos iv.	$\frac{1}{20}$ viii.	viii. − ix.	iii. + x.	2·34 × v.	*c* for Mar. × ii.
h.	h.	h.	h.		°			m.				m.	m.	m.
23		+ 0·73												
0	+ 0·20		− 1·04	− 2·29	− 10	+ 0·40	+ 2	+ 0·4	− 2·26	− 11	− 2·15	−4·4	+ 0·9	+ 3·8
1		− 0·31	− 1·02	− 2·24	+ 20	− 0·77	− 4	− 0·8	− 2·11	− 11	− 2·00	− 4·2	− 1·8	+ 3·8
2	− 0·82		− 1·00	− 2·20	+ 50	− 1·69	− 8	− 1·8	− 1·41	− 7	− 1·34	− 3·5	− 4·0	+ 3·7
3		− 1·31	− 0·96	− 2·11	+ 80	− 2·09	− 10	− 2·2	− 0·37	− 2	− 0·35	− 2·5	− 4·9	+ 3·6
&c.	&c.		&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.
22	+ 1·27		− 1·09	− 2·40	− 70	+ 2·26	+ 11	+ 2·4	− 0·82	− 4	− 0·78	− 3·2	+ 5·3	+ 4·0
23		+ 0·73	− 1·07	− 2·35	− 40	+ 1·51	+ 8	+ 1·6	− 1·80	− 9	− 1·71	− 4·1	+ 3·5	+ 4·0
0	+ 0·20													

March.—Heights.

	i.		$2\,dh/dT$. ii.	iii.	iv.	v.	vi.	P″. vii.	viii.	ix.	x.	− Q″. xi.	∂h. xii.	$\mathfrak{h}'$. xiii.
	h.		Diff. i.	$\frac{11}{300}$ × ii.	Θ for Mar.	iii. sin iv.	$\frac{1}{20}$ v.	v. + vi.	iii. cos iv.	$\frac{1}{20}$ viii.	viii. − ix.	iii. + x.	2·34 × v.	*d* for Mar. × ii.
h.	ft.	ft.			°			ft.				ft.	ft.	ft.
23		2·53												
0	2·69		+·26	+·0095	− 10	−·0017	− 1	−·002	+·0094	+ 5	+·0089	+·018	−·004	−·016
1		2·79	+·12	+ 44	+ 20	+ 15	+ 1	+ 2	+ 41	+ 2	+ 39	+ 8	+ 4	− 7
2	2·81		−·08	− 29	+ 50	− 22	− 1	− 2	− 19	− 1	− 18	− 5	− 5	+ 5
3		2·71	−·32	− 117	+ 80	− 115	− 6	− 12	− 20	− 1	− 19	− 14	− 27	+ 20
&c.	&c.		&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.
22	2·28		+·56	+ 206	− 70	− 194	− 10	− 20	+ 71	+ 4	+ 67	+ 27	− 45	− 35
23		2·53	+·41	+ 150	− 40	− 96	− 5	− 10	+ 115	+ 6	+ 109	+ 26	− 23	− 25
0	2·69													

Additional Values of Intervals and Heights.

Where the intervals change largely between one column and the next, it would add much to the accuracy if additional values were calculated. Thus, in the calculation for March further values between 75° or 5^h and 90° or 6^h, and again, between 90° or 6^h and 105° or 7^h, would be desirable. The like is true for August, where a column between 105° or 7^h and 120° or 8^h would be useful. I choose this last case for my example.

If INMAN'S traverse table be used, interpolation may be made for $112\frac{1}{2}°$ without much difficulty, but I think it is better to interpolate for an even number of additional degrees, and to compute a column for 113°, found by adding 8° to 105°.

It is proposed then to add a new column between the 8th and 9th.

We begin by interpolating in the sequences of angles. In each sequence we have to find the 8th entry for August; then, if it is semi-diurnal, add 16°; if diurnal add 8° for K_1 and P, and subtract 8° for O; and, if quater-diurnal, add 32°. In the sequence for Θ we add 16°, since it is similar to a semi-diurnal term. The calculation runs thus :—

	K_2.	S_2.	O.	K_1.	P.	S_4.
8th entry . .	$\pi - 46°$	$\pi + 19°$	$- 88°$	$\pi + 63°$	$\pi - 45°$	$\pi + 83°$
Add	+ 16	+ 16	− 8	+ 8	+ 8	+ 32
ϑ	$\pi - 30$	$\pi + 35$	$\pi + 84$	$\pi + 71$	$\pi - 37$	$- 75$

Also $\vartheta_m = 0$, $\vartheta_{2m} = \pi - 36°$, as before.

The 8th entry of Θ is $\pi - 40°$, to which we add 16°, and find $\Theta = \pi - 24°$. With this value of Θ, compute δT, δM, δN. The interpolation amongst the sequences of angles for the parallactic terms is done in the same way. With these new values, and with the former H, Hp^2, Hp we now compute new A's, F's, G's, and are then in a position to compute a new column corresponding to 113° or $7^h\ 32^m$.

In computing δI, δh, M″, N″, P″, Q″, ι, ϑ′, column ii., for intervals, or $2dI/d\mathrm{T}$, must be put equal to $2(I_{120} - I_{105})$; and similarly, column ii. for heights, or $2dh/d\mathrm{T}$, must be put equal to $2(h_{120} - h_{105})$.

This interpolation would be especially valuable in the case of M, N, P, Q, which change abruptly.

Some interpolation of the kind has been done in my example, but I do not reproduce the work.

The Calculation of a Low Water Table (referred to in general rule VII.).

This may be done almost independently of the H. W. table by replacing the ϑ's by θ's and using the rules given in § 12. The calculation may, however, be materially abridged, and I will now go over the several steps of the calculation noting the mode of transition from one case to the other.

The new A, G, F will be distinguished from the old by enclosing the new ones in square parentheses.

Semi-diurnal $[A_2]$, $[G_2]$, $[F_2]$.

These sequences are derivable directly from the old ones by the rule

$$[A_2] = A_2 + \tfrac{1}{9}F_2, \qquad [G_2] = G_2 + \tfrac{1}{9}F_2, \qquad [F_2] = F_2 - \tfrac{1}{9}(A_{,,} + A_s).$$

Diurnal $[A_1]$, $[G_1]$, $[F_1]$.

The rule is here more complex :—

$$\begin{array}{llll} [A_o] = -\{A_o - \tfrac{1}{8}F_o\}, & [G_o] = -\{G_o - \tfrac{1}{35}F_o\}, & [F_o] = -\{F_o + \tfrac{1}{35}A_o\}, \\ [A_,] = A_, + \tfrac{1}{9}F_,, & [G_,] = G_, + \tfrac{1}{32}F_,, & [F_,] = F_, - \tfrac{1}{32}A_,, \\ [A_p] = A_p + \tfrac{1}{10}F_p, & [G_p] = G_p + \tfrac{1}{38}F_p, & [F_p] = F_p - \tfrac{1}{38}A_p; \end{array}$$

and in all these sequences *shift the list of months* in the margin *six places downwards*, so that in the O and K_1 sequences March stands where June stood, and in the P sequence March stands where December stood.

If it be agreed to neglect the terms involving $\frac{1}{8}$, $\frac{1}{35}$, &c., the rule is simply to shift the months and change the signs of the O sequences; but at Aden where the diurnal tide is very large, this would lead to a sensible error.

After the new sequences for O, K_1, P have been found, they are combined to find $[A_1]$, $[G_1]$, $[F_1]$ just as for H. W.

Quater-diurnal

$$[A_4] = -A_4, \qquad [G_4] = -G_4, \qquad [F_4] = -F_4;$$

that is to say, simply change signs.

Semi-annual and Mean Water and Annual Tide.

The old calculation serves again.

Mean Interval and its Parallactic Correction.

Here we subtract 180° from κ_m; thus

$$\begin{array}{rr} \kappa_m - \pi = & 49^\circ \\ \frac{1}{30}(\kappa_m - \pi) = & 1{\cdot}633 \\ \frac{21}{20}\cdot\frac{1}{30^2}(\kappa_m - \pi) = & 57 \\ i = & 1{\cdot}690 \end{array}$$

Parallactic correction $= +\ 0^{m}{\cdot}08\, i = +\ 0^{m}{\cdot}14.$

It may be well to warn the computer that i may be negative, that is to say L. W. may occur on the average earlier than moon's transit.

Parallactic Corrections.

α, α', β, β' are unchanged.

The rules are

$$[{}_m\mathrm{A}] = {}_m\mathrm{A} - \tfrac{1}{17}\,{}_m\mathrm{F}, \qquad [{}_{''}\mathrm{A}] = {}_{''}\mathrm{A} + \tfrac{2}{9}\,{}_{''}\mathrm{F}$$
$$[{}_m\mathrm{G}] = {}_m\mathrm{G} - \tfrac{1}{17}\,{}_m\mathrm{F}, \qquad [{}_{''}\mathrm{G}] = {}_{''}\mathrm{G} + \tfrac{1}{4}\,{}_{''}\mathrm{F}$$
$$[{}_m\mathrm{F}] = {}_m\mathrm{F} + \tfrac{1}{17}\,{}_m\mathrm{A}, \qquad [{}_{''}\mathrm{F}] = {}_{''}\mathrm{F} - \tfrac{1}{4}\,{}_{''}\mathrm{A}$$

We then compute $[z_2]$, $[m_2]$, $[l_2]$ by the same rules as before.

In the diurnal terms compute by the following rules :—

$$[{}_o\mathrm{A}] = -\{{}_o\mathrm{A} - \tfrac{1}{3}\,{}_o\mathrm{F}\}, \qquad [{}_{'}\mathrm{A}] = {}_{'}\mathrm{A} + \tfrac{2}{9}\,{}_{'}\mathrm{F},$$
$$[{}_o\mathrm{G}] = -\{{}_o\mathrm{G} - \tfrac{1}{14}\,{}_o\mathrm{F}\}, \qquad [{}_{'}\mathrm{G}] = {}_{'}\mathrm{G} + \tfrac{1}{16}\,{}_{'}\mathrm{F},$$
$$[{}_o\mathrm{F}] = -\{{}_o\mathrm{F} + \tfrac{1}{14}\,{}_o\mathrm{A}\}, \qquad [{}_{'}\mathrm{F}] = {}_{'}\mathrm{F} - \tfrac{1}{16}\,{}_{'}\mathrm{A},$$

the list of months in the margin being shifted six places downwards, so that March stands where June stood. The values of $[z_1]$, $[m_1]$, $[l_1]$ are then computed by the same rule as before.

Nodal Corrections.

We may, with sufficient accuracy, take a_2, b_2, c_2, d_2, e_2, f_2 to be unchanged. Referring to the instructions for the computation of a_1, b_1, &c., the new rule may be stated thus :—

$[a_1]$ is (ii.) − (i.), $[b_1]$ is (iv.) − (iii.), $[c_1]$ is (vi.) − (v.),
$[d_1]$ is (viii.)−(vii.), $[e_1]$ is (x.)−(ix.), and $[f_1]$ is (xii.) − (xi.),

and the list of months in the margin is pushed down six places, so that March stands where June stood.

The corrections δT, δM, δN remain unchanged.

When the L. W. sequences have been formed the calculation follows the lines of H. W. calculation precisely, save in three respects—first, in the "heights" h is to be subtracted from B_o, and δh is to be then subtracted from $\mathrm{B}_o - h$, instead of the corresponding additions in the H. W. calculation; secondly, the signs of P, Q, S are to be changed as a last step in the calculation of those quantities, in order that the corrections to the heights may be additive instead of subtractive as they would be if we left off exactly as in the case of H. W.; thirdly, after the final table for $\mathfrak{h}'$ has been made, its values must be subtracted from the annual tide.

The reader will easily understand the necessity for these changes when it is remarked that h, δh, $\mathfrak{h}'$ have been estimated as depressions below mean water, whereas B_o and the annual tide are estimated as elevations above the adopted datum; in the result we require, of course, to estimate heights with reference to the datum.

It may be well to warn the computer that $i + I$ may often be negative.

It will be unnecessary to refer henceforth to L. W., since the instructions for H. W. serve also for L. W.

Interpolation.

The sun's longitude increased by 5° is indicated by ⊙, and the months March, April, May, &c., really mean the dates when ⊙ is 0°, 30°, 60°, &c.—that is to say, about the middle of the months. The dates which, on the average, fall the nearest to these times, are given in Table VI.

The 12 columns for any month, headed 0^h, 1^h, 2^h, ... 11^h contain on the left the 12 values of $\mathfrak{J}$, $\mathfrak{H}$, M, N, &c., corresponding to moon's transit at 0^h, 1^h, 2^h, ... 11^h, and they contain on the right the 12 values for moon's transit at 12^h, 13^h, ... 23^h. This applies to the month named at the head of the table. But these values also appertain to the opposite month (*i.e.*, September opposite to March, October to April, and so on) by reading the right hand entries as appertaining to 0^h, 1^h, ... 11^h, and the left to 12^h, 13^h, ... 23^h. The same is true of $\mathfrak{i}$ and $\mathfrak{h}'$ (see *Corrections for reference to moon's transit*), except that here the values change sign in the opposite month ; thus the values of $\mathfrak{i}$ and $\mathfrak{h}'$ which we have computed for March must be taken with the opposite sign when applied for September.

Now it is required to form interpolated tables for every 10° of ⊙, and in all the 18 tables (of which 6 will be originally computed and 12 interpolated) to interpolate for every 20^m of moon's transit.

These interpolations may be done graphically, and I find with millimetre-square paper a convenient scale for ⊙ is 1 mm. to 1°, and for time of moon's transit 15 mm. to 1^h. These will be set out horizontally as abscissæ, and the ordinates will be time in treating $\mathfrak{J}$, and height in treating $\mathfrak{H}$.

A convenient time scale for $\mathfrak{J}$ is 30 mm. to the hour. In the case of $\mathfrak{H}$ the scale must depend on the range of tide at the place—for Aden (with a small range) I have found 50 mm. to the foot convenient.

I will begin with interpolation for ⊙ (*i.e.*, for time of year), and will only refer to $\mathfrak{J}$, since $\mathfrak{H}$ follows the same plan. Write March, April, May, ... January, February, March, at 0, 30, ... 360 mm. along a horizontal line, corresponding to the same number of degrees of ⊙. It may be well to repeat February before the first March, and April after the last March. Set off as an ordinate the left-hand entries from column 0^h for the six months March, April, ... August, and from the right-hand entries of 0^h for the same six months set off ordinates for their opposite months, *e.g.*, the right-hand $\mathfrak{J}$ of 0^h for March, affords $\mathfrak{J}$ of 0^h for September. Through

the tops of these ordinates draw a smooth curve of 0^h.* Proceed similarly to form curves of 1^h, 2^h, &c., twelve in all. If the figures get confused we may have two or more, and confusion may often be avoided by drawing parts of the curve with upward or downward shift, so as to make things clear where a number of curves go through nearly the same point.

We now start a fresh figure with time of moon's transit as horizontal line, and $\mathfrak{J}$ as ordinate corresponding to March (or $\odot = 0°$). These 24 (computed) values of $\mathfrak{J}$, joined by a smooth curve, enable us to read off the values of $\mathfrak{J}$ for $\odot = 0°$ for every 20^m of moon's transit, *i.e.*, on the adopted scale, at every 5 mm. of horizontal space.

We now set off from the previous figure the 24 (interpolated) values of $\mathfrak{J}$ corresponding to $\odot = 10°$, of which the first 12 are found at "March + 10 mm." and the last 12 at "September + 10 mm." These 24 values being joined by a curve, give $\mathfrak{J}$ for $\odot = 10°$, and for every 20^m of moon's transit.

We next set off 24 values of $\mathfrak{J}$ corresponding to $\odot = 20°$, of which the first 12 are found in the preceding figure at "March + 20 mm.," and the last 12 at "September + 20 mm." These are treated the same way.

The next in the series are the computed April ($\odot = 30°$) $\mathfrak{J}$'s which are set off like the March ones, joined by a curve and read off to each 20^m of moon's transit.

We then take 24 (interpolated) values of $\mathfrak{J}$ from "April + 10 mm." and "October + 10 mm.," to give $\mathfrak{J}$ for $\odot = 40°$. The 24 from "April + 20 mm." and "October + 20 mm." afford $\mathfrak{J}$ for $\odot = 50°$. The next is the computed May ($\odot = 60°$) series, and we so pass on through six months, the last in the set being derived from "August + 20 mm." and "February + 20 mm." corresponding to $\odot = 170°$.

If one person reads the numbers from the figure, whilst another writes, the tabulation may be done very rapidly.

* The following rule is probably known, but I do not know where it has been stated, except in a note of my own in the 'Messenger of Mathematics.' I have found it very useful in drawing good curves.

Rule for Graphical Interpolation half-way between Computed Ordinates.

Draw the polygon (A) joining the tops of a number of equidistant ordinates, and draw the two polygons (B) joining the tops of alternate ordinates. Then every ordinate has marked on it an intercept or sagitta where a side of polygon (B) cuts it. On the half-way ordinates next on each side of a sagitta, set off one-fourth of the sagitta from the points where the two sides of polygon (A) cut those half-way ordinates; the set-off is to be in the direction in which the sagitta would shoot if it were an arrow. When all the quarter sagittas are thus set off, every half-way ordinate (except the first or last of the series) has two points marked on it.

The interpolated curve passes half-way between the pairs of marked points, except in the case of the first and last half-way ordinate, when it passes through the single marked points.

This rule is correct to fourth differences, except in the case of the first and last ordinate, when it is correct only to third differences. In the cases in the text the computed values are cyclical, and there are therefore no first or last.

By means of proportional compasses set to 4, the quarter sagittas may be set off rapidly, and the bisection of the pairs of marked points may be made by eye.

The same process is applied for tabulation of the $\mathfrak{M}$'s. We should, in strictness, do the same by M, N, P, Q, R, S, $\mathfrak{i}$, $\mathfrak{h}'$, but it appears unnecessary to work with so much accuracy.

I have done much of the interpolation by simply writing out the computed values of the quantity to be tabulated in a chess-board table with blanks for the interpolated values. If sixteen squares be considered, a computed value will stand at each corner. Then a great many of the interpolated values may be put in by inspection of the march of the quantity in the two directions. In other parts I make a pencil curve, on millimetre-square paper, of four or five adjacent values, and pass a freehand curve through them to fill in the interpolated values; I rub out the curve when used. It must be remembered that close accuracy in these terms would be mere pedantry.

M, N are computed to the decimal of a minute, and the decimal part may be useful for drawing the pencil interpolation curves, but the result should be tabulated only to the nearest minute. Similarly the third decimal in P and Q may be dropped.

The interpolation of $\mathfrak{i}$ and $\mathfrak{h}'$ follows the same plan, but it must be borne in mind that these functions change sign in opposite months, and this consideration is important when we come to interpolate for $\odot$ = August + 10°, and + 20°.

When there is an evanescent tide (as in the case of March, 18^h) the corrections M, N, R become infinite. As a practical solution this is absurd, and the fact is that there may or may not be a H. W. according to the values of $\mathfrak{s}$ and Π. Again, in other parts of this and other lunations there may be no H. W., although the tide-table predicts one. In all such cases there is a long period of four or five hours duration of nearly slack water, and it is accordingly almost a matter of indifference whether or not a small H. W. is predicted. It would necessitate very laborious computations to make correct predictions in these cases, and the result would not be worth the labour. I have adopted, therefore, a makeshift, and have replaced H. W. by the height of water and time when the rate of change of water level is a minimum.

It has been proposed above that P, Q, S shall be computed with a conjectural $\Delta = + 90°$, and this is better than the plan which I actually adopted in my experimental table for Aden, of which a sample is given below.* The practical point to consider in the present instructions is the manner of treatment of M, N, R about the time of evanescence. I propose then that the gap in the values shall be bridged by a conjectural curve, and that the values be only given in round numbers. For example, for March we have the following values of M:—$(16^h) + 9^m{\cdot}3$, $(17^h) + 28^m{\cdot}5$ (18^h) blank $(19^h) + 45^m{\cdot}0$ $(20^h) + 17^m{\cdot}7$, &c. By drawing a curve I conjecture $+ 50^m$ for the missing value at 18^h.

A comparison with the corresponding complete curves for February and April helps us in filling the gap.

* Thus if any one seeks to verify my table, he will not get *exactly* my values for P, Q, S in the neighbourhood of 18^h.

After the complete table for $\mathfrak{h}'$ is formed we proceed to add to it the values computed in the table of the annual tide for every 10° of ☉, and so form a table of $\mathfrak{h}$. For example, the first five values of $\mathfrak{h}'$ are — ·02*, — ·01, — ·01, — ·01*, ·00 (of which those marked * are computed), and to these we add ·390, the computed annual tide for March, and obtain ·37, ·38, ·38, ·38, ·39.

The final results are then arranged in a table.* If the L.W. were also computed I should propose that the L.W. and H.W. should be given alternately. The following is a sample of the table computed only for H.W. at Aden :—

PORT of Aden; High Water Tide-Table.

Times of moon's transit for—		For March 15th (*i.e.*, from sun's long. 350° to 0°), and for September 17th (*i.e.*, from sun's long. 170° to 180°). The upper signs of $\mathfrak{i}$ and $\mathfrak{h}$ apply to March, the lower to September.								
March.	Sept.	$\mathfrak{J}$.	$\mathfrak{i}$.	$\mathfrak{H}$.	$\mathfrak{h}$.	M.	N.	R.	P.	Q.	S.
h. m.	h. m.	h. m.	m.	ft.	ft.	m.	m.	m.	ft.	ft.	ft.
0 0	12 0	8 5	± 4	6·49	± ·37	0	+ 9	— 1·3	+ ·02	— ·08	+ ·14
20	20	7 55	± 4	·54	± ·38	0	+ 9	— 1·0	+ ·03	— ·08	·14
40	40	46	± 4	·57	± ·38	— 1	+ 9	— 0·6	+ ·04	— ·09	·14
1 0	13 0	7 37	± 4	6·59	± ·38	— 1	+ 9	— 0·2	+ ·05	— ·09	·14
20	20	28	± 4	·60	± ·39	— 2	+ 9	+ 0·1	+ ·06	— ·09	·14
40	40	18	± 4	·60	± ·39	— 2	+ 9	+ 0·4	+ ·07	— ·09	·15
&c.		&c.		&c.		&c.					
17 0	5 0	8 29	∓ 14	4·21	± ·37	+ 28	— 11	+ 2·3	— ·16	— ·01	·09
20	20	9 52	∓ 15	·21	± ·36	+ 35	— 2	— 7·3	— ·10	— ·03	·10
40	40	11 2	∓ 14	·27	± ·35	+ 43	+ 20	— 16	— ·05	— ·05	·10
18 0	6 0	11 17	∓ 10	4·38	± ·35	+ 50	+ 30	— 20	·00	— ·06	·09
20	20	15	∓ 6	·53	± ·34	+ 54	+ 25	— 30	+ ·05	— ·05	·08
40	40	7	∓ 1	·71	± ·33	+ 50	+ 20	— 30	+ ·10	— ·04	·06
&c.		&c.		&c.		&c.					

If ϖ be the moon's parallax at moon's transit in minutes of arc, and ☊ be the longitude of moon's node, the interval is

$$\mathfrak{J} + \mathfrak{i} + M \cos ☊ + N \sin ☊ + (\varpi - 57') R,$$

and the height is

$$\mathfrak{H} + \mathfrak{h} + P \cos ☊ + Q \sin ☊ + (\varpi - 57') S.$$

After the table has been completed the computer should test the correctness of the prediction by computing two or three tides in each month, and comparing the results either with the observations from which the harmonic constants were originally derived, or with other known values of high and low water.

* I have found that it is convenient to cut the constituent tables into strips and paste them together again, so as to save much copying and verification.

Examples of the Use of the Table.

From and after the year 1887, the datum for the tide-tables of the Indian Government for Aden has been 0·37 ft. lower than that used in my table; as I am going to compare my results for 1889 with those of the Indian Government, 0·37 ft. will be added to my heights to make the two comparable.

(A) The moon crossed the meridian at Aden on March 17th, 1889, at $0^h\ 11^m$, Aden M. T. Aden is in $3^h\ 0^m$ E. long., and therefore this is about 21^h, March 16th, G. M. T.; whence from the 'Nautical Almanac' we find ϖ the moon's parallax at Aden transit was 58′·2, and $\varpi - 57' = +1{\cdot}2$. The longitude of moon's node was 108°, and $\cos\Omega = -{\cdot}3$, $\sin\Omega = +1{\cdot}0$.

Then referring to our table and interpolating between $0^h\ 0^m$ and $0^h\ 20^m$, and taking the upper signs of $\mathfrak{i}$ and $\mathfrak{h}$, we find

$$\mathfrak{J} + \mathfrak{i} = 8^h\ 4^m, \quad \mathfrak{H} + \mathfrak{h} = 6{\cdot}89 \text{ ft.}, \quad M = 0, \quad N = +9^m, \quad R = -1^m{\cdot}2,$$
$$P = +{\cdot}03 \text{ ft.}, \quad Q = -{\cdot}08 \text{ ft.}, \quad S = +{\cdot}14 \text{ ft.}$$

Hence

$$M\cos\Omega + N\sin\Omega + R(\varpi - 57) = +8^m,$$

and

$$P\cos\Omega + Q\sin\Omega + S(\varpi - 57) + 0{\cdot}37 = +0{\cdot}45 \text{ ft.}$$

Therefore the interval is $8^h\ 12^m$, and the time of H. W. $8^h\ 12^m + 0^h\ 11^m$ or $8^h\ 23^m$ p.m., March 17th; and the height is 7·34 ft. or 7 ft. 4 in.

The Indian tide-table gives as time $8^h\ 12^m$ p.m., and as height 7 ft. 4 in.

(B) On September 17th, 1889, the moon crossed the meridian at $18^h\ 36^m$, ϖ was 54′·2 or $\varpi - 57' = -2{\cdot}8$, $\Omega = 98°$, $\cos\Omega = -{\cdot}14$, $\sin\Omega = +1{\cdot}0$.

Interpolating in our table between $18^h\ 20^m$ and $18^h\ 40^m$, and taking the lower signs of $\mathfrak{i}$ and $\mathfrak{h}$, we find

$$\mathfrak{J} - \mathfrak{i} = 11^h\ 22^m, \quad \mathfrak{H} - \mathfrak{h} = 4{\cdot}34, \quad M = +50^m, \quad N = +20^m, \quad R = -30^m,$$
$$P = +{\cdot}10, \quad Q = -{\cdot}04, \quad S = +{\cdot}06.$$

Hence

$$M\cos\Omega + N\sin\Omega + R(\varpi - 57) = +97^m = +1^h\ 37^m,$$

and

$$P\cos\Omega + Q\sin\Omega + S(\varpi - 57) + 0{\cdot}37 = +0{\cdot}15 \text{ ft.}$$

Therefore the interval is $12^h\ 59^m$, and the time of H. W. $12^h\ 59^m + 18^h\ 36^m$ or $31^h\ 35^m$ September 17th, *i.e.*, $7^h\ 35^m$ p.m. September 18th, and the height is 4·49 or 4 ft. 6 in.

The Indian tide-table gives "no inferior H. W."

This example shows our table at its worst, for it is clear that a nominally small correction to the time which amounts to an hour and forty minutes must give unsatisfactory results. At this time of year mean water has a height of 3 ft. 10 in.,

hence our prediction only shows a rise of 8 inches. Although there was probably in reality no maximum (as predicted in the Indian table), I should expect that the water stood at about 4 ft. 6 in. at half-past seven of the evening of September 18th, 1889.

Part III.—Comparison and Discussion.

Comparison.

As stated in the Introduction, Mr. Allnutt computed a complete H. W. tide-table for 1889 for Aden in order to compare the results with the Indian tide-tables, which are made with the tide-predicting instrument. When this comparison was made our tables had not been brought into exactly the form given above, and Mr. Allnutt's work was considerably more laborious than it would have been if undertaken later.

The mechanical predictions were probably made with constants which are the means of the results derived from eight years of observation, whereas the constants used in our tables are derived from only four years. Mr. Roberts has supplied me with six weeks of prediction for the year 1887, worked mechanically in duplicate, namely, with the eight-year and the four-year constants. In the latter case the times of H. W. seem to run about 6^m later than in the former, but the difference often rises to 10^m, occasionally to 15^m, and at rare intervals to 20^m. The two sets of constants also give a systematic difference in the heights amounting to about 2 inches, but the difference often rises to 3 inches or falls to 1 inch, and occasionally reaches 4 inches and zero. It follows, therefore, that a sensible part of the discrepancy, or error as it may be called, of our computation is due to the difference of constants. But not nearly all of the "error" can be set down to this cause; it is due to a combination of causes, namely, flaws in the interpolation, imperfect representation of the elliptic tides, and partial inclusion in our method of all the lunar inequalities which are totally neglected in the machine. The principal cause of "error," however, is the imperfect correction for longitude of moon's node and parallax about the time in each lunation when there is partial or total evanescence of the inferior H. W.

Omitting, as we must do, the cases of evanescence, there were 689 H. W. computed by Mr. Allnutt, and he finds "the probable errors" in the time and height to be 9^m and 1·2 inches.

In the course of the year there were thirty-two occasions on which the time "error" amounted to 30^m or more. All of these occur in the inferior H. W. at the times of approximate evanescence, where the nodal and parallactic corrections are large. At these times there is always a period of four or five hours of nearly slack water, and the time at which a small maximum occurs is of no importance from a navigational point of view. If these thirty-two cases be taken away the probable error falls to 7^m A cursory inspection of the table shows also that nearly all the "errors" of 25^m to 30^m fall about the time of evanescence and are therefore unimportant. We are

accordingly justified in saying that our predictions do not differ materially from the Indian tables.

It has been already mentioned that I have six weeks of mechanical prediction for 1887, made with the identical constants used in our table. I have, therefore, taken a month, or 58 H. W., out of these six weeks, and compared them with my predictions, made without cross interpolation for date. I find that the errors of time are 27 from 0^m to 5^m, 13 from 5^m to 10^m, 10 from 10^m to 15^m, 4 from 15^m to 20^m, one error of 26^m, and one of 34^m. These give a probable error of 7^m. All the large errors fall on the inferior H. W., at the time when it is very small, and they are, therefore, practically unimportant.

In the heights there are 16 cases of agreement, 22 errors of 1 in., 15 of 2 in., and 5 of 3 in. These give a probable error of 1·0 inch.

The errors of 3 inches all fall about the time when the moon's parallax was small, and I also observe that this was commonly a time when the height errors rose to their greatest in 1889. This is probably due to the imperfect representation of the elliptic tides, which, as shown in § 3, is inherent to our method.

The concordance between the two is good enough, but less perfect in the heights than I expected.

The last comparison is between our predictions and actuality. The observed times and heights for part of 1884 have been furnished me by Colonel Hill, R.E., by the direction of Colonel Strahan, R.E., Deputy Surveyor-General in India. For the purpose of testing the present method, I have computed a H. W. tide-table from 10th March to 9th April, and again from 12th November to 12th December, 1884. In these periods, there are 117 actual and one evanescent H. W. The observation of one H. W. is missing through an accident to the tide-gauge; there are, therefore, 117 H. W. for the purpose of comparison.*

The following is a table of errors, regardless of signs :—

Time.		Height.	
Magnitude of error.	No. of H. W.	Magnitude of error.	No. of H. W.
m. m.		Inches.	
0 to 5	35	0	15
5 „ 10	32	1	48
10 „ 15	19	2	28
15 „ 20	19	3	14
20 „ 25	5	4	11
26 , 28	2		
33 , 36	2		
56 , 57	2		
Evanescence	1	Evanescence	1
	117		117

* [I originally compared my computations with some results sent me some years ago by Colonel Baird, R.E., but as it appeared that these estimates of actuality had probably been biassed in favour of the Indian predictions, I asked to have the tide-curves re-measured.—March 21, 1891.]

Omitting the case of evanescence and assuming the errors to conform in distribution to the normal exponential law (which is not accurately the case), the probable error is about 9^m in time, and 1·4 inch in height.

When the rise from the higher L. W. to the lower H. W. is *nil* there is evanescence, and when that rise is small there is approximate evanescence. I have accordingly examined the 12 cases in which the error of time is equal to or greater than 20^m, and the following table gives the result.

Rise from L. W. to H. W.	Time errors.
Nil.	. . .
6^{in} to 8^{in}	22, 26, 28, 56, 57 minutes
13^{in}	36 minutes
17^{in}	22 ,,
19^{in}	33 ,,
$2^{ft}\ 10^{in}$	22 ,,
$3^{ft}\ 9^{in}$	23 ,,
$3^{ft}\ 11^{in}$	20 ,,

It appears that where the errors of time were 56^m and 57^m the tide was very nearly evanescent, and that the two other considerable errors of time, viz., 36^m and 33^m, pertain to very small tides.

It has been already pointed out that in such cases a considerable error in the time is of no importance, and it is justifiable, in testing the calculations, to set aside the nine tides in which the rise is less or equal to 19 inches.

There remain 108 H. W., and the greatest error in the times amounts to only 23 minutes. In 58 cases the error is 7 minutes or less, and in 51 cases it is 6 minutes or less; as the half of 108 is 54, it follows that the probable error is a little over 6 minutes.

The Indian predictions maintain their standard of excellence fairly well through the periods of approximate evanescence, but out of 116 tides there are 59 cases with time errors of 10 minutes or less; as the half of 116 is 58, the probable error is about 10 minutes.

Turning now to the heights I find that both mine and the Indian predictions present 63 out of 116 H. W. with zero errors or errors of 1 inch. We may take it then that the probable error for both modes of prediction is about 1 inch. The Indian predictions have, however, the disadvantage that several errors of 5, 6, 7 inches occur. On the other hand, the 11 cases of 4 inches of error which occur in mine have a systematic character; they are all positive (actuality the greater), and all but one affect the higher H. W. about the time when the moon's parallax is small. This defect is doubtless due to the imperfect representation of the elliptic, evectional, and variational tides inherent to my method.

The slight superiority shown over the mechanical prediction must be attributed to the fact that I have used better values of the tidal constants than were available in 1883, when the Indian predictions must have been made.

I learn from Colonel HILL that two independent observers reading the same tide-curve will frequently differ by 5^m and sometimes by 10^m in their estimate of the time, and by 1 and sometimes by 2 inches in the height. Accordingly, predictions which agree with a reading of a tide-curve with probable errors of $6\frac{1}{2}^m$ in time and 1 inch in height may claim to possess a high order of accuracy.

I conclude from the preceding discussion that with good values of the tidal constants the present method leads to excellent predictions, and that they are even better than are required for nautical purposes.

Discussion.

It is probable that methods may be invented by which some abridgement of the computations may be made, but I am, of course, unable to suggest such improvements.

The last-mentioned comparison seemed to show that but little accuracy would be lost if P, Q, were entirely omitted, and if $\mathfrak{h}'$ were treated as zero, so that $\mathfrak{h}$ would consist simply of the annual tide. Indeed, the only advantage gained by the retention of P, Q, $\mathfrak{h}'$ appeared to be the avoidance of a few considerable errors in the inferior H. W. about the time of approximate evanescence. Experience must decide whether the computation and the tables may be lightened by the omission of these quantities.

The advantage gained from M and N is marked, but as these quantities arise almost entirely from the diurnal tides, I am inclined to think that, at places where the diurnal tide is not extremely large, a very fair tide-table might be made without them.

The present method will probably be applied to ports of second rate importance, where there are not sufficient data for very accurate determination of the tidal constants. In such cases it will be best to omit the computation of P, Q, $\mathfrak{h}'$, and to postpone that of M, N, and perhaps also of R and $\mathfrak{i}$, until the simple tide-table has been tested as to its adequacy for navigational purposes. At most places the annual tide is so large that $\mathfrak{h}$ cannot be omitted, and it is impossible to dispense with the value of S. But it is possible that it might suffice to attribute to S a constant value,* although this would certainly cause very perceptible error in the heights of the lower H. W. and higher L. W. A tide-table which only gave $\mathfrak{J}$, $\mathfrak{H}$, $\mathfrak{h}$, and a constant S would be fairly short, even if computed for every ten days in the year; and this would be a great gain.

The question of how far to go in each case must depend on a variety of circum-

* I may suppose the elliptic tides unknown, and I should then take $S = \frac{1}{19}H_m + \cdot 036H_{//}$. For Aden this would give $S = \cdot 129$, or say $\frac{1}{8}$.

stances. The most important consideration is, I fear, likely to be the amount of money which can be spent on computation and printing; and after this will come the trustworthiness of the tidal constants and the degree of desirability of an accurate tide-table.

My aim has been to reduce the tables to a simple form, and if, as I imagine, the mathematical capacity of an ordinary ship's captain will suffice for the use of the tables, whether in full or abridged, I have attained the principal object in view.

VI. *On Ellipsoidal Harmonics.*

By W. D. Niven, *F.R.S.*

Received October 23,—Read December 11, 1890.

1. The theory of Ellipsoidal Harmonics is usually based upon solutions of Laplace's Equation suitably expressed in terms of ellipsoidal coordinates as independent variables. This manner of treating the subject, introduced by Lamé, has received its most complete exposition at the hands of Heine in his 'Kugelfunctionen' and is probably the most direct and effective for all practical purposes. The Cartesian forms of the harmonics seem, however, to possess many advantages in point of clearness and suggestiveness. I have therefore been led by a perusal of Green's Memoir on Ellipsoids of varying densities and by Thomson and Tait's investigation of Spherical Harmonics to attempt the development of the subject by Cartesian processes.

In dealing with the general case of the ellipsoid of three unequal axes it is convenient to consider, in the first place (§§ 2–15), the propositions and theorems which relate to the forms applicable to the inside, and afterwards (§§ 16–21) those applicable to the outside of the ellipsoid. The main feature of my investigation of internal harmonics is the establishment of relations between the harmonics of the ellipsoid and certain harmonics of the sphere which have a close correspondence with them. By means of these relations it is possible, in the first place, to obtain readily the Cartesian forms of ellipsoidal harmonics (§ 15), and, in the next place, to determine an expansion, in ellipsoidal harmonics, having arbitrarily assigned values at the surface of the ellipsoid (§§ 8–10). As a particular case, I have taken the arbitrarily assigned value to be a homogeneous function of the coordinates x, y, z (§ 11), and it is an easy transition from this case to that of any function capable of expansion by Taylor's theorem in ascending powers of x, y, z (§§ 12, 13). I have also expanded the reciprocal of the distance between two points, one of which is on the surface (§ 14).

The leading proposition in External Harmonics, upon which much of the work of this paper is based, relates to the expression of those harmonics in terms of differential operations upon the potential at an external point due to an ellipsoid of variable density. The proposition referred to, while possessing some peculiarities and difficulties of its own, is obtained in a manner analogous to that employed by Clerk Maxwell in finding from the Theory of Attractions a physical interpretation of a Spherical Harmonic (§§ 17–20).

As illustrations of the use of the harmonics pertaining to an ellipsoid of three

unequal axes I have considered briefly the problem of the magnetism induced in a body of ellipsoidal form (§§ 22–25). In §§ 25–27 the potential has been found at an external point, due to a thin shell, bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point, while § 28 contains a proposition in electrical capacity.

In §§ 30–52 I have investigated at some length various inside and outside forms of harmonics applicable to prolate and oblate spheroids, deducing them from the more general expressions of the earlier parts of the paper. The corresponding spherical harmonics, to which reference has already been made, reduce in the case of spheroids to the ordinary canonical conjugate system (§§ 30–32). From this system the next step is to investigate the forms of the internal harmonics of prolate spheroids (§§ 34–36), and the expressions for the external harmonics are then discussed in §§ 37–42. The corresponding forms for oblate spheroids are stated in §§ 43–49. In § 50 I have shown how the harmonics of the prolate spheroid reduce to those of the circular cylinder, and in § 51 to those of the paraboloid of revolution. The remainder of the paper is occupied with proofs of additional theorems for ellipsoidal harmonics, spherical harmonics of the second kind, and BESSEL's functions.

Notation and Preliminary Investigations.

2. The notation employed by Dr. FERRERS in Chapter VI. of his 'Treatise on Spherical Harmonics,' besides being well known, seems so suitable that it will be convenient to adopt it as far as possible in the following investigations.

The semi-axes of the ellipsoid of reference being a, b, c, the ellipsoidal coordinates of any point xyz will be denoted by $\epsilon\nu\nu'$, so that

$$x^2 = \frac{(a^2+\epsilon)\,(a^2+\nu)\,(a^2+\nu')}{(a^2-b^2)\,(a^2-c^2)},$$

with similar expressions for y^2 and z^2.

The following quantities and operators will be of so frequent occurrence that special abbreviations are necessary.

$\dfrac{x^2}{a^2+\theta_r}+\dfrac{y^2}{b^2+\theta_r}+\dfrac{z^2}{c^2+\theta_r}$ will be denoted by K_{θ_r} or K_r.

K_r-1 . . . . . ,, ,, Θ_r.

$a^2\dfrac{\partial^2}{\partial x^2}+b^2\dfrac{\partial^2}{\partial y^2}+c^2\dfrac{\partial^2}{\partial z^2}$. ,, ,, D^2.

$\dfrac{1}{a^2+\theta_r}\dfrac{\partial^2}{\partial x^2}+\dfrac{1}{b^2+\theta_r}\dfrac{\partial^2}{\partial y^2}+\dfrac{1}{c^2+\theta_r}\dfrac{\partial^2}{\partial z^2}$,, $\mathrm{E}_{\theta_r}{}^2$ or $\mathrm{E}_r{}^2$.

$\dfrac{a^2}{{}^2+\theta_r}\dfrac{\partial^2}{\partial x^2}+\dfrac{b^2}{b^2+\theta_r}\dfrac{\partial^2}{\partial y^2}+\dfrac{c^2}{c^2+\theta_r}\dfrac{\partial^2}{\partial z^2}$,, $\mathfrak{E}_r{}^2$.

3. All the types of interior harmonics subject to certain restrictions as to the values of θ, which will be presently investigated, are included under the following general scheme :—

$$\left\{\begin{array}{cccc} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{array}\right\} \Theta_1\Theta_2 \ldots \Theta_n \quad \ldots\ldots\ldots \quad (1),$$

where any of the quantities inside the bracket may be taken as a multiplier of the product outside. The first column gives a harmonic of degree $2n$, the second $2n+1$, the third $2n+2$, the fourth $2n+3$. Beginning with simple cases we see that Θ will satisfy LAPLACE'S equation provided

$$\frac{1}{a^2+\theta} + \frac{1}{b^2+\theta} + \frac{1}{c^2+\theta} = 0.$$

Again, the function $\Theta_1\Theta_2$, when entered in LAPLACE'S equation, produces

$$\left(\frac{1}{a^2+\theta_1} + \frac{1}{b^2+\theta_1} + \frac{1}{c^2+\theta_1}\right)\Theta_2 + \left(\frac{1}{a^2+\theta_2} + \frac{1}{b^2+\theta_2} + \frac{1}{c^2+\theta_2}\right)\Theta_1 + 4\left\{\frac{x^2}{(a^2+\theta_1)(a^2+\theta_2)} + \frac{y^2}{(b^2+\theta_1)(b^2+\theta_2)} + \frac{z^2}{(c^2+\theta_1)(c^2+\theta_2)}\right\} = 0.$$

Observing that the last line may be thrown into the form

$$4\frac{\Theta_2-\Theta_1}{\theta_1-\theta_2},$$

we see that, when LAPLACE'S equation is satisfied by the product $\Theta_1\Theta_2$, we must have

$$\left.\begin{aligned} \psi_1 \equiv \frac{1}{a^2+\theta_1} + \frac{1}{b^2+\theta_1} + \frac{1}{c^2+\theta_1} + \frac{4}{\theta_1-\theta_2} = 0 \\ \psi_2 \equiv \frac{1}{a^2+\theta_2} + \frac{1}{b^2+\theta_2} + \frac{1}{c^2+\theta_2} + \frac{4}{\theta_2-\theta_1} = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (2).$$

The equations (2) we may call the characteristic equations of $\Theta_1\Theta_2$.

In like manner it can be proved that, if LAPLACE'S equation is satisfied by $\Theta_1\Theta_2 \ldots \Theta_n$, we must have

$$\left.\begin{aligned} \psi_1 \equiv \frac{1}{a^2+\theta_1} + \frac{1}{b^2+\theta_1} + \frac{1}{c^2+\theta_1} + \frac{4}{\theta_1-\theta_2} + \ldots + \frac{4}{\theta_1-\theta_n} = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \psi_n \equiv \frac{1}{a^2+\theta_n} + \frac{1}{b^2+\theta_n} + \frac{1}{c^2+\theta_n} + \frac{4}{\theta_n-\theta_1} + \ldots + \frac{4}{\theta_n-\theta_{n-1}} = 0 \end{aligned}\right\} \quad . \quad (3).$$

The characteristic equations corresponding to $x\Theta_1 \ldots \Theta_n$ are the same as the equations $\psi_1, \ldots, \psi_n$ just found, with the exception of the terms in the first column which have to be multiplied by 3; *i.e.*, for all values of the suffix we have $3/(a^2+\theta)$ in place of $1/(a^2+\theta)$. Similar changes take place in the second and third columns if y and z are taken in place of x. Again, if the function be $yz\,\Theta_1 \ldots \Theta_n$, the characteristic equations are the same as in (3) with the exception of the second and third columns, which have each to be multiplied by 3; *i.e.*, we must have, for all values of the suffix, $3/(b^2+\theta) + 3/(c^2+\theta)$ instead of $1/(b^2+\theta) + 1/(c^2+\theta)$. Similar changes have to be made when the factors are zx and xy. Finally, in the case of $xyz\,\Theta_1 \ldots \Theta_n$, the first three columns are each to be multiplied by 3.

4. It should be remarked at this point that the system of expressions given by the scheme

$$\left\{\begin{array}{cccc} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{array}\right\} \mathrm{K}_1 \ldots \mathrm{K}_n \quad \ldots\ldots\ldots \quad (4)$$

are spherical harmonics for precisely the same values of θ which make the corresponding Θ expressions ellipsoidal harmonics.

The harmonics in the schemes (1) and (4) may be said to correspond when they involve the same values of θ.

The designation *type* will be used to denote any particular member of the group (1) or (4). The *degree* of a harmonic is the degree of its highest terms in x, y, z. Since there are $2n+1$ independent harmonics of degree n (§ 5 below), if we suppose them arranged according to any definite system, the number indicating the position of any particular harmonic is its *order*. If, following Clerk Maxwell, we denote a solid spherical harmonic of degree n and order σ by H_n^{σ} and suppose the type to be according to the scheme (4), the corresponding ellipsoidal harmonic or Green's harmonic of scheme (1) may be denoted by G_n^{σ}.

The outside harmonic corresponding to G_n^{σ} may be denoted by $\mathfrak{G}_n^{\sigma}$ so that at any external point xyz, $\mathfrak{G}_n^{\sigma}(xyz) = \mathrm{G}_n^{\sigma}\,\mathrm{I}_n^{\sigma}(xyz)$ where I_n^{σ} is an integral of the form

$$\int_{\epsilon}^{\infty} \frac{d\lambda}{(\theta_1-\lambda)^2(\theta_2-\lambda)^2 \ldots \mathrm{L}}$$

in which θ_1, $\theta_2, \ldots$ are the characteristics of G_n^{σ} and L is an abbreviation for $\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}$.

At the surface of the ellipsoid we may write $\mathrm{I}_n^{\sigma}(0)$ as the value of the integral.

When $n=0$ the integral $\int_{\epsilon}^{\infty} d\lambda/\mathrm{L}$ (Ferrers's α) may be denoted by $\mathrm{I}_0(xyz)$, and at the surface of the ellipsoid by $\mathrm{I}_0(0)$.

5. In the equations (3) we observe that the first equation $\psi_1 = 0$ is of degree $n+1$ in θ_1 and of degree 1 in all the other quantities $\theta_2 \ldots \theta_n$. The same is true of

the second in regard to θ_2 and so on. Hence all these quantities satisfy an equation whose degree is $n\ (n+1)$. But since the equations are such that if $n-1$ of the θ's be given the n^{th} is uniquely determined from them by each of $n-1$ equations, the roots must form $n+1$ groups of what we shall presently be enabled to describe as n conjugates. Each group gives rise to an ellipsoidal harmonic of the type $\Theta_1 \ldots \Theta_n$ so that there are, on the whole, $n+1$ independent varieties of this particular type. Since the introduction in one or more of the first three columns of equations (3) of the factor 3 does not affect this reasoning there will be $n+1$ distinct harmonics of any one of the types given by the general scheme (1).

6. Let $f(\theta)=0$ be an equation of the n^{th} degree, satisfied by all the characteristics in (3) § 3, for any particular harmonic; then by the properties of roots the first equation may be thrown into the form

$$\frac{1}{a^2+\theta_1}+\frac{1}{b^2+\theta_1}+\frac{1}{c^2+\theta_1}+2\frac{d^2f(\theta_1)}{d\theta_1^2}\Big/\frac{df(\theta_1)}{d\theta_1}=0\,,$$

or, if the suffix be dropped, all the roots satisfy

$$\tfrac{1}{2}\{(b^2+\theta)(c^2+\theta)+\ldots\}\frac{df}{d\theta}+(a^2+\theta)(b^2+\theta)(c^2+\theta)\frac{d^2f}{d\theta^2}=0\,,$$

an equation whose degree is higher by 1 than $f(\theta)=0$. The left-hand side must therefore be identically the same as

$$\{n(n+\tfrac{1}{2})\theta+r\}f(\theta)$$

where r is a constant to be determined.

The characteristics belonging to other types may be dealt with in a similar manner.

It is worthy of remark, as connecting the method of this paper with that expounded by FERRERS, that the equation just found may be thrown into the form

$$\sqrt{\{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}}\,\frac{d}{d\theta}\sqrt{\{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}}\,\frac{d}{d\theta}f=\{n(n+\tfrac{1}{2})\theta+r\}f,$$

and that all the equations for the other types may also be transformed into the same form. For instance, take the type $yz\,\Theta_1 \ldots \Theta_n$, and let $F(\theta)=0$ be the equation of its characteristics. Put $\sqrt{\{(b^2+\theta)(c^2+\theta)\}}\,F(\theta)=f(\theta)$; then the differential equation for $f(\theta)$ will be of the same type as the above, the equation to determine r being however different and $n+1$ taking the place of n. (See FERRERS, VI., 6, 7, 8.)

Further consideration of the roots of the characteristic equation will most appropriately be taken when their positions and values have to be determined for ellipsoids of revolution.

Conjugacy of the K-*functions.*

7. All the ellipsoidal harmonics of the same degree are conjugates, in the sense that if G, G′ be any two of them, the integral $\iint GG'pd\mathrm{S}$ is 0, where p is the perpendicular from the centre to the tangent plane of the element $\delta\mathrm{S}$, and the integrations are taken over the ellipsoid. This has been proved by Dr. Ferrers in his Treatise VI., 14, and we shall deduce from it that $\iint \mathrm{HH}'\,ds = 0$ where the integrations are taken over a concentric sphere.

For instance, let $\mathrm{G} = \Theta_1 \ldots \Theta_n$ and $\mathrm{G}' = \Phi_1 \ldots \Phi_n$; then, remembering that at the surface any factor Θ may be put equal to

$$-\theta\left\{\frac{x^2}{a^2(a^2+\theta)}+\frac{y^2}{b^2(b^2+\theta)}+\frac{z^2}{c^2(c^2+\theta)}\right\},$$

or,

$$-\theta\left(\frac{x_1^2}{a^2+\theta}+\frac{y_1^2}{b^2+\theta}+\frac{z_1^2}{c^2+\theta}\right),$$

where $x_1y_1z_1$ on a sphere of unit radius corresponds to xyz on the ellipsoid, we may put

$$\mathrm{G}(x,y,z) = (-1)^n\,\theta_1 \ldots \theta_n \mathrm{H}\,(x_1, y_1, z_1);$$
$$\mathrm{G}'(x,y,z) = (-1)^n\,\phi_1 \ldots \phi_n \mathrm{H}'(x_1, y_1, z_1);$$
$$pd\mathrm{S} = abc\,ds.$$

Hence, $\iint \mathrm{HH}'\,ds = 0$, the integrations being taken over the surface of the sphere.

This proves that the functions H, H′, corresponding to G, G′, are conjugate spherical harmonics, and the same proposition may readily be established for any two such functions whatever be their types.

The system of conjugates thus found, though possessing great generality on account of the endless variety of values which may be assigned to a, b, c, do not appear to take simple forms, except when two of the quantities, a, b, c, are equal, say $a = b$, in which case they reduce, as we shall afterwards see, to the canonical spherical forms. We can show that the poles, in Clerk Maxwell's sense, of any one of the systems lie in one or other of the principal planes of the ellipsoid. Take the harmonic $\mathrm{K}_1 \ldots \mathrm{K}_n$ for example. It can readily be proved that, if $f(x, y, z)$ be a spherical harmonic of degree n, then

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = (-1)^n \frac{(2n)!}{2^n\, n!}\frac{f(x,y,z)}{r^{2n+1}}.$$

Hence

$$K_1 \ldots K_n = \frac{2^{2n}(2n)!}{(4n)!} r^{4n+1} \Pi \left(\frac{1}{a^2+\theta} \frac{\partial^2}{\partial x^2} + \ldots \right) \frac{1}{r}.$$

Now the n values of θ lie either between $-a^2$ and $-b^2$, or between $-b^2$ and $-c^2$ (see § 33 below). Suppose there are σ of the former and $n-\sigma$ of the latter. Then for the former values any single factor of the operator, in virtue of the relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{dz^2} \right) \frac{1}{r}$$

being zero, becomes

$$-\frac{1}{c^2+\theta} \left\{ \frac{a^2-c^2}{a^2+\theta} \frac{\partial^2}{\partial x^2} + \frac{b^2-c^2}{b^2+\theta} \frac{\partial^2}{\partial y^2} \right\},$$

and for the latter

$$\frac{1}{a^2+\theta} \left\{ \frac{a^2-b^2}{b^2+\theta} \frac{\partial^2}{\partial y^2} + \frac{a^2-c^2}{c^2+\theta} \frac{\partial^2}{\partial z^2} \right\}.$$

Both of these operators can be put into real factors : there are, therefore, 2σ poles in the plane of xy, and $2n-2\sigma$ in the plane of yz.

Expansions having assigned Values at the surface of the Ellipsoid.

8. Let $v \equiv f(x, y, z)$ represent an assigned value at the surface of the ellipsoid, it is required to find a series of ellipsoidal harmonics which shall agree with it at the surface.

We may put $v = f(ax_1, by_1, cz_1)$, and an expansion in spherical harmonics can at once be found by LAPLACE's well-known formula, viz. :—

$$4\pi v = \iint v\, ds + \ldots + (2n+1) \iint v\, Q_n\, ds + \ldots$$

where Q_n is a zonal harmonic, whose pole is at $x_1 y_1 z_1$, and the integrations are taken over the sphere. This expansion has the value v at the surface of the sphere.

Since, according to the preceding article, the harmonics H_n^{σ} form a system of conjugates, Q_n may be expressed, as we shall presently see, in $2n+1$ terms of the form $A_n^{\sigma} H_n^{\sigma}(X, Y, Z) H_n^{\sigma}(x_1, y_1, z_1)$ where A_n^{σ} is a function of n and σ, and σ may have $(2n+1)$ different values. The point XYZ is the centre of the element ds.

Hence

$$4\pi v = \iint v\, ds + \ldots + (2n+1) \Sigma A_n^{\sigma} H_n^{\sigma}(x_1, y_1, z_1) \iint H_n^{\sigma}(X, Y, Z)\, v\, ds + \ldots$$

Or, if we express the series in terms of ellipsoidal harmonics and integrations over the ellipsoid

$$4\pi abc\, v = \iint vp\, d\mathrm{S} + \ldots + (2n+1)\,\Sigma\, \mathrm{B}_n^{\sigma}\, \mathrm{G}_n^{\sigma}(x, y, z) \iint \mathrm{G}_n^{\sigma}\, vp\, d\mathrm{S} + \ldots$$

In this expansion, xyz are the coordinates of the point at which it is required to express the value v in harmonics, and G_n^{σ}, v, p under the integral, are functions of the coordinates of the element dS.

The quantities A_n^{σ} and B_n^{σ}, which are seen to be connected by the equation $\mathrm{A}_n^{\sigma} = \{\Pi(\theta)\}^2 \mathrm{B}_n^{\sigma}$, will now be determined.

Expansion of Q_n *in* K-*Functions.*

9. Clerk Maxwell, in the chapter on "Spherical Harmonics" in the 'Electricity and Magnetism,' 2nd edition, vol. 1, p. 186, has introduced a theorem of great generality relating to any two spherical harmonics Y_n, Y_n' of the same degree n, viz., that

$$\iint \mathrm{Y}_n \mathrm{Y}_n'\, ds = \frac{4\pi}{2n+1} \frac{1}{n!} \frac{d^n}{dh_1 \ldots dh_n} \left(\mathrm{Y}_n' r^n \right)$$

where the integrations are taken over a sphere of radius unity, and the differentiations correspond to the n poles of the harmonic Y_n.

The result may be more conveniently stated thus: let $f(x, y, z)$ and $\mathrm{F}(x, y, z)$ be any two solid spherical harmonics of degree n; then

$$\iint f(x, y, z)\, \mathrm{F}(x, y, z)\, ds = 4\pi \frac{2^n n!}{(2n+1)!} f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \mathrm{F}(x, y, z).$$

This theorem offers a very simple criterion for the conjugacy of two functions, f and F; for, if they be conjugate, the right-hand side must vanish. Further, when a solid spherical harmonic of any degree is expressed in terms of a system of conjugates, each multiplied by a constant to be determined, the readiest way to determine the constant is to operate on both sides with the operator pertaining to the conjugate having that particular constant as multiplier. Clerk Maxwell has pointed out this simple method of procedure in the chapter already cited (p. 200).

10. The pole of the harmonic Q_n being $x_1 y_1 z_1$ and XYZ, any other point on the sphere we may put

$$\mathrm{Q}_n(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$$
$$= \frac{(2n)!}{2^n n!\, n!} \left\{ (\mathrm{X}x_1 + \mathrm{Y}y_1 + \mathrm{Z}z_1)^n - \frac{n(n-1)}{2(2n-1)} (\mathrm{X}x_1 + \mathrm{Y}y_1 + \mathrm{Z}z_1)^{n-2} (\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) + \ldots \right\}.$$

Put this equal to $\Sigma\, h_n^{\sigma}\, \mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ as we are entitled to do by § 7, and operate on both sides with

$$\mathrm{H}_n^{\sigma} \left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}} \right).$$

Now

$$\begin{aligned}
\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{Q}_n(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) &= \mathrm{Q}_n\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}), \\
&= \frac{(2n)!}{2^n n!\, n!}\left(x_1 \frac{\partial}{\partial \mathrm{X}} + y_1 \frac{\partial}{\partial \mathrm{Y}} + z_1 \frac{\partial}{\partial \mathrm{Z}}\right)^n \mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}), \\
&= \frac{(2n)!}{2^n n!\, n!}\, \mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right)(x_1 \mathrm{X} + y_1 \mathrm{Y} + z_1 \mathrm{Z})^n, \\
&= \frac{(2n)!}{2^n n!}\, \mathrm{H}_n^{\sigma}(x_1, y_1, z_1).
\end{aligned}$$

The coefficient of h_n^{σ} is therefore determined, and we may write

$$\mathrm{Q}_n = \frac{(2n)!}{2^n n!} \sum^{2n+1} \frac{\mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})\, \mathrm{H}_n^{\sigma}(x_1, y_1, z_1)}{\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})},$$

or, adopting the result of the preceding article,

$$\frac{4\pi}{2n+1} \sum^{2n+1} \frac{\mathrm{H}_n^{\sigma}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})\, \mathrm{H}_n^{\sigma}(x_1, y_1, z_1)}{\iint (\mathrm{H}_n^{\sigma})^2\, ds}.$$

Comparing this last with the assumed series in § 8 it appears that

$$\mathrm{A}_n^{\sigma} = \frac{4\pi}{2n+1} \frac{1}{\iint (\mathrm{H}_n^{\sigma})^2\, ds} \quad \text{and} \quad \mathrm{B}_n^{\sigma} = \frac{4\pi}{2n+1} \frac{abc}{\iint (\mathrm{G}_n^{\sigma})^2\, p\, d\mathrm{S}}.$$

The required expansion, having the value v at the surface of the ellipsoid, will therefore be

$$\frac{1}{4\pi abc} \iint vp\, d\mathrm{S} + \ldots + \Sigma \frac{\mathrm{G}_n^{\sigma}(x, y, z) \iint \mathrm{G}_n^{\sigma} vp\, d\mathrm{S}}{\iint (\mathrm{G}_n^{\sigma})^2\, p\, d\mathrm{S}} + \&\mathrm{c}.$$

Case of a Homogeneous Function of the Coordinates.

11. Let $v \equiv f(x, y, z)$ be a homogeneous function of degree p.
If we write $u = f(\xi, \eta, \zeta)$, then will

$$\begin{aligned}
v &= \frac{1}{p!}\left(x \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \eta} + z \frac{\partial}{\partial \zeta}\right)^p u \\
&= \frac{1}{p!}\left(a x_1 \frac{\partial}{\partial \xi} + b y_1 \frac{\partial}{\partial \eta} + c z_1 \frac{\partial}{\partial \zeta}\right)^p u.
\end{aligned}$$

It will be convenient to expand this first by one of the formulæ of the preceding article in a series of K-Functions, viz. :—

$$v = \frac{1}{4\pi}\iint v\,ds + \ldots + \frac{(2n+1)!}{2^n n!}\,\Sigma\,\frac{\mathrm{H}_n^{\sigma}(x_1, y_1, z_1)\iint \mathrm{H}_n^{\sigma} v\,ds}{\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^{\sigma}(x, y, z)} + \ldots$$

In regard to the integral $\iint \mathrm{H}_n^{\sigma} v\,ds$ it is to be observed that p and n must both be even or both odd, otherwise the integral will manifestly vanish: let, therefore, $p = n + 2m$. In Part 1 of the 'Philosophical Transactions' for 1879, I have shown how to evaluate integrals of this class, and the result in the present instance may be presented in the form

$$h_n^{m}\,\mathrm{H}_n^{\sigma}\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\mathrm{D}^{2m}u, \quad \text{where} \quad \mathrm{D}^2 = a^2\frac{\partial^2}{\partial\xi^2} + \ldots, \ (\S\,2)$$

h_n^{m} being

$$\frac{4\pi}{(2n+2m+1)!}\,\frac{(m+n)!}{m!}\,2^m.$$

Reverting now to the expansion for v given above and observing that

$$\frac{(2n+1)!}{2^n n!}\,h_n^{m} \quad \text{reduces to} \quad \frac{4\pi}{2^m m!\,(2m+2n+1)(2m+2n-1)\ldots(2n+3)},$$

the required expansion may be presented under the form

$$\Sigma\Sigma\,\mathfrak{h}_n^{m}(m, n, \sigma)\,\mathrm{H}_n^{\sigma}(x_1, y_1; z_1)$$

where

$$(m, n, \sigma) = \frac{\mathrm{H}_n^{\sigma}\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\mathrm{D}^{2m}u}{\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^{\sigma}(x, y, z)}$$

and

$$1/\mathfrak{h}_n^{m} = 2^m m!\,(2n+2m+1)\,(2n+2m-1)\ldots(2n+3).$$

The double Σ means that σ may have $2n+1$ different values for a given value of n, and n may have as many integral values as may be obtained from the equation $n = p - 2m$ by giving integral values to m. It only remains to express $\mathrm{H}_n^{\sigma}(x_1, y_1, z_1)$ in terms of $\mathrm{G}_n^{\sigma}(x, y, z)$, according to the relation of § 7.

Case of any Function capable of expansion by TAYLOR'S *Theorem.*

12. Consider now any function $v = f(\xi - x, \eta - y, \zeta - z)$ where x, y, z is on the surface, ξ, η, ζ outside, and let it be required to find a series of harmonics in x, y, z agreeing with v. If we assume that v may be expanded in ascending powers of x, y, z we may put

$$v = \Sigma (-1)^p \frac{1}{p!} \left(x \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \eta} + z \frac{\partial}{\partial \zeta} \right)^p u,$$

where $u = f(\xi, \eta, \zeta)$. The foregoing reasoning then shows that the coefficient of $H_n^{\sigma}(x_1, y_1, z_1)$, when we transform v into sphere coordinates, will be

$$(-1)^n \Sigma \mathfrak{h}_n^m (m, n, \sigma),$$

in which the Σ means that every possible integral value of m is to be taken from 0 upwards. In fact, if we put

$$u' = \left(1 + \frac{D^2}{2(2n+3)} + \frac{D^4}{2^2 2!(2n+5)(2n+3)} + \ldots \right) u,$$

the coefficient of $H_n^{\sigma}(x_1, y_1, z_1)$ will become

$$(-1)^n \frac{H_n^{\sigma}\left(a \frac{\partial}{\partial \xi}, b \frac{\partial}{\partial \eta}, c \frac{\partial}{\partial \zeta} \right) u'}{H_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) H_n^{\sigma}(x, y, z)}.$$

13. Should the function $f(x, y, z)$ satisfy LAPLACE'S equation, the expansion just found for $f(\xi - x, \eta - y, \zeta - z)$ will hold, not only at the surface of the ellipsoid, but at all points in its interior, and is, moreover, capable of simplification; for in that case

$$\frac{a^2 \frac{\partial^2}{\partial \xi^2}}{a^2 + \theta} + \ldots = -\theta \left(\frac{\frac{\partial^2}{\partial \xi^2}}{a^2 + \theta} + \ldots \right),$$

and therefore

$$H_n^{\sigma}\left(a \frac{\partial}{\partial \xi}, b \frac{\partial}{\partial \eta}, c \frac{\partial}{\partial \zeta} \right) = \Pi(-\theta) H_n^{\sigma}\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta} \right),$$

where $\Pi(-\theta)$ may include a factor of any of the forms a, b, c, bc, ca, ab, abc. Joining on the factor $\Pi(-\theta)$ to the harmonic H_n^{σ}, we finally obtain

$$f(\xi - x, \eta - y, \zeta - z) = \Sigma\Sigma(-1)^n \frac{H_n^\sigma\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) u'}{H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H_n^\sigma(x, y, z)} G_n^\sigma(x, y, z).$$

We may deduce from this result a practical rule for expressing any function of the form $f(\xi - x, \eta - y, \zeta - z)$, satisfying LAPLACE's equation in ellipsoidal harmonics. First express f in spherical K-conjugate functions, then substitute u' for u, and change H_n^σ into G_n^σ.

In the particular case, when the function is $\{(\xi - x)^2 + (\eta - y)^2 + (\zeta + z)^2\}^{-1/2}$ the above result may be put into a more concise form. The foregoing proof, however, depending as it does upon an expansion by TAYLOR's theorem, which will not be convergent unless $\xi\eta\zeta$ is further from the centre than xyz, can only be accepted without ambiguity when the point $\xi\eta\zeta$ is at a greater distance from the centre than the longest semi-axis. I shall, therefore, give another proof of the important case in question, against which this objection cannot be urged.

GREEN's *Function.*

14. Let E $(\xi\eta\zeta)$ and P (xyz) be two points, of which the former is outside of, and the latter is on, the ellipsoid, and let 1/EP be expanded in a series of ellipsoidal harmonics by the formula at the end of § 10, viz.:—

$$\frac{1}{4\pi abc}\iint\frac{1}{EP}\,p\,dS + \ldots + \Sigma\frac{G_n^\sigma(x, y, z)\iint\frac{1}{EP}G_n^\sigma(f, g, h)\,p\,dS}{\iint(G_n^\sigma)^2\,p\,dS} + \ldots$$

where f, g, h are the coordinates of the element dS.

Now it can be easily shown that density $pG_n^\sigma(f, g, h)$ on the surface produces potential at an outside point $\xi\eta\zeta$ equal to

$$2\pi abc\,\{\Pi(\theta)\}^2\,\mathfrak{G}_n^\sigma(\xi, \eta, \zeta.) \quad \text{(See § 4.)}$$

The expansion of 1/EP is therefore

$$\tfrac{1}{2}I_0(\xi,\eta,\zeta) + \ldots + 2\pi\frac{G_n^\sigma(x, y, z)\,\mathfrak{G}_n^\sigma(\xi, \eta, \zeta)}{\iint(H_n^\sigma)^2\,ds} + \ldots$$

If P (xyz) be now supposed to be outside of the ellipsoid, GREEN'S function at P due to the electricity induced by unit charge at E, or at E due to unit charge at P, is

$$-\tfrac{1}{2}\left\{\frac{\mathrm{I}_0(xyz)\,\mathrm{I}_0(\xi,\eta,\zeta)}{\mathrm{I}_0(0)}+\ldots+4\pi\frac{\mathfrak{G}_n{}^{\sigma}(\xi,\eta,\zeta)\,\mathfrak{G}_n{}^{\sigma}(x,y,z)}{\mathrm{I}_n{}^{\sigma}(0)\iint(\mathrm{H}_n{}^{\sigma})^2\,ds}+\ldots\right\}.$$

Connexions between corresponding Ellipsoidal and Spherical Harmonics.

15. In § 13 it was shown that the expansion of a function satisfying LAPLACE'S equation, in ellipsoidal harmonics, might be obtained from the expansion of the same function in corresponding spherical harmonics by the adoption of certain changes. To render this method complete, it is necessary to show how an internal ellipsoidal harmonic can be expressed most readily in a series of rational algebraic functions. The present article is intended to effect this, and the results obtained will be found useful in the case of ellipsoids of revolution.

If we denote $\mathrm{K}_1\ldots\mathrm{K}_n$ by S_n and the sum of the products of the K's taken together r at a time by S_r then

$$\Theta_1\ldots\Theta_n=(\mathrm{K}_1-1)\ldots(\mathrm{K}_n-1)=\mathrm{S}_n-\mathrm{S}_{n-1}+\&\mathrm{c}.$$

It will now be shown how the S functions may all be derived from S_n by successive differentiation.

Theorem i. With the values of $\theta_1\ldots\theta_n$ given in (3) § 3 corresponding to the functions $\Theta_1\ldots\Theta_n$, if the operator D^2 be applied to S_r the result will be

$$(2n-2r+2)\,(2n+2r-1)\,\mathrm{S}_{r-1}.$$

To show this let the result of the operation D^2 upon the product of two factors $\mathrm{K}_1\mathrm{K}_2$ be first found. We shall have

$$\left(\frac{2a^2}{a^2+\theta_1}+\frac{2b^2}{b^2+\theta_1}+\frac{2c^2}{c^2+\theta_1}\right)\mathrm{K}_2+\left(\frac{2a^2}{a^2+\theta_2}+\frac{2b^2}{b^2+\theta_2}+\frac{2c^2}{c^2+\theta_2}\right)\mathrm{K}_1$$
$$+\frac{8a^2x^2}{(a^2+\theta_1)(a^2+\theta_2)}+\frac{8b^2y^2}{(b^2+\theta_1)(b^2+\theta_2)}+\frac{8c^2z^2}{(c^2+\theta_1)(c^2+\theta_2)}.$$

Now the last line is readily seen to be equal to

$$\frac{8}{\theta_1-\theta_2}(\theta_1\mathrm{K}_1-\theta_2\mathrm{K}_2).$$

Hence the coefficient of K_2 is

$$\frac{2a^2}{a^2+\theta_1}+\frac{2b^2}{b^2+\theta_1}+\frac{2c^2}{c^2+\theta_1}-\frac{8\theta_2}{\theta_1-\theta_2}.$$

Let attention be now fixed upon one particular term in S_{r-1} after the operation D^2 is supposed to have been performed upon S_r, say the term $K_{n-r+2} \ldots K_n$. This term will obviously arise from differentiations upon $(K_1 + \ldots + K_{n-r+1})\, K_{n-r+2} \ldots K_n$ and from none other. The resulting coefficient written at length will be

$$\begin{aligned} &\sum_1^{n-r+1}\left(\frac{2a^2}{a^2+\theta_s}+\frac{2b^2}{b^2+\theta_s}+\frac{2c^2}{c^2+\theta_s}\right) \\ &-8\,\frac{\theta_{n-r+2}}{\theta_1-\theta_{n-r+2}}-\ldots-8\,\frac{\theta_n}{\theta_1-\theta_n} \\ &\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots \\ &-8\,\frac{\theta_{n-r+2}}{\theta_{n-r+1}-\theta_{n-r+2}}-\ldots-8\,\frac{\theta_n}{\theta_{n-r+1}-\theta_n} \end{aligned}$$

Putting $a^2/(a^2+\theta) = 1 - \theta/(a^2+\theta)$ in each case, and bearing in mind the equations (3), § 3, we find

$$\begin{aligned} &6\,(n-r+1)+8\,(n-r+1)(r-1) \\ &+8\theta_1\left(\frac{1}{\theta_1-\theta_2}+\ldots+\frac{1}{\theta_1-\theta_{n-r+1}}\right) \\ &\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots \\ &+8\theta_{n-r+1}\left(\frac{1}{\theta_{n-r+1}-\theta_1}+\ldots+\frac{1}{\theta_{n-r+1}-\theta_{n-r}}\right). \end{aligned}$$

The terms after the first line now combine in two's, the sum of each pair, involving the same two θ's, being 8. Hence the final result is

$$\begin{aligned} &6\,(n-r+1)+8\,(n-r+1)(r-1)+4\,(n-r+1)(n-r). \\ &\qquad=(2n-2r+2)(2n+2r-1). \end{aligned}$$

The next three results are given without proof.

ii. For a function of the type $x\Theta_1 \ldots \Theta_n$ the operation $D^2 x\, S_r$ gives

$$(2n-2r+2)(2n+2r+1)\, x\, S_{r-1}.$$

iii. Of the type $yz\,\Theta_1 \ldots \Theta_{n-1}$, $D^2 yz\, S_{r-1} = (2n-2r+2)(2n+2r-1)\, yz\, S_{r-2}$.

iv. Of the type $xyz\,\Theta_1 \ldots \Theta_{n-1}$, $D^2 xyz\, S_{r-1} = (2n-2r+2)(2n+2r+1)\, xyz\, S_{r-2}$.

The equalities of the coefficients in i. and iii. and in ii. and iv. should not escape notice. They show that ellipsoidal harmonics of even and odd degrees may be expressed respectively as follows :—

$$G_{2n} = \left(1 - \frac{D^2}{2(4n-1)} + \frac{D^4}{2^2 2!\,(4n-1)(4n-3)} - \&c.\right) H_{2n}$$

$$G_{2n+1} = \left(1 - \frac{D^2}{2(4n+1)} + \frac{D^4}{2^2 2!\,(4n+1)(4n-1)} - \&c.\right) H_{2n+1}.$$

Both cases may be combined and the result exhibited in the form

$$G_n = \left(1 - \frac{D^2}{2(2n-1)} + \ldots + (-1)^m \frac{D^{2m}}{2^m m!\,(2n-1)(2n-3)\ldots(2n-2m+1)} + \ldots\right) H_n,$$

where the greatest value of m is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

This result may be expressed concisely by means of a definite integral, viz.,

$$G_n = \frac{1}{(2n)!}\int_0^\infty (u^2 - D^2)^n e^{-u}\, du.\, H_n$$

or

$$\frac{1}{(2n)!}\int_0^\infty \left(\frac{d}{du}\right)^n (u^2 - D^2)^n e^{-u} du.\, H_n$$

$$= \frac{2^n n!}{(2n)!} D^{2n} \int_0^\infty P_n(uD^{-1})\, e^{-u}\, du.\, H_n,$$

P_n being a zonal harmonic of degree n.

Expansion of an External Ellipsoidal Harmonic in Spherical Harmonics.

16. Comparing the coefficient of $G_n^\sigma(x, y, z)$ in the expansion of

$$\{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{-1/2}$$

in § 14 with that in § 13, we find

$$\frac{2\pi \mathfrak{G}_n^\sigma(\xi, \eta, \zeta)}{\iint (H_n^\sigma)^2\, ds} = \frac{(-1)^n H_n^\sigma\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) u'}{H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H_n^\sigma(x, y, z)}$$

where

$$u' = \left\{1 + \frac{D^2}{2(2n+3)} + \ldots\right\} \frac{1}{\sqrt{(\xi^2 + \eta^2 + \zeta^2)}},\quad D^2 = a^2 \frac{\partial^2}{\partial \xi^2} + \ldots$$

Hence

$$\mathfrak{G}_n^\sigma(\xi, \eta, \zeta) = (-1)^n \frac{2^{n+1} n!}{(2n+1)!} H_n^\sigma\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) u'.$$

This result, however, as has been already pointed out, cannot be accepted unless $\sqrt{(\xi^2 + \eta^2 + \zeta^2)}$ is greater than the longest semi-axis. It also follows from an

investigation, to which I shall now proceed, undertaken with the object of establishing a physical interpretation of ellipsoidal harmonics from the Theory of Attractions similar to that given by Clerk Maxwell in the case of the sphere.

Physical Interpretation of an Ellipsoidal Harmonic.

17. We consider here the potentials and derivates therefrom of ellipsoids whose layers of equal density are similar and similarly situated to the bounding surface.

Let the density at any internal point fgh be of the form $n\left(1-\frac{f^2}{a^2}-\frac{g^2}{b^2}-\frac{h^2}{c^2}\right)^{n-1}$, then it is shown in Thomson and Tait's 'Natural Philosophy,' vol. 1, part II., p. 526, that the potential at any external point $\xi\eta\zeta$ or $\epsilon\nu\nu'$ is

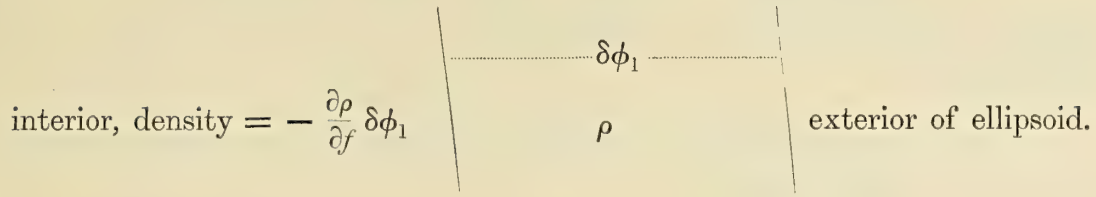
$$\pi abc\int_{\epsilon}^{\infty}\left(1-\frac{\xi^2}{a^2+\lambda}-\frac{\eta^2}{b^2+\lambda}-\frac{\zeta^2}{c^2+\lambda}\right)^{n}\frac{d\lambda}{\mathrm{L}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

I shall denote this potential by $\pi abc\mathrm{V}_n$.

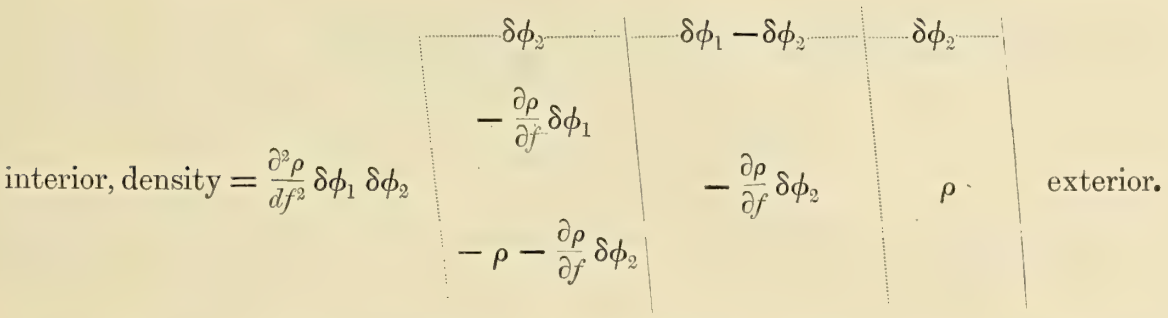
Suppose now an ellipsoid of any law of internal density ρ, and let U be the potential at an external point $\xi\eta\zeta$; then if we superpose upon this ellipsoid another of density $-\rho$ with its axes in the same directions, and its centre at the point $-\delta\phi_1$, 0, 0, we shall have an outside potential $-\frac{\partial \mathrm{U}}{\partial \xi}\delta\phi_1$ and the density at any internal point fgh will be $-\frac{\partial \rho}{\partial f}\delta\phi_1$. At the surface, however, we shall have a shell the volume density of which at any point $f_1g_1h_1$ will still be ρ.

I remark here that as the density we are to consider is of the form $n\left(1-\frac{f^2}{a^2}-\frac{g^2}{b^2}-\frac{h^2}{c^2}\right)^{n-1}$ where n is an integer, and, as this vanishes at the surface, it can only give rise, by repeated applications of the above process, to expressions which are not evanescent at the surface by being differentiated at least $n-1$ times. In $n-1$ successive differentiations, therefore, in which the surface shell only is to be considered, we may omit consideration of all terms which arise in the differentiation, except those only in each case where the above vanishing expression for the density loses in its index.

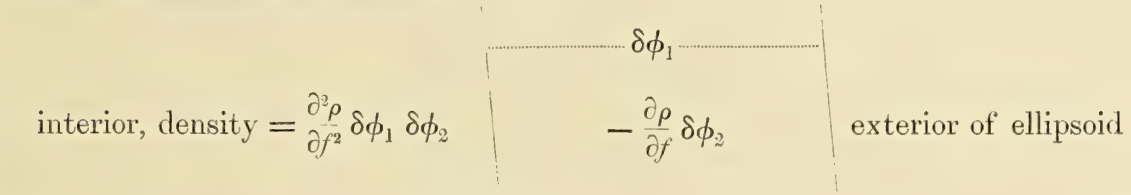
Let us now make a second displacement like the first, parallel to the axis of x, through a distance of $\delta\phi_2$, which, for the sake of clearness, we shall suppose less than $\delta\phi_1$. The outside potential will become $\frac{\partial^2 \mathrm{U}}{\partial \xi^2}\delta\phi_1\,\delta\phi_2$, and the inside density will be $\frac{\delta^2\rho}{\delta f^2}\delta\phi_1\,\delta\phi_2$, while at the surface there will be a shell of a somewhat complicated character, the nature of which may be best gathered from the following scheme. After the first displacement the state of things in the neighbourhood of any point of the surface in the positive octant is represented by



After the second displacement,



There are thus three shells which a normal to the surface would perforate before reaching the interior, but it is to be remarked that, according to the explanation given above, the density ρ of the first shell may be disregarded, as it can contribute nothing to the final result. In like manner $-\left(\rho + \frac{\partial \rho}{\partial f}\delta\phi_1\right)$ in the third shell, which is equal to the first shell geometrically, can contribute nothing, for this quantity is equal to $-\rho$ at a point, $\delta\phi_1$, further along the direction of the axis of x in the first shell. The practical effect of eliminating the first shell and the part referred to of the third will be to leave this state of things—



that is to say, exactly as after the first displacement, provided we substitute $-\frac{\partial \rho}{\partial f}\delta\phi_2$ in place of ρ.

By giving exactly equal displacements parallel to the axis of y, we should find for the outside potential $\frac{\partial^2 U}{\partial \eta^2}\delta\phi_1\,\delta\phi_2$, for the internal density $\frac{\partial^2 \rho}{\partial f^2}\delta\phi_1\,\delta\phi_2$, and a similar system of shells, the effective density of which is $-\frac{\partial \rho}{\partial g_1}\delta\phi_2$, and whose breadth as measured parallel to the axis of y is $\delta\phi_1$.

Similar results hold also as regards the axis of z.

With this explanation we may now suppose the three systems multiplied respectively by $a^2/a^2 + \theta_1$, $b^2/b^2 + \theta_1$, $c^2/c^2 + \theta_1$, and superposed.

The following statement may then be made:—

The potential

$$\left(\frac{a^2}{a^2+\theta_1}\frac{\partial^2}{\partial\xi^2}+\frac{b^2}{b^2+\theta_1}\frac{\partial^2}{\partial\eta^2}+\frac{c^2}{c^2+\theta_1}\frac{\partial^2}{\partial\zeta^2}\right)\mathrm{U}.\delta\phi_1,$$

arises from an internal density

$$\left(\frac{a^2}{a^2+\theta_1}\frac{\partial^2}{\partial f^2}+\frac{b^2}{b^2+\theta_1}\frac{\partial^2}{\partial g^2}+\frac{c^2}{c^2+\theta_1}\frac{\partial^2}{\partial h^2}\right)\rho.\delta\phi_1,$$

and from three shells whose volume densities are respectively

$$-\frac{a^2}{a^2+\theta_1}\frac{\partial\rho}{\partial f_1},\ -\frac{b^2}{b^2+\theta_1}\frac{\partial\rho}{dg_1},\ -\frac{c^2}{c^2+\theta_1}\frac{\partial\rho}{\partial h_1}.$$

18. If we make a series of successive operations of the same kind, we shall arrive at this result:—The potential

$$(-1)^m\,\theta_1\ldots\theta_m\,\mathrm{E}_1^2\ldots\mathrm{E}_m^2\,\mathrm{U}.\delta\phi_1 \quad (\text{see § 2}) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

arises from internal density

$$\mathfrak{E}_1^2\ldots\mathfrak{E}_m^2\,\rho.\delta\phi_1 \quad (\text{see § 2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and from three shells of densities

$$-\frac{a^2}{a^2+\theta_1}\,\mathfrak{E}_2^2\ldots\mathfrak{E}_m^2\,\frac{\partial\rho}{\partial f_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$-\frac{b^2}{b^2+\theta_1}\,\mathfrak{E}_2^2\ldots\mathfrak{E}_m^2\,\frac{\partial\rho}{\partial g_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$-\frac{c^2}{c^2+\theta_1}\,\mathfrak{E}_2^2\ldots\mathfrak{E}_m^2\,\frac{\partial\rho}{\partial h_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

19. Let us now make the supposition that, in the expression for the law of density in § 17, n is even and equal to $2m$, then we shall show that the expression (3) of § 18 vanishes, provided $\theta_1\ldots\theta_m$ are the characteristics of an ellipsoidal harmonic $\Theta_1\ldots\Theta_m$.

The vanishing of (3) is obviously the same thing as the vanishing of

$$\mathrm{E}_1^2\ldots\mathrm{E}_m^2\,(x^2+y^2+z^2-1)^{2m-1}$$

for the same values of θ.

To prove that this vanishes, I shall avail myself of a theorem given by Dr. Ferrers in a paper on a kindred subject in the 'Quarterly Journal of Mathematics' (vol. 14,

p. 8). With the notation of this paper, the theorem in question may be stated as follows :—

$$\text{“}(\lambda x + \mu y + \nu z)^{n-1} = A_{n-1}\left(\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}\right)^{n-1}(x^2 + y^2 + z^2 - 1)^{n-1}$$
$$+ A_{n-2}(\lambda^2 + \mu^2 + \nu^2)\left(\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}\right)^{n-3}(x^2 + y^2 + z^2 - 1)^{n-2}$$
$$+ \&c.,$$

where A_{n-1}, A_{n-2}, . . . are numerical multipliers.”

If we differentiate this again by means of the operator

$$\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}$$

we find

$$(n-1)(\lambda^2 + \mu^2 + \nu^2)(\lambda x + \mu y + \nu z)^{n-2}$$
$$= A_{n-1}\left(\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}\right)^{n}(x^2 + y^2 + z^2 - 1)^{n-1} + \&c.$$

Now this result is true for all possible values of the ratios $\lambda : \mu : \nu$. The coefficients of the various powers and products of λ, μ, ν, when all the expansions are made and the terms collected, must therefore be separately zero. We may accordingly substitute in place of λ, μ, ν differential operators $\frac{\partial}{\partial f}$, $\frac{\partial}{\partial g}$, $\frac{\partial}{\partial h}$ upon a function $M(f, g, h)$, which satisfies Laplace's equation.

When this is done, we obtain

$$\left(\frac{\partial}{\partial f}\frac{\partial}{\partial x} + \frac{\partial}{\partial g}\frac{\partial}{\partial y} + \frac{\partial}{\partial h}\frac{\partial}{dz}\right)^{n} M(f, g, h)(x^2 + y^2 + z^2 - 1)^{n-1} = 0.$$

In this result M is quite arbitrary, subject only to the foregoing condition. We may, therefore, put

$$M(f, g, h) = \left(\frac{f^2}{a^2 + \theta_1} + \frac{g^2}{b^2 + \theta_1} + \frac{h^2}{c^2 + \theta_1}\right) \cdots \left(\frac{f^2}{a^2 + \theta_m} + \frac{g^2}{b^2 + \theta_m} + \frac{h^2}{c^2 + \theta_m}\right).$$

It is easy then to show that the result finally reduces to

$$E_1^2 \ldots E_m^2 (x^2 + y^2 + z^2 - 1)^{n-1} = 0.$$

Hence (3) of § 18 vanishes.

20. We resume the consideration of the three shells, the expressions for the volume densities of which are given in § 18, (4), (5), and (6). If we bear in mind the remark

contained in § 17 as to what is to be omitted in successive differentiations, the process of reducing the expressions for the densities is easy. Take, for example, $E_2^2\rho$; we have to find

$$\left(\frac{a^2}{a^2+\theta_2}\frac{\partial^2}{\partial f_1^2}+\frac{b^2}{b^2+\theta_2}\frac{\partial^2}{\partial g_1^2}+\frac{c^2}{c^2+\theta_2}\frac{\partial^2}{\partial h_1^2}\right) n\left(1-\frac{f_1^2}{a^2}-\frac{g_1^2}{b^2}-\frac{h_1^2}{c^2}\right)^{n-1}.$$

This, with the omissions alluded to, becomes

$$2^2 n(n-1)(n-2)\left\{\frac{f_1^2}{a^2(a^2+\theta_2)}+\text{two similar terms}\right\}\left\{1-\frac{f_1^2}{a^2}-\text{two similar terms}\right\}^{n-3}.$$

If we continue this process for (4) we shall arrive at

$$2^{2m-1}n!\,(-1)^{m+1}\frac{\Theta_2\ldots\Theta_m(f_1,g_1,h_1)}{\theta_2\ldots\theta_m}\frac{f_1}{a^2+\theta_1}.$$

This represents the volume density of the shell indicated by the operation (4), and since, according to the process explained in § 17, the whole of the matter of this shell is included within the boundaries of what is practically the first shell of the series in § 17 the thickness of which at $f_1g_1h_1$ was $\frac{p_1f_1}{a^2}\delta\phi_1$, where p_1 is the perpendicular from the centre on the tangent plane at $f_1g_1h_1$, the surface density for (4) at this point is

$$2^{2m-1}(2m)!\,(-1)^{m+1}\frac{\Theta_2\ldots\Theta_m}{\theta_2\ldots\theta_m}\frac{p_1f_1^2}{a^2(a^2+\theta_1}\delta\phi_1.$$

A similar treatment applying to the reductions of (5) and (6), we may now superpose (4), (5), and (6), and, observing that $U=\pi abc\,V_{2m}$, present the result in the following form :—

The potential

$$E_1^2\ldots E_m^2\,V_{2m}$$

is due to a shell of surface density

$$2^{2m}(2m)!\,\frac{p_1}{2\pi abc}\,\frac{\Theta_1\ldots\Theta_m(f_1,g_1,h_1)}{\theta_1^2\ldots\theta_m^2}.$$

But this density produces a potential at outside points $\xi\eta\zeta$ or $\epsilon\nu\nu'$

$$=2^{2m}(2m)!\,\Theta_1\ldots\Theta_m(\xi,\eta,\zeta)\int_\epsilon^\infty\frac{d\lambda}{(\lambda-\theta_1)^2\ldots\mathrm{L}}$$

and at internal points the same expression with ϵ put $=0$.

In the notation described in § 4, this is

$$\mathfrak{G}_{2m}{}^{\sigma}(\xi, \eta, \zeta) \text{ or } \mathrm{G}_{2m}{}^{\sigma}\mathrm{I}_{2m}{}^{\sigma}(\xi, \eta, \zeta) = \frac{1}{2^{2m}(2m)!}\mathrm{H}_{2m}{}^{\sigma}\left(\frac{\partial}{\partial\xi}, \frac{\partial}{d\eta}, \frac{\partial}{d\zeta}\right)\mathrm{V}_{2m}.$$

The proof has been established for one particular type of harmonic of even degree, but it is almost identical for every variety of case included in the scheme of § 3. The general result is

$$\mathfrak{G}_{n}{}^{\sigma} = (-1)^{n}\frac{1}{2^{n}n!}\mathrm{H}_{n}{}^{\sigma}\left(\frac{\partial}{\partial\xi}, \frac{\partial}{d\eta}, \frac{\partial}{d\zeta}\right)\mathrm{V}_{n}.$$

When $n = 0$, which is an exceptional case, we may take

$$\mathfrak{G}_{0} = \mathrm{I}_{0} = \int_{\epsilon}^{\infty}\frac{d\lambda}{\mathrm{L}}.$$

21. The expansion for the potential due to a simple shell, whose semi-axes are a, b, c, at a sufficiently distant external point $\xi\eta\zeta$, is easily shown to be

$$\text{mass of shell}\left(1 + \frac{\mathrm{D}^2}{3!} + \ldots + \frac{\mathrm{D}^{2m}}{(2m+1)!} + \ldots\right)u,$$

where

$$u^{-2} = \xi^2 + \eta^2 + \zeta^2.$$

Hence, the potential due to any one of the simple shells into which the ellipsoid may be divided, with the law of density of § 17, is

$$4\pi abc\,\theta^2\partial\theta\,.\,n(1-\theta^2)^{n-1}\left(1 + \frac{\theta^2\mathrm{D}^2}{3!} + \ldots\right)u.$$

Integrating this between the limits 0 and 1, and remembering that the result is $\pi abc\mathrm{V}_n$ we find

$$\mathrm{V}_n = 2\Gamma(n+1)\sum_{m=0}^{m=\infty}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(2m+2)\,\Gamma(m+n+\frac{3}{2})}\mathrm{D}^{2m}u$$

$$= \frac{2^{2n+1}n!\,n!}{(2n+1)!}\sum_{m=0}^{m=\infty}\frac{\mathrm{D}^{2m}u}{2^{m}m!\,(2n+2m+1)(2n+2m-1)\ldots(2n+3)}.$$

On substituting this expression for V_n, in the last line of § 20, we obtain the same result as in § 16. It may further be remarked that V_n may be exhibited in the form

$$\int_{0}^{1}(1-\theta^2)^{n}\left(e^{\theta\mathrm{D}} + e^{-\theta\mathrm{D}}\right)d\theta\,.\,u,$$

Applications.—1. *Induced Magnetism.*

22. To show what calculations are required in solving physical problems involving boundary conditions at the surface of an ellipsoid, it will be sufficient to take the problem of finding the magnetism induced in a solid ellipsoid under given magnetic forces.

If $\xi\eta\zeta$ be the magnetic centre of any external magnet, and xyz any point inside of the ellipsoid, we may represent the potential at the latter point due to the outside magnet in the form

$$\left(\mathrm{A}\frac{\partial}{\partial\xi}+\mathrm{B}\frac{\partial}{\partial\eta}+\mathrm{C}\frac{\partial}{\partial\zeta}\ldots\right)\frac{1}{r},$$

where r is the distance between $\xi\eta\zeta$ and xyz (Clerk Maxwell's 'Electricity and Magnetism,' vol. 2, § 391). Hence, taking the expansion of $1/r$, given in § 14 above, we see that for any known system of magnets we may always express the potential due to them in the form of a series, the general term of which is $\alpha_n{}^\sigma \mathrm{G}_n{}^\sigma(xyz)$ where $\alpha_n{}^\sigma$ is a known constant.

23. If we assume corresponding series for the potential due to the induced magnetism it will be necessary, in order to satisfy the condition of continuity of the magnetic induction at the surface, to find thereat the rates of normal variation of $\mathrm{G}_n{}^\sigma(x, y, z)$ and $\mathfrak{G}_n{}^\sigma(x, y, z)$. Take first

$$\frac{\partial \mathrm{G}_n{}^\sigma}{\partial n}=p\left(\frac{x}{a^2}\frac{\partial}{\partial x}+\frac{y}{b^2}\frac{\partial}{\partial y}+\frac{z}{c^2}\frac{\partial}{\partial z}\right)\mathrm{G}_n{}^\sigma,$$

and observe that $\mathrm{G}_n{}^\sigma$ will contain a series of factors Θ, upon any one of which the operator within brackets gives at the surface $-2\Theta/\theta$. We may therefore write

$$\frac{\partial \mathrm{G}_n{}^\sigma}{\partial n}=l_n{}^\sigma \,.\, p\mathrm{G}_n{}^\sigma,$$

where

$$l_n{}^\sigma=-2\Sigma\frac{1}{\theta}+\frac{p}{a^2}+\frac{q}{b^2}+\frac{r}{c^2};$$

the quantities p, q, r being either unity or zero, according as $\mathrm{G}_n{}^\sigma$ does or does not contain x, y, z as factors.

In like manner, if Π denote the product of all the values of θ, as well as a, b, c corresponding to factors x, y, z, we shall have at the surface

$$\frac{\partial}{\partial n}\mathfrak{G}_n{}^\sigma=\frac{\partial}{\partial n}(\mathrm{G}_n{}^\sigma \mathrm{I}_n{}^\sigma)=\left(l_n{}^\sigma \mathrm{I}_n{}^\sigma(0)-\frac{2}{abc\,\Pi^2}\right)p\mathrm{G}_n{}^\sigma.$$

24. The inside and outside potentials due to the induced magnetism may readily be found from the usual conditions in the following forms, viz. : $A_n{}^\sigma G_n{}^\sigma I_n{}^\sigma(0)$ and $A_n{}^\sigma \mathfrak{G}_n{}^\sigma$, where

$$\frac{4\pi\kappa\alpha_n{}^\sigma}{A_n{}^\sigma} + 4\pi\kappa I_n{}^\sigma(0) + \frac{2}{abc\,\Pi^2 l_n{}^\sigma} = 0,$$

κ being the magnetic susceptibility of the ellipsoid.

The potential due to the magnetism induced in a solid sphere admits of a compact expression. If V denote the potential due to the inducing system, then at any point in the interior of the sphere the potential due to the induced magnetism is

$$-\frac{2\pi\kappa}{1+2\pi\kappa} r^{-(2+4\pi\kappa)^{-1}} \int_0^r r^{(2+4\pi\kappa)^{-1}} \frac{\partial V}{\partial r}\, dr.$$

The outside potential may be expressed in a similar form.

Applications.—2. Potential due to a thin shell bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point.

25. First let the fixed point $\xi\eta\zeta$ or $\epsilon\nu\nu'$ be outside of the ellipsoid, and let r denote the distance from it to a point fgh on the surface : consider then the integral

$$\iint \frac{G_n{}^\sigma(f,g,h)}{r}\, p\, dS = 2\pi abc\, \{\Pi(\theta)\}^2 G_n{}^\sigma I_n{}^\sigma(\xi,\eta,\zeta).$$

If we operate on both sides of this relation by means of the operator

$$\frac{\xi}{a^2}\frac{\partial}{\partial\xi} + \frac{\eta}{b^2}\frac{\partial}{\partial\eta} + \frac{\zeta}{c^2}\frac{\partial}{\partial\zeta},$$

we may arrange the result of the left-hand operation thus :

$$\iint \frac{G_n{}^\sigma(f,g,h)}{r^3}\left\{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)-\left(\frac{f(f-\xi)}{a^2}+\frac{g(g-\eta)}{b^2}+\frac{h(h-\zeta)}{c^2}\right)\right\} p\; dS,$$

which, if we put $1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2} = u$, is the same as

$$u\iint \frac{G_n{}^\sigma p\, dS}{r^3} + \iint G_n{}^\sigma \frac{\partial}{\partial n}\frac{1}{r}\, .\, dS,$$

If we now take the expansion $1/r$ given in § 14, and, observing that at the surface,

$\frac{\partial}{\partial n} G_m{}^\sigma = l_m{}^\sigma . pG_m{}^\sigma$ (§ 23), enter it in the second integral, it will be seen that all the terms but one disappear, and we find

$$\iint G_n{}^\sigma \frac{\partial}{\partial n} \frac{1}{r}\, dS = 2\pi G_n{}^\sigma I_n{}^\sigma (\xi, \eta, \zeta)\, l_n{}^\sigma abc \{\Pi(\theta)\}^2.$$

On the right-hand side we have to find

$$2\pi abc \{\Pi(\theta)\}^2 \left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \frac{\eta}{b^2}\frac{\partial}{\partial \eta} + \frac{\zeta}{c^2}\frac{\partial}{\partial \zeta}\right) G_n{}^\sigma I_n{}^\sigma (\xi, \eta, \zeta).$$

Suppose Θ one of the factors of $G_n{}^\sigma (\xi, \eta, \zeta)$, the operator upon Θ alone would give

$$\frac{2\xi^2}{a^2(a^2+\theta)} + \frac{2\eta^2}{b^2(b^2+\theta)} + \frac{2\zeta^2}{c^2(c^2+\theta)} \quad \text{or} \quad -\frac{2}{\theta}(\Theta + u).$$

Hence,

$$\left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \ldots\right) G_n{}^\sigma (\xi, \eta, \zeta) = \left(l_n{}^\sigma - 2u\Sigma \frac{1}{\theta\Theta}\right) G_n{}^\sigma (\xi, \eta, \zeta),$$

and

$$\left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \ldots\right) I_n{}^\sigma (\xi, \eta, \zeta) = -\frac{1}{\{\Pi(\theta-\epsilon)\}^2 \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}} \left(\frac{\xi}{a^2}\frac{d\epsilon}{d\xi} + \ldots\right)$$

$$= \frac{2\varpi^2 u}{\{\Pi(\theta-\epsilon)\}^2 \epsilon \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}},$$

ϖ being the perpendicular from the centre on the tangent plane to the confocal ellipsoid at $\xi\eta\zeta$.

Performing the complete operation on the right-hand side, and comparing the result with that on the left, the term containing $l_n{}^\sigma$ cancels, and we obtain

$$\iint \frac{G_n{}^\sigma\, p dS}{r^3} = \frac{4\pi\varpi^2 abc \{\Pi(\theta)\}^2 G_n{}^\sigma (\xi, \eta, \zeta)}{\{\Pi(\theta-\epsilon)\}^2 \epsilon \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}} - 4\pi abc \{\Pi(\theta)\}^2 u\Sigma \frac{1}{\theta\Theta} G_n{}^\sigma I_n{}^\sigma (\xi, \eta, \zeta).$$

Now, by § 10, if r on the left-hand side denote the distance between $\xi\eta\zeta$ and fgh and, on the right, between $\xi\eta\zeta$ and any element dS on the surface, then

$$\frac{1}{r^3} = \frac{1}{4\pi abc} \iint \frac{p\, dS}{r^3} + \ldots + \Sigma \frac{G_n{}^\sigma (f, g, h) \iint \frac{G_n{}^\sigma}{r^3} p\, dS}{\iint (G_n{}^\sigma)^2 p\, dS} + \ldots$$

Hence, on entering the values of the integrals just found,

$$\frac{1}{r^3} = \frac{\varpi^2}{\epsilon\sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}}\left\{1 + \ldots + 4\pi abc\left(\frac{\Pi(\theta)}{\Pi(\theta-\epsilon)}\right)^2 \frac{G_n^{\sigma}(f,g,h)\,G_n^{\sigma}(\zeta,\eta,\zeta)}{\iint (G_n^{\sigma})^2\, p\, dS} + \ldots\right\}$$
$$- 4\pi \sum_{n=2}^{n=\infty} \frac{G_n^{\sigma}(f,g,h)\,\Sigma\,\dfrac{1}{\theta\Theta}\,G_n^{\sigma}I_n^{\sigma}(\xi,\eta,\zeta)}{\iint (H_n^{\sigma})^2\, ds}.$$

If we multiply both sides of the last found relation by p, the perpendicular from the centre to the tangent plane at fgh, we may find the potentials outside and inside the shell due to density p/r^3, for we know the potentials which $p\,G_n^{\sigma}(f, g, h)$ will produce.

It will be observed that the second of the two series expressing $1/r^3$ begins with harmonics of the second degree, and contains those harmonics only which have at least one factor of the form Θ. From this circumstance, by taking the first four terms of the outside potential, viz., as far as harmonics of the first degree inclusive, it may be shown that, to the same degree of approximation, the potential at any outside point x, y, z is the same as if the mass were collected at a certain internal point.* If we denote by I_0, xI_1', yI_1'', zI_1''' the first four harmonics at an external point xyz, then the potential at this point is

$$\frac{2\pi abc\varpi^2}{\epsilon\sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}}\left\{I_0 + \frac{3a^2\xi}{a^2+\epsilon}\frac{xI_1'}{a^2} + \frac{3b^2\eta}{b^2+\epsilon}\frac{yI_1''}{b^2} + \frac{3c^2\zeta}{c^2+\epsilon}\frac{zI_1'''}{c^2}\right\}.$$

But, by § 14, this to the same degree of approximation is the same as if the mass were collected at the point $\alpha\beta\gamma$, such that

$$\frac{\alpha}{a^2} = \frac{\xi}{a^2+\epsilon}, \qquad \frac{\beta}{b^2} = \frac{\eta}{b^2+\epsilon}, \qquad \frac{\gamma}{c^2} = \frac{\zeta}{c^2+\epsilon}.$$

26. It may be pointed out that the two points $\xi\eta\zeta$ and $\alpha\beta\gamma$ possess a remarkable geometrical property which may be stated as follows :—

If xyz be any point on the surface of the ellipsoid, then

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$
$$+ \epsilon\left\{\left(\frac{x-\alpha}{a}\right)^2 + \left(\frac{y-\beta}{b}\right)^2 + \left(\frac{z-\gamma}{c}\right)^2\right\}.$$

By means of this geometrical theorem it is easy to demonstrate the property proved above, that $\alpha\beta\gamma$ is the centre of inertia of the shell just discussed. The geometrical theorem occurs, though in a different form, in a paper by CAYLEY in the 'Proceedings of the London Mathematical Society' (vol. 6, p. 58).

* The property referred to was set as a problem by Mr. TEMPERLEY in the Mathematical Tripos, 1880.

27. If the point $\xi\eta\zeta$ be inside of the shell, the foregoing expressions and series will take different forms.

The proper expansion of $1/r$ will then be

$$\tfrac{1}{2}\mathrm{I}_0(f, g, h) + \ldots + 2\pi\Sigma \frac{\mathrm{G}_n^{\sigma}(\xi, \eta, \zeta)\,\mathrm{G}_n^{\sigma}\mathrm{I}_n^{\sigma}(f, g, h)}{\iint(\mathrm{H}_n^{\sigma})^2\,ds} + \ldots$$

and $\dfrac{\partial}{\partial n}\dfrac{1}{r}$ will be

$$2\pi\Sigma \frac{\mathrm{G}_n^{\sigma}(\xi\eta\zeta)}{\iint(\mathrm{H}_n^{\sigma})^2\,ds}\left(l_n^{\sigma}\mathrm{I}_n^{\sigma}(0) - \frac{2}{abc\{\Pi(\theta)\}^2}\right) p\,\mathrm{G}_n^{\sigma}(f, g, h).$$

The left-hand side in this case will, therefore, be

$$u\iint\frac{\mathrm{G}_n^{\sigma}\,p\,d\mathrm{S}}{r^3} + 2\pi\,\mathrm{G}_n^{\sigma}(\xi\eta\zeta)\left[abc\,\{\Pi(\theta)\}^2\,l_n^{\sigma}\mathrm{I}_n^{\sigma}(0) - 2\right].$$

The right-hand side will be

$$2\pi abc\,\{\Pi(\theta)\}^2\left(l_n^{\sigma} - 2u\,\Sigma\frac{1}{\theta\Theta}\right)\mathrm{G}_n^{\sigma}(\xi\eta\zeta)\,\mathrm{I}_n^{\sigma}(0).$$

Hence

$$u\iint\frac{\mathrm{G}_n^{\sigma}}{r^3}p d\mathrm{S} = 4\pi\mathrm{G}_n^{\sigma}(\xi\eta\zeta) - 4\pi abc\,\{\Pi(\theta)\}^2\,u\,\Sigma\frac{1}{\theta\Theta}\,\mathrm{G}_n^{\sigma}(\xi\eta\zeta)\,\mathrm{I}_n^{\sigma}(0),$$

and

$$\frac{u}{r^3} = \frac{1}{abc} + \ldots + 4\pi\frac{\mathrm{G}_n^{\sigma}(fgh)\,\mathrm{G}_n^{\sigma}(\xi\eta\zeta)}{\iint(\mathrm{G}_n^{\sigma})^2\,p d\mathrm{S}} - 4\pi u\sum_{n=2}^{n=\infty}\frac{\mathrm{G}_n^{\sigma}(fgh)\,\mathrm{I}_n^{\sigma}(0)\,\Sigma\dfrac{1}{\theta\Theta}\,\mathrm{G}_n^{\sigma}(\xi\eta\zeta)}{\iint(\mathrm{H}_n^{\sigma})^2\,d\mathrm{s}}.$$

The centre of inertia of this shell is the point $\xi\eta\zeta$ itself; and its mass is clearly given by a different expression to what it was in the former case.

If unit of electricity be placed at an internal point $\xi\eta\zeta$ and σ denote the density at fgh, then $-4\pi\sigma$ is equal to the first of the two last series multiplied by p. The above result, therefore, shows how far $4\pi\sigma$ is different to $-pu/r^3$, which corresponds to the similar expression for the sphere.

Applications.—3. Electrical Capacity.

28. The theorem which forms the subject of this article is not a special illustration of the use of ellipsoidal harmonics, except in so far that GREEN'S function can be determined for an ellipsoid in terms of those functions. I shall now show that the determination of GREEN'S function for any surface suffices also to determine in a simple form the capacity of the inverted surface.

If R be the radius of inversion, r the radius vector from the electrified point to an element $\delta\mathrm{S}$ of the conductor, at which the density is σ; r', $\delta\mathrm{S}'$, σ' similar quantities for the inverted surface: then, by well-known results,

$$\sigma' \delta S' = \left(\frac{R}{r'}\right)^3 \sigma \left(\frac{r'}{r}\right)^2 \delta S = \frac{R}{r} \sigma . \delta S.$$

$$\therefore \quad \iint \sigma' \, dS' = R \iint \frac{\sigma \, dS}{r}.$$

On the left-hand side, the integrations cover the whole of the inverted surface, and on the right the whole of the original surface. We have supposed the electrified point to be unity. If we take it equal to R, then the potential of the freely electrified inverted surface will be unity, and the above result shows that its capacity is $R^2 u$ where $-u$ is the potential at the electrified point, of unit magnitude, due to the charge induced by it in the original conductor. For instance, let unit of electricity be placed at a distance f from the centre of a spherical conductor of radius a, and let the sphere be inverted into itself. Then

$$R^2 = f^2 - a^2 \qquad \text{and} \qquad u = \frac{a}{f} \Big/ \left(f - \frac{a^2}{f}\right).$$

Therefore $R^2 u = a$, as it ought to be.

29. It may be pointed out that the foregoing theorem is suitable for the determination of the capacity of the inverted surfaces derived from

$$V \equiv \frac{1}{r} - \frac{a}{f} \frac{1 + e \tan^n \tfrac{1}{2}\theta \cos n\phi}{r'} = 0,$$

where n is integral, r, r' are the radii vectores from fixed points A and B to any point P ; PBA $= \theta$; and the longitudinal angle of the plane PBA $= \phi$. We shall suppose n even and $f > a(1 + e)$, the distance AB being, as before, $f - a^2/f$. Then so long as the expression

$$1 + e \tan^n \tfrac{1}{2}\theta \cos n\phi$$

is negative, it is impossible to find points for which $V = 0$. If, however, it is positive, there are real points until it reaches the value f/a, when r and r' become infinite. Hence, when $\cos n\phi$ is positive, the equation

$$1 + e \tan^n \tfrac{1}{2}\theta \cos n\phi = f/a$$

is that of a cone parallel to the conical asymptote to the surface $V = 0$, the vertex of which is A. The inverted surface, with respect to A, will therefore have a conical protuberance, terminating at the point A, with n corrugations symmetrically ranged upon it.

If the radius of inversion be, as before, $\sqrt{(f^2 - a^2)}$, the capacities of all such inverted surfaces will be a.

Inverted figures obtained from ellipsoids present many varieties according to their degrees of prolateness or oblateness, and the position of the point of inversion. The expression for GREEN's function in the general case of the ellipsoid with three unequal axes is, however, of too complicated a character to admit of interesting results.

Ellipsoids of Revolution.

Reduction of the K-*Functions.*

30. We here consider the case in which two axes of the ellipsoid are equal, say $a = b$.

It will be convenient to take one particular form for discussion; let us, therefore, choose $\Theta_1 \ldots \Theta_n$, and find to what the corresponding function $K_1 \ldots K_n$ reduces when all the characteristics $= -a^2$.

In the equations (3) § 3 write $b^2 + \theta_1 = (a^2 - b^2)(-q_1)$, $a^2 + \theta_1 = (a^2 - b^2)(1 - q_1)$, and similarly for $\theta_2 \ldots \theta_n$; multiply by $a^2 - b^2$ throughout, and put $a = b$. We obtain

$$\frac{1}{q_1 - 1} + \frac{1}{q_1} + \frac{4}{q_1 - q_2} + \&c. = 0, \&c.$$

Hence all the roots satisfy an equation $f(q) = 0$ such that

$$\frac{1}{q-1} + \frac{1}{q} + 2\frac{d^2f}{dq^2} \Big/ \frac{df}{dq} = 0.$$

This equation being of the same degree as f itself, we must have, identically,

$$q(q-1)\frac{d^2f}{dq^2} + \tfrac{1}{2}(2q-1)\frac{df}{dq} = n^2 f,$$

or

$$\surd\{q(1-q)\}\frac{d}{dq}\surd\{q(1-q)\}\frac{df}{dq} + n^2 f = 0,$$

or

$$\frac{d^2f}{d\chi^2} + 4n^2 f = 0,$$

where $q = \sin^2\chi$.

Hence $f = \cos 2n\chi$ or $\sin 2n\chi$, the former value being appropriate to the harmonic under consideration, the latter to the form $xy\,\Theta_1 \ldots \Theta_n$ which leads by transformation (§ 6) to the same differential equation.

Since any factor K in the case we have supposed is proportional to

$$x^2(-q) + y^2(1-q),$$

i.e., to

$$(x^2 + y^2)\left(\frac{y^2}{x^2 + y^2} - q\right) \text{ or } (x^2 + y^2)(\sin^2\phi - q)$$

if we disregard unessential constants, we have $K_1 \ldots K_n$ proportional to

$$(x^2 + y^2)^n (\sin^2\phi - \sin^2\chi_1) \ldots (\sin^2\phi - \sin^2\chi_n)$$

where $\chi_1 \ldots \chi_n$ satisfy $\cos 2n\chi = 0$.

The harmonic, therefore, becomes proportional to

$$(x^2 + y^2)^n \cos 2n\phi,$$

or, as it is usual to write it, $\frac{1}{2}(\xi^{2n} + \eta^{2n})$, where $\xi = x + jy$, $\eta = x - jy$.

It is noteworthy that, in this particular case, $\Theta_1 \ldots \Theta_n$ reduces to the same result.

31. Next let $\Theta_1 \ldots \Theta_n$ have σ values equal to $-a^2$ and $n - \sigma$ between $-a^2$ and $-c^2$.

It will be found that the equation to determine the equal roots is identically the same as in the preceding article, except that σ takes the place of n. One factor of $K_1 \ldots K_n$ will therefore be $\cos 2\sigma\phi$.

Let each of the remaining $n - \sigma$ roots satisfy an equation $f(\theta) = 0$: then, by the equations (3), § 3, each of the roots in question must satisfy

$$\frac{4\sigma + 2}{a^2 + \theta} + \frac{1}{c^2 + \theta} + 2\frac{d^2f}{d\theta^2}\Big/\frac{df}{d\theta} = 0.$$

Write herein $c^2 + \theta = -(a^2 - c^2)\,p$, $a^2 + \theta = (a^2 - c^2)\,(1 - p)$; then

$$-2p(1-p)\frac{d^2f}{dp^2} + \{(4\sigma + 3)\,p - 1\}\frac{df}{dp} = 0,$$

and this being of the same degree in p as f itself the left-hand side must be identically the same as $(n - \sigma)(2n + 2\sigma + 1)f$.

Now put $p = \mu^2$; then the equation to determine the form of f is

$$(1 - \mu^2)\frac{d^2f}{d\mu^2} - 2(2\sigma + 1)\mu\frac{df}{d\mu} + (2n - 2\sigma)(2n + 2\sigma + 1)f = 0.$$

But this equation (see FERRERS, IV., § 9) is satisfied by

$$f = (1 - \mu^2)^\sigma \frac{d^{2\sigma}}{d\mu} \cdot P_{2n}(\mu).$$

If, now, we omit unessential constants, a K-factor becomes $z^2 - (x^2 + y^2 + z^2)\,p$. Hence the $n - \sigma$ factors produce, when multiplied together,

$$(x^2 + y^2 + z^2)^{n-\sigma}\frac{d^{2\sigma}}{d\mu} P_{2n}(\mu).$$

The complete K-spherical harmonic is therefore

$$(\xi^{2\sigma} + \eta^{2\sigma})(x^2 + y^2 + z^2)^{n-\sigma}\frac{d^{2\sigma}}{d\mu} P_{2n}(\mu).$$

32. We have thus investigated one particular case of the scheme (1), § 3, but all the other cases may be similarly treated. It will be found that functions of the forms $(1, z, x, xz)\ \mathrm{K}_1 \ldots \mathrm{K}_n$ have $\xi^\sigma + \eta^\sigma$ as their longitudinal factor, σ being even for the first two forms and odd for the second two, and that $(y, yz, xy, xyz)\ \mathrm{K}_1 \ldots \mathrm{K}_n$ have $\xi^\sigma - \eta^\sigma$, σ being odd for the first two and even for the second two.

The general form to which the K-harmonic of n^{th} degree reduces is

$$r^n \begin{matrix}\cos \sigma\phi \\ \sin \sigma\phi\end{matrix} (1 - \mu^2)^{\sigma/2} \frac{d^\sigma}{d\mu} \mathrm{P}_n(\mu),$$

and is seen to be that of the ordinary tesseral and sectorial system.

33. Reverting to the particular type $\Theta_1 \ldots \Theta_n$ we may arrange the solutions as follows :—First with the n values of θ each $= -a^2$; next with $n-1$ values $= -a^2$ and one between $-a^2$ and $-c^2$; and so on; the total number of values, $= -a^2$, used in the process, being $n + (n-1) + \ldots + 1 = \frac{1}{2} n(n+1)$, *i.e.*, half the whole number of the values of θ (§ 5). This shows in what manner the values of θ are distributed in the case of the ellipsoid of three unequal axes. A harmonic of the type in question may, in the general case, have r values of θ between $-a^2$ and $-b^2$ and $n-r$ between $-b^2$ and $-c^2$, where r is any integer from 0 up to n.

Internal Harmonics of a Prolate Spheroid.

34. It is desirable at this stage to effect an alteration of meaning in two of the symbols, viz., H_n^σ and G_n^σ, as used to express corresponding internal spherical and spheroidal harmonics. It will now be convenient to adopt the spherical harmonic system in Thomson and Tait's 'Natural Philosophy,' and Clerk Maxwell's 'Electricity and Magnetism,' and to express corresponding spheroidal harmonics in conformity therewith. The new meaning of H_n^σ may be taken as expressed by any one of the following well-known forms, collected together here for future reference :—

i. $$\frac{(n+\sigma)!}{2^{2\sigma} n!\, \sigma!} (\xi^\sigma + \eta^\sigma) \left\{ z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{4(\sigma+1)} z^{n-\sigma+2}(x^2+y^2) + \ldots \right\}.$$

ii. $$\frac{(2n)!}{2^{n+\sigma} n!\, n!} (\xi^\sigma + \eta^\sigma) \left\{ z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)} z^{n-\sigma+2}(x^2+y^2+z^2) + \ldots \right\}.$$

iii. $$\frac{(n+\sigma)!\, \sigma!}{(2\sigma)!\, n!} (\xi^\sigma + \eta^\sigma) \frac{1}{\pi} \int_0^\pi (z + j\rho\cos\theta)^{n-\sigma} \sin^{2\sigma}\theta\, d\theta.$$

iv. $$2^{-\sigma} (-j)^\sigma \frac{(n+\sigma)!\, (n-\sigma)!}{n!\, n!} 2\cos\sigma\phi \frac{1}{\pi} \int_0^\pi (z + j\rho\cos\theta)^n \cos\sigma\theta\, d\theta.$$

If $\sigma = 0$ these expressions must be divided by 2 to express $r^n \mathrm{P}_n(\mu)$. Should there

be occasion to specially denote the corresponding harmonics when the longitudinal factor is $\xi^\sigma - \eta^\sigma$ in place of $\xi^\sigma + \eta^\sigma$ we shall use the symbol $H'_\sigma{}^n$.

35. According to § 15 the corresponding spheroidal harmonic is given by

$$\begin{aligned} G_n{}^\sigma &= \left(1 - \frac{c^2 - a^2}{2\,(2n-1)}\frac{\partial^2}{\partial z^2} + \ldots\right) H_n{}^\sigma \\ &= H_n{}^\sigma + A H_{n-2}{}^\sigma + B H_{n-4}{}^\sigma + \ldots, \end{aligned}$$

where A, B, . . . are functions of n and σ arising in the process of differentiation.

It will conduce to brevity if we introduce a new symbol, defined by the relation

$$H_n{}^\sigma = r^n . 2\cos\sigma\phi\,(1-\mu^2)^{\sigma/2}\chi_n{}^\sigma(\mu),$$

so that

$$\begin{aligned} (1-\mu^2)^{\sigma/2}\chi_n{}^\sigma(\mu) &= 2^{-\sigma}\frac{(n-\sigma)!}{n!}(1-\mu^2)^{\sigma/2}\frac{d^\sigma}{d\mu}P_n(\mu) \\ &= 2^{-\sigma}(-j)^\sigma\frac{(n+\sigma)!\,(n-\sigma)!}{n!\;n!}\frac{1}{\pi}\int_0^\pi(\mu + j\nu\cos\theta)^n\cos\sigma\theta\,d\theta. \end{aligned}$$

Taking now the case of a prolate spheroid, *i.e.*, supposing $c > a$, and writing $c^2 - a^2 = \gamma^2$, we see that $G_n{}^\sigma$ may be thrown into either of the following forms :—

$$\text{i.}\quad \frac{2^{n+\sigma}(n+\sigma)!\;n!\;\sigma!}{(2n)!\;(2\sigma)!}\gamma^{n-\sigma}(\xi^\sigma + \eta^\sigma)\frac{1}{\pi}\int_0^\pi \chi_n{}^\sigma\left(\frac{z + j\rho\cos\theta}{\gamma}\right)\sin^{2\sigma}\theta\,d\theta,$$

$$\text{ii.}\quad (-j)^\sigma\frac{2^{n-\sigma}(n+\sigma)!\;(n-\sigma)!}{(2n)!}\gamma^n 2\cos\sigma\phi\frac{1}{\pi}\int_0^\pi P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right)\cos\sigma\theta\,d\theta,$$

and

$$G_n = \frac{2^n\,n!\;n!}{(2n)!}\gamma^n\frac{1}{\pi}\int_0^\pi P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right)d\theta.$$

36. The expressions just found enable us to express $G_n{}^\sigma$ in terms of ellipsoidal coordinates.

In the formulæ

$$(c^2 - a^2)\,z^2 = (c^2 + \epsilon)(c^2 + \upsilon'), \qquad (c^2 - a^2)\,\rho^2 = -(a^2 + \epsilon)(a^2 + \upsilon')$$

put

$$\begin{aligned} (c^2 - a^2)\,\mu^2 &= c^2 + \epsilon & (c^2 - a^2)\,\nu^2 &= -(a^2 + \epsilon) \\ (c^2 - a^2)\,\mu_1{}^2 &= c^2 + \upsilon' & (c^2 - a^2)\,\nu_1{}^2 &= -(a^2 + \upsilon'). \end{aligned}$$

Then

$$\begin{aligned} P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right) &= P_n\{\mu\mu_1 + \nu\nu_1\cos(\pi - \theta)\} \\ &= P_n(\mu)\,P_n(\mu_1) + \ldots \\ &\quad + (-1)^\sigma 2^{2\sigma+1}\frac{n!\;n!}{(n+\sigma)!\;(n-\sigma)!}\nu^\sigma\nu_1{}^\sigma\chi_n{}^\sigma(\mu)\,\chi_n{}^\sigma(\mu_1)\cos\sigma\theta + \&c. \end{aligned}$$

First expressing $\nu^\sigma \chi_n^\sigma$ in terms of ellipsoidal coordinates according to the formula of the preceding article, and then entering the series just obtained in the second expression for G_n^σ (ii. in § 35), we may exhibit the final result, after reduction, in the form

$$G_n^\sigma = \alpha_n^\sigma . 2 \cos \sigma\phi . f_n^\sigma (\epsilon) f_n^\sigma (\upsilon'),$$

where

$$\alpha_n^\sigma = (-j)^\sigma 2^{n-\sigma} \frac{\{(n+\sigma)!\ (n-\sigma)!\}^2}{(2n)!\ n!\ n!}$$

$$f_n^\sigma (\epsilon) = \frac{1}{\pi} \int_0^\pi \{\sqrt{(c^2+\epsilon)} - \sqrt{(a^2+\epsilon)} \cos\theta\}^n \cos\sigma\theta\, d\theta$$

$$f_n^\sigma (\upsilon') = \frac{1}{\pi} \int_0^\pi \{\sqrt{(c^2+\upsilon')} - \sqrt{(a^2+\upsilon')} \cos\theta\}^n \cos\sigma\theta\, d\theta.$$

External Harmonics of a Prolate Spheroid.

37. Since the form of G_n^σ is completely determined in the foregoing articles, the present part of this investigation may be considered as a method of evaluating the integral I_n^σ.

Two preliminary remarks seem necessary:—(1) The result established in § 20, expressing $\mathfrak{G}_n^\sigma$ or $G_n^\sigma I_n^\sigma$ in terms of a differential operation upon V_n, is true for the altered meanings of H_n^σ and G_n^σ, for the change merely amounts to the removal of a constant from the two sides of an equation. In like manner the various forms of expansion investigated in §§ 10 and 15 are clearly unaffected by the change. (2) Since, by § 15, G_n^σ is determined from H_n^σ by an operator $f(D^2)$ where D^2 may be written $(a^2-c^2)\frac{\partial^2}{\partial x^2} + (b^2-c^2)\frac{\partial^2}{\partial y^2}$, in which the differences of the squares of the semi-axes only appear, and since I_n^σ and V_n are both independent of any particular confocal, it is clear that we may select that confocal of the system which is the most readily dealt with. This will obviously be the confocal ellipse to which we will suppose the ellipsoid contracted, retaining in its contraction, however, the same law of internal density. When this is done the result may be stated thus:—

$$\frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \iint \left(1 - \frac{f^2}{a^2-c^2} - \frac{g^2}{b^2-c^2}\right)^{n-\frac{1}{2}} \frac{df\,dg}{\sqrt{\{(x-f)^2+(y-g)^2+z^2\}}} = \pi \sqrt{\{(a^2-c^2)(b^2-c^2)\}}\, V_n(xyz),$$

V_n retaining the same meaning as before.

If the ellipsoid be a prolate spheroid it ultimately contracts to the line joining the two foci, the length of which is $2\sqrt{(c^2-a^2)}$ or 2γ. We shall then have, by an easy integration,

$$V_n(x, y, z) = \frac{1}{\gamma}\int_{-\gamma}^{\gamma} \frac{\left(1 - \frac{u^2}{\gamma^2}\right)^n}{\sqrt{\{x^2 + y^2 + (z-u)^2\}}}\, du.$$

38. Confining attention, as before, to the case of the prolate spheroid, we shall enter the expression for V_n just found in the relation of § 20, viz.,

$$G_n{}^\sigma I_n{}^\sigma(x, y, z) = \frac{(-1)^n}{2^n n!} H_n{}^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V_n.$$

The form of $H_n{}^\sigma$ to be used is ii. in § 34, and it will be observed that, in making the substitutions for x, y, z in differential operators, $\xi^\sigma + \eta^\sigma$ must be replaced by

$$2^\sigma\left(\frac{\partial^\sigma}{\partial \xi} + \frac{\partial^\sigma}{\partial \eta}\right).$$

Hence

$$G_n{}^\sigma I_n{}^\sigma = (-1)^n \frac{(2n)!}{2^{2n}(n!)^3}\left(\frac{\partial^\sigma}{\partial \xi} + \frac{\partial^\sigma}{\partial \eta}\right)\frac{\partial^{n-\sigma}}{\partial z} V_n$$

$$= (-1)^{n+\sigma} \frac{(2n)!\,(2\sigma)!}{2^{2n+2\sigma}(n!)^3\,\sigma!}(\xi^\sigma + \eta^\sigma)\frac{1}{\gamma}\frac{\partial^{n-\sigma}}{\partial z}\int_{-\gamma}^{\gamma} \frac{\left(1 - \frac{u^2}{\gamma^2}\right)^n du}{\{x^2 + y^2 + (z-u)^2\}^{\sigma + 1/2}}.$$

Enter on the left-hand side of this result, viz., in $G_n{}^\sigma I_n{}^\sigma$, the first form of $G_n{}^\sigma$ given in § 35; cut out $\xi^\sigma + \eta^\sigma$; and then put $x, y = 0$.

We then obtain

$$\frac{2^{n-\sigma}(n+\sigma)!\,n!}{(2n)!\,\sigma!}\gamma^{n-\sigma}\chi_n{}^\sigma\left(\frac{z}{\gamma}\right) I_n{}^\sigma = (-1)^{n+\sigma}\frac{(2n)!\,(2\sigma)!}{2^{2n+2\sigma}(n!)^3\,\sigma!}\frac{1}{\gamma^{2n+1}}\frac{d^{n-\sigma}}{dz}\int_{-\gamma}^{\gamma}\frac{(\gamma^2 - u^2)^n}{(z-u)^{2\sigma+1}}\, du.$$

If we write herein $u = \gamma v$ and put $z = \mu\gamma$, then the limits of the integral $I_n{}^\sigma$ will be ∞ and ϵ, where $\gamma^2 + \epsilon = z^2$, and we shall have

$$\chi_n{}^\sigma(\mu) I_n{}^\sigma = \frac{\{(2n)!\}^2}{2^{3n+\sigma}(n!)^4}\frac{1}{\gamma^{2n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu - v)^{n+\sigma+1}}\, dv.$$

When $\sigma = 0$

$$P_n(\mu) I_n = \frac{\{(2n)!\}^2}{2^{3n}(n!)^4}\frac{1}{\gamma^{2n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu - v)^{n+1}}\, dv.$$

39. The symbol $I_n{}^\sigma$ was originally introduced as an abbreviation for

$$\int_\epsilon^\infty \frac{d\theta}{(\theta - \theta_1)^2 \ldots \sqrt{\{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)\}}}.$$

In the case of prolate spheroids, in which $a = b$, the unequal values of θ are given by

$$\frac{d^\sigma}{d\mu} \mathrm{P}_n(\mu) = \frac{(2n)!}{2^n n!\,(n-\sigma)!} \left\{ \mu^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)} \mu^{n-\sigma-2} + \ldots \right\} = 0,$$

in which $(c^2 - a^2)\mu^2 = c^2 + \theta$.

If $n - \sigma$ be even, there will be $\frac{1}{2}(n-\sigma)$ unequal values of θ and either $\frac{1}{2}\sigma$ values $= -a^2$, or $\frac{1}{2}(\sigma - 1)$ along with a factor x or y in the harmonic.

If $n - \sigma$ be odd, there will be $\frac{1}{2}(n - \sigma - 1)$ unequal values of θ with a factor z and either $\frac{1}{2}\sigma$ values $= -a^2$, or $\frac{1}{2}(\sigma - 1)$ with a factor x or y. All cases, however, are included in

$$\mathrm{I}_n{}^\sigma = \int_\epsilon^\infty \frac{d\theta}{(\theta + a^2)^\sigma (\theta - \theta_p)^2 \ldots (\theta + a^2)\sqrt{(\theta + c^2)}},$$

where $\theta_p, \ldots$ are the unequal values of θ, and, therefore,

$$(\theta - \theta_p)^2 \ldots = \left\{ \frac{2^n n!\,(n-\sigma)!}{(2n)!} \frac{d^\sigma}{d\lambda} \mathrm{P}_n(\lambda) \right\}^2 \gamma^{2n-2\sigma}.$$

In this expression $c^2 + \theta = \lambda^2\gamma^2$, and therefore $a^2 + \theta = (\lambda^2 - 1)\gamma^2$. Hence, recollecting that $c^2 + \epsilon = z^2 = \mu^2\gamma^2$, we obtain for $\mathrm{I}_n{}^\sigma$ the value

$$\left\{ \frac{(2n)!}{2^n n!\,(n-\sigma)!} \right\}^2 \frac{2}{\gamma^{n+1}} \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \left\{ \frac{d^\sigma}{d\lambda} \mathrm{P}_n(\lambda) \right\}^2}$$

or

$$\left\{ \frac{(2n)!}{2^{n+\sigma} n!\, n!} \right\}^2 \frac{2}{\gamma^{n+1}} \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \{\chi_n{}^\sigma(\lambda)\}^2}.$$

Substituting this value in the expressions found in § 38, we find

$$\chi_n{}^\sigma(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \{\chi_n{}^\sigma(\lambda)\}^2} = \frac{1}{2^{n-\sigma+1}} \int_{-1}^1 \frac{(1 - v^2)^n}{(\mu - v)^{n+\sigma+1}}\, dv$$

and

$$\mathrm{P}_n(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)\{\mathrm{P}_n(\lambda)\}^2} = \frac{1}{2^{n+1}} \int_{-1}^1 \frac{(1-v^2)^n}{(\mu - v)^{n+1}}\, dv.$$

40. These results, for spherical harmonics of the second kind, are probably in the simplest forms for determining the expansions in powers of $1/\mu$. To determine, however, the expression in finite terms, take for simplicity the case of the zonal harmonic, and write $\mu - v = \xi$; then

$$\begin{aligned}(v^2 - 1)^n &= \{(\mu - \xi)^2 - 1\}^n \\ &= (\mu^2 - 1)^n + \ldots + \frac{(-1)^r}{r!} \frac{d^r}{d\mu} (\mu^2 - 1)^n \,.\, \xi^r + \ldots\end{aligned}$$

Dividing by ξ^{n+1} we see that the part of the integral which produces the logarithm is

$$\frac{1}{2^{n+1}n!}\frac{d^n}{d\mu}(\mu^2-1)^n \cdot \int_{\mu-1}^{\mu+1}\frac{d\xi}{\xi},$$

or,

$$\tfrac{1}{2}\,\mathrm{P}_n(\mu)\,\log\frac{\mu+1}{\mu-1}.$$

The other terms may be grouped in pairs, viz.:—

$$(-1)^{n-s}\frac{d^{n-s}}{d\mu}(\mu^2-1)^n\int\xi^{-s}\,d\xi+(-1)^{n+s}\frac{d^{n+s}}{d\mu}(\mu^2-1)^n\int\xi^s\,d\xi.$$

Hence, by RODRIGUES' theorem (FERRERS, II., § 8) the resulting integration of the pair is

$$(-1)^{n+s}\frac{d^{n+s}}{d\mu}(\mu^2-1)^n\left\{\frac{(\mu+1)^{s+1}-(\mu-1)^{s+1}}{s+1}+\frac{(\mu^2-1)\,[(\mu+1)^{s-1}-(\mu-1)^{s-1}]}{s-1}\right\}.$$

41. The last grouping, and the forms of the expressions before integration suggest, in the general case, the substitution

$$\mu-v=\sqrt{(\mu^2-1)}e^{-\eta},$$

so that

$$dv/(\mu-v)=d\eta\,,$$

and

$$\frac{1-v^2}{\mu-v}=2\,\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\},$$

while, for

$$v=\pm1\,,\quad e^{\pm\eta}=\sqrt{\left(\frac{\mu+1}{\mu-1}\right)},\quad \text{say } e^{\pm\eta_1}.$$

Hence,

$$\frac{1}{2^{n-\sigma+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu-v)^{n+\sigma+1}}\,dv,$$

which we may denote by T_n^{σ}

$$=\frac{2^{\sigma-1}}{(\mu^2-1)^{\sigma/2}}\int_{-\eta_1}^{\eta_1}\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\}^n e^{\sigma\eta}\,d\eta$$

$$=\frac{2^{\sigma}}{(\mu^2-1)^{\sigma/2}}\int_{0}^{\eta_1}\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\}^n\cosh\sigma\eta\,d\eta\,.$$

This integral is given in HEINE'S 'Kugelfunctionen' (vol. 1, p. 224), and can be readily transformed into others by HEINE'S methods, founded on JACOBI'S transformations for harmonics of the first kind. Thus, if $\cosh\eta=z$, then

$$\cosh\sigma\eta=\frac{1}{1\,.\,3\,\ldots\,(2\sigma-1)}\frac{d^{\sigma}}{dz}(z^2-1)^{\sigma-\frac{1}{2}}\sin h\eta\,.$$

Transforming then the above integral from η to z, we obtain

$$\frac{2^\sigma}{(\mu^2-1)^{\sigma/2}}\frac{1}{1\,.\,3\ldots(2\sigma-1)}\int_1^{z_1}\{\mu - z\sqrt{(\mu^2-1)}\}^n\frac{d^\sigma}{dz}(z^2-1)^{\sigma-\frac{1}{2}}\,dz\,,$$

where $z_1 = \mu/\sqrt{(\mu^2-1)}$.

Performing σ integrations by parts we obtain

$$\frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1\,.\,3\ldots(2\sigma-1)}\int_1^{z_1}\{\mu - z\sqrt{(\mu^2-1)}\}^{n-\sigma}(z^2-1)^{\sigma-\frac{1}{2}}dz.$$

Now write

$$\mu - z\sqrt{(\mu^2-1)} = \frac{1}{\mu+\sqrt{(\mu^2-1)}\cosh\zeta},$$

observing that

$$dz = \frac{\sin h\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^2} \quad\text{and}\quad z^2-1 = \frac{\sin^2 h\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^2}.$$

The integral T_n^σ then becomes

$$\frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1\,.\,3\ldots 2\sigma-1}\int_0^\infty\frac{\sin^{2\sigma}h\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+\sigma+1}}$$

$$=\frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1\,.\,3\ldots(2\sigma-1)}\int_1^\infty\frac{(z^2-1)^{\sigma-\frac{1}{2}}dz}{\{\mu+\sqrt{(\mu^2-1)}\,z\}^{n+\sigma+1}}$$

$$=\frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\frac{2^\sigma}{1\,.\,3\ldots(2\sigma-1)}(\mu^2-1)^{-\sigma/2}\int_1^\infty\frac{\frac{d^\sigma}{dz}(z^2-1)^{\sigma-\frac{1}{2}}\,.\,dz}{\{\mu+\sqrt{(\mu^2-1)}\,z\}^{n+1}}$$

$$=\frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}2^\sigma(\mu^2-1)^{-\sigma/2}\int_0^\infty\frac{\cosh\sigma\zeta\,.\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}.$$

(See Heine's 'Kugelfunctionen,' vol. 1, p. 223).

The zonal harmonic of the second kind takes the comparatively simple form

$$\int_0^\infty\frac{d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}.$$

42. Combining the expression for $\chi_n^\sigma(\mu)\,I_n^\sigma$ given in § 38 with the result of the preceding article, we have

$$(\mu^2-1)^{\sigma/2}\chi_n^\sigma(\mu)\,I_n^\sigma = \frac{(2n)!\,(2n)!\,\gamma^{-2n-1}}{2^{2n+\sigma-1}\,n!\,n!\,(n+\sigma)!\,(n-\sigma)!}\int_0^\infty\frac{\cosh\sigma\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}$$

where $\mu\gamma = \sqrt{(\gamma^2+\epsilon)}$.

Now $\mathfrak{G}_n^\sigma = G_n^\sigma I_n^\sigma$, and this by §§ 35 and 36 will be found equal to

$$\gamma^n \cdot 2\cos\sigma\phi \cdot \frac{2^{n+\sigma}\, n!\, n!}{(2n)!}(1+\mu_1^2)^{\sigma/2}\chi_n^\sigma(\mu_1)(\mu^2-1)^{\sigma/2}\chi_n^\sigma(\mu)\, I^\sigma{}_n,$$

(by § 41) $$= \frac{2^{-n+1}(2n)!}{(n+\sigma)!\,(n-\sigma)!}\, 2\cos\sigma\phi\,(1-\mu_1^2)^{\sigma/2}\chi_n^\sigma(\mu_1)\int_0^\infty \frac{\cosh\sigma\zeta\, d\zeta}{\{\sqrt{(\gamma^2+\epsilon)}+\sqrt{\epsilon}\cosh\zeta\}^{n+1}}$$

$$= \beta_n^\sigma \cdot 2\cos\sigma\phi \cdot f_n^\sigma(\nu')\, F_n^\sigma(\epsilon), \text{ suppose,}$$

where

$$\beta_n^\sigma = (-j)^\sigma \frac{2^{-n-\sigma+1}(2n)!}{n!\,n!},$$

$f_n^\sigma(\nu')$ is defined in § 36; and

$$F_n^\sigma(\epsilon) = \int_0^\infty \frac{\cosh\sigma\zeta\, d\zeta}{\{\sqrt{(\gamma^2+\epsilon)}+\sqrt{\epsilon}\cosh\zeta\}^{n+1}}.$$

Harmonics of Oblate Spheroids.

43. *Internal Harmonics.*—If we put $a^2 - c^2 = \alpha^2$, and in place of the harmonic functions $\chi_n^\sigma(\lambda)$, $P_n(\lambda)$ we substitute two others ${}_1\chi_n^\sigma(\lambda)$ and ${}_1P_n(\lambda)$ the same as the former, except that the sign of λ^2 is everywhere changed unless in the single factor λ should the degree be odd, then the internal harmonics of oblate spheroids are precisely of the same forms as those given in § 35, i. and ii., provided we substitute α for γ, ${}_1\chi_n^\sigma$ for χ_n^σ, and ${}_1P_n$ for P_n.

In § 36 the internal harmonic for a prolate spheroid was expressed in terms of spheroidal coordinates. That article would require to be recast for the oblate spheroid. Without, however, entering upon the details, I shall merely state the result, viz.:—

$$G_n^\sigma = \alpha_n^\sigma \cdot 2\cos\sigma\phi\, f(\epsilon) f_1(\nu),$$

where

$$\alpha_n^\sigma = j^\sigma 2^{n-\sigma}\frac{\{(n+\sigma)!\,(n-\sigma)!\}^2}{(2n)!\,n!\,n!},$$

$$f(\epsilon) = \frac{1}{\pi}\int_0^\pi \{\sqrt{(c^2+\epsilon)} - \sqrt{(a^2+\epsilon)}\cos\theta\}^n \cos\sigma\theta\, d\theta,$$

$$f_1(\nu) = \frac{1}{\pi}\int_0^\pi \{\sqrt{(-c^2-\nu)} - \sqrt{(-a^2-\nu)}\cos\theta\}^n \cos\sigma\theta\, d\theta.$$

44. *External Harmonics.*—If $\alpha^2\mu^2 = \epsilon$ we shall have

$${}_1\chi_n^\sigma(\mu)\, I_n^\sigma = \frac{(2n!)\,(2n)!}{2^{3n+\sigma}\,(n!)^4}\frac{1}{\alpha^{2n+1}}\int_{-1}^{1}\frac{(1-\nu^2)^n}{(\mu - j\nu)^{n+\sigma+1}}\, d\nu.$$

I have deduced this expression from applying the operator

$$\frac{(-1)^n}{2^n n!}\,\frac{(2n)!}{2^n n!\,n!}\left(\frac{\partial^\sigma}{\partial\xi}+\frac{\partial^\sigma}{\partial\eta}\right)\frac{\partial^{n-\sigma}}{\partial z}$$

to the expression for V_n given in § 17, viz., in this case :—

$$\int_\epsilon^\infty\left(1-\frac{z^2}{\lambda}-\frac{\xi\eta}{\alpha^2+\lambda}\right)^n\frac{d\lambda}{(\alpha^2+\lambda)\sqrt{\lambda}},$$

and equating the result to the expression for $G_n{}^\sigma I_n{}^\sigma$, then cutting out $\xi^\sigma+\eta^\sigma$ and making $x, y=0$.

It may be observed that the formula of this article might also with great ease be deduced from the expression for V_n noticed in § 21, viz. :—

$$V_n=\int_0^1(1-\theta^2)^n\,(e^{\theta D}+e^{-\theta D})\,\frac{1}{\sqrt{(x^2+y^2+z^2)}}\,d\theta,$$

or,

$$\int_{-1}^1(1-\theta^2)^n\,e^{\theta D}\,\frac{1}{\sqrt{(x^2+y^2+z^2)}}\,d\theta$$

$$=\int_{-1}^1\frac{(1-\theta^2)^n}{\sqrt{\{x^2+y^2+(z+j\alpha\theta)^2\}}}\,d\theta$$

As, however, this was only established in cases where $x^2+y^2+z^2>\alpha^2$, the above result would not necessarily have been true for points where this condition was not fulfilled.

45. It will be sufficient to state the result of applying the method of this paper to the following expression for V_n, viz. :—

$$V_n=\frac{1}{\pi\alpha^2}\,\frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\iint\left(1-\frac{R^2}{\alpha^2}\right)^{n-\frac{1}{2}}\frac{df\,dg}{\sqrt{\{(f-x)^2+(g-y)^2+z^2\}}}$$

R^2 being put for f^2+g^2.

We thus obtain

$$\chi_n{}^\sigma\left(\frac{z}{\alpha}\right)I_n{}^\sigma=(-1)^{n+\sigma}\,\frac{1}{2^{n+\sigma-1}}\,\frac{(2n)!}{n!\,n!\,(n+\sigma)!}\,\frac{1}{\alpha^{n-\sigma+2}}\,\frac{d^{n+\sigma}}{dz}\int_0^\alpha\frac{\left(1-\frac{R^2}{\alpha^2}\right)^{n-\frac{1}{2}}R\,dR}{\sqrt{(z^2+R^2)}}$$

or, by putting $R=\alpha\sin\theta$, $z=\alpha\mu$,

$$\chi_n{}^\sigma(\mu)\,I_n{}^\sigma=\frac{1}{2^{n+\sigma-1}}\,\frac{(2n)!}{n!\,n!}\,\frac{1}{\alpha^{2n+1}}\,\frac{1}{\pi}\int_0^\pi\int_0^\pi\frac{\cos^{2n}\theta\sin\theta\,d\theta\,d\psi}{(\mu+j\sin\theta\cos\psi)^{n+\sigma+1}}.$$

The formulæ of this and the preceding article include, as particular cases, expressions for ${}_1P_n(\mu)\,I_n{}^\sigma$.

46. The value found for ${}_1\chi_n{}^\sigma(\mu)\,\mathrm{I}_n{}^\sigma$ in § 44 leads by the same process as in § 39 to

$$ {}_1\mathrm{T}_n{}^\sigma = {}_1\chi_n{}^\sigma(\mu)\int_\mu^\infty \frac{d\lambda}{(\lambda^2+1)^{\sigma+1}\{{}_1\chi_n{}^\sigma(\lambda)\}^2} = \frac{1}{2^{n-\sigma+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu-jv)^{n+\sigma+1}}\,dv. $$

$$ {}_1\mathrm{P}_n(\mu)\int_\mu^\infty \frac{d\lambda}{(\lambda^2+1)\{{}_1\mathrm{P}_n(\lambda)\}^2} = \frac{1}{2^{n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu-jv)^{n+1}}\,dv. $$

47. The function ${}_1\mathrm{T}_n{}^\sigma$ may be expressed as an integral analogous to that in § 41, viz.:—

$$ \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\,2^\sigma(\mu^2+1)^{-\sigma/2}\int_0^\infty \frac{\cosh\sigma\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2+1)}\cosh\zeta\}^{n+1}} $$

(Heine's 'Kugelfunctionen,' vol. 2, p. 129).

48. The external harmonic may also be expressed in terms of spheroidal coordinates in the form

$$ \mathfrak{G}_n{}^\sigma = \beta_n{}^\sigma \,.\, 2\cos\sigma\phi \,.\, \mathrm{F}(\epsilon)\,f_1(v), $$

where

$$ \beta_n{}^\sigma = j^\sigma\frac{2^{-n-\sigma+1}(2n)!}{n!\,n!}, $$

$f_1(v)$ is defined in § 43, and

$$ \mathrm{F}(\epsilon) = \int_0^\infty \frac{\cosh\sigma\zeta}{\{\sqrt{\epsilon}+\sqrt{(\alpha^2+\epsilon)}\cosh\zeta\}^{n+1}}\,. $$

Expressions for the Reciprocal of the Distance between Two Points.

49. The meanings of $\mathrm{H}_n{}^\sigma$ and $\mathrm{G}_n{}^\sigma$ having been changed, the expansion of the reciprocal of the distance between an outside point *fgh* and an inside *xyz* requires re-statement.

Referring to § 34, series i. and ii., and to the remark made in § 38, we have

$$ \mathrm{H}_n\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\mathrm{H}_n{}^\sigma(x,y,z) = \frac{(2n)!}{2^n n!\,n!}\,\frac{(n+\sigma)!}{2^{2\sigma}n!\,\sigma!}\left(\frac{\partial^\sigma}{\partial\xi}+\frac{\partial^\sigma}{\partial\eta}\right)\frac{\partial^{n-\sigma}}{\partial z}(\xi^\sigma+\eta)\,z^{n-\sigma} $$

$$ = \frac{(2n)!}{2^n n!\,n!}\,\frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma-1}n!}; $$

therefore, by § 9

$$ \iint(\mathrm{H}_n{}^\sigma)^2 ds = \frac{4\pi}{2n+1}\,\frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma-1}\,n!\,n!}. $$

The required expansion is therefore

$$ \tfrac{1}{2}\,\mathrm{I}_0(f,g,h) + \ldots + 2^{2\sigma-2}(2n+1)\,\frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\,\mathrm{G}_n{}^\sigma(x,y,z)\,\mathrm{G}_n{}^\sigma(f,g,h) + \ldots $$

in which may be entered any of the foregoing forms investigated for inside and outside harmonics.

By §§ 36 and 42, for example, this expansion in the case of the prolate spheroid for $\epsilon_1 v_1' \phi_1$ and an inner point $\epsilon_2 v_2' \phi_2$ may be thrown into the form:—

$$\mathrm{F}_0(\epsilon_1) + \ldots + (2n+1)\frac{(n+\sigma)!\,(n-\sigma)!}{n!\,n!} 2\cos\sigma(\phi_1 - \phi_2) f_n^{\sigma}(v_1') f_n^{\sigma}(v_2') f_n^{\sigma}(\epsilon_2) \mathrm{F}_n(\epsilon_1) + \ldots$$

Cylindrical Harmonics.

50. The harmonics pertaining to an elliptic cylinder may be derived from those of the ellipsoid. I shall, however, confine myself to the case of the circular cylinder, which is a limiting form of the prolate spheroid.

Referring to § 36 and observing that when n becomes infinite $(n+\sigma)!\,(n-\sigma)! : n!\,n!$ is ultimately a ratio of equality, and that $2^{2n}\,n!\,n!/(2n)!$ is ultimately $\sqrt{(\pi n)}$, if we make $a = 0$, $c = \gamma = \infty$, put $\epsilon = \rho^2$, $\gamma^2 + v' = z^2$ and assume $n = \gamma\lambda$, we shall have

$$f_n^{\sigma}(\epsilon) = \gamma^n \frac{1}{\pi}\int_0^{\pi} e^{-\lambda\rho\cos\theta}\cos\sigma\theta\, d\theta = \gamma^n \mathrm{J}_{\sigma}(j\lambda\rho)$$

$$\mathrm{F}_n^{\sigma}(\epsilon) = \gamma^{-n-1}\int_0^{\infty} e^{-\lambda\rho\cosh\zeta}\cosh\sigma\zeta\, d\zeta = \gamma^{-n-1}\,\mathrm{K}_{\sigma}(j\lambda\rho).$$

In regard to $f(v')$ let us suppose n and σ both even or both odd, then comparing this integral with the expressions ii. and iv. in § 34, and inverting the series ii. we may put

$$f_n^{\sigma}(v') = \mathrm{A}\left\{1 - \frac{(n-\sigma)(n+\sigma+1)}{1.2}\frac{z^2}{\gamma^2} + \frac{(n-\sigma)(n-\sigma-2)(n+\sigma+1)(n+\sigma+3)}{1.2.3.4}\frac{z^4}{\gamma^4}\,\&c.\right\}$$

where

$$\mathrm{A} = (-j)^n \frac{1}{\pi}\int_0^{\pi}\cos^n\theta\cos\sigma\theta\, d\theta,$$

the limit of which when n is infinite is $\sqrt{(2/n\pi)}$.

The function $f(v')$ in this case therefore reduces to

$$\gamma^n \sqrt{\frac{2}{\pi n}}\cos\lambda z.$$

In like manner, if $n - \sigma$ be odd, we should find

$$\gamma^n \sqrt{\frac{2}{\pi n}}\sin\lambda z.$$

Substituting these various expressions in the formula of § 49 we find for the general term

$$\frac{8}{\pi\gamma}\cos\sigma(\phi_1-\phi_2)\cos\lambda z_1\cos\lambda z_2\,\mathrm{K}_\sigma(j\lambda\rho_1)\,\mathrm{J}_\sigma(j\lambda\rho_2)$$

or, since the origin is arbitrary,

$$\frac{4}{\pi\gamma}\,2\cos\sigma(\phi_1-\phi_2)\cos\lambda(z_1-z_2)\,\mathrm{K}_\sigma(j\lambda\rho_1)\,\mathrm{J}_\sigma(j\lambda\rho_2).$$

Since $n=\gamma\lambda$, if we suppose $n+1=\gamma(\lambda+\partial\gamma)$ we shall have $+1/\gamma=\partial\lambda$, and the reciprocal of the distance between the two points is

$$\Sigma\frac{4}{\pi}\,2\cos\sigma(\phi_1-\phi_2)\int_0^\infty\cos\lambda(z_1-z_2)\,\mathrm{K}(j\lambda\rho_1)\,\mathrm{J}(j\lambda\rho_2)\,d\lambda.$$

When $\sigma=0$, the term must be divided by 2.

Paraboloidal Harmonics.

51. Any given harmonic of the circular cylinder, considered as a limiting case of a harmonic of the prolate spheroid, is such that, of the characteristics θ in § 3 pertaining to it, some are infinite and others finite. For the infinite values the quantity $\gamma^2+\theta$ may be infinite or finite, but there will be an infinite number of each variety, and the total number of θ's is, as we have seen, of the same order of infinity as γ.

The characteristics which pertain to any given paraboloidal harmonic are in like manner partly finite and partly infinite, there being an infinite number of each, but the order of infinity of the total number is of the same order as $\sqrt{\gamma}$.

By moving the origin of coordinates from the centre to the lower focus of the prolate spheroid, the axis of which is supposed to be vertical, and putting $\epsilon=2\gamma\epsilon'$, $\upsilon'=-2\gamma\upsilon'$, then making γ infinite, we find for the equation of any paraboloid of the system

$$x^2+y^2=4\epsilon'(\epsilon'+z),$$

and for the orthogonals

$$x^2+y^2=4\upsilon'(\upsilon'-z).$$

The forms of the harmonics may be readily deduced from the expressions for G_n^σ and $\mathfrak{G}_n^\sigma$ contained in §§ 36 and 42. Thus, when γ is large, we may put

$$f(\epsilon)=f(2\gamma\epsilon')=\gamma^n\frac{1}{\pi}\int_0^\pi\left(1-\sqrt{\frac{2\epsilon'}{\gamma}}\cos\theta\right)^n\cos\sigma\theta\,d\theta,$$

or, if we write herein $\kappa^2\gamma=2n^2$, the expression ultimately becomes

$$\gamma^n\frac{1}{\pi}\int_0^\pi e^{-\kappa\sqrt{\epsilon'}\cos\theta}\cos\sigma\theta\,d\theta=\gamma^n\,\mathrm{J}_\sigma(j\kappa\sqrt{\epsilon'}).$$

Similarly

$$f(v) = \gamma^n \, \mathrm{J}_\sigma(\kappa \sqrt{v'}),$$

and

$$\mathrm{F}(\epsilon) = \gamma^{-n-1} \int_0^\infty e^{-\kappa \sqrt{\epsilon'} \cosh \zeta} \cosh \sigma\zeta \, d\zeta,$$
$$= \gamma^{-n-1} \, \mathrm{K}_\sigma(j\kappa \sqrt{\epsilon'}).$$

The expansion for the reciprocal of the distance becomes

$$\Sigma \, 4 \cos \sigma(\phi_1 - \phi_2) \frac{n}{\gamma} \mathrm{J}_\sigma(j\kappa \sqrt{\epsilon_2'}) \, \mathrm{J}_\sigma(\kappa \sqrt{\nu_2'}) \, \mathrm{K}_\sigma(j\kappa \sqrt{\epsilon_1'}) \, \mathrm{J}_\sigma(\kappa \sqrt{\nu_1'}).$$

When $\sigma = 0$, this term must be divided by 2.

Since $\kappa^2\gamma = 2n^2$, we may put $(\kappa + \delta\kappa)^2 \gamma = 2(n+1)^2$, and, therefore, $n/\gamma = \frac{1}{2} \kappa \, \delta\kappa$. Hence the required reciprocal is

$$\Sigma \, 2 \cos \sigma(\phi_1 - \phi_2) \int_0^\infty \mathrm{J}_\sigma(j\kappa \sqrt{\epsilon_2'}) \, \mathrm{J}_\sigma(\kappa \sqrt{\nu_2'}) \, \mathrm{K}_\sigma(j\kappa \sqrt{\epsilon_1'}) \, \mathrm{J}_\sigma(\kappa \sqrt{\nu_1'}) \, \kappa \, d\kappa,$$

the term, in which $\sigma = 0$, being divided by 2.

Electrical Capacities.

52. The electrical capacity of a conductor in the form of the surface obtained by inverting a prolate spheroid, with regard to one of its foci, is readily shown to be

$$\frac{\mathrm{R}^2}{2\gamma} \left\{ \log \frac{c+\gamma}{c-\gamma} + 3 \left(\log \frac{c+\gamma}{c-\gamma} - 2 \frac{\gamma}{c} \right) + 5 \left(\log \frac{c+\gamma}{c-\gamma} - \frac{6c\gamma}{3c^2 - \gamma^2} \right) + \&c. \right\}.$$

This series is ultimately convergent, but when γ is nearly equal to c it is inappropriate. When $c = \gamma$ we come to paraboloids, and it may be shown from the expression obtained in the preceding article that the capacity of a cardioide of revolution $r = 2a(1 - \cos\theta)$ is

$$4a^2 \int_0^\infty \frac{\mathrm{K}_0(j\kappa \sqrt{a})}{\mathrm{J}_0(j\kappa \sqrt{a})} \kappa \, d\kappa.$$

Green has worked out completely the case of a body freely charged, somewhat resembling the cardioide of revolution, but with a conical hollow whose vertex is not a cusp, and it will be found that the capacity is very nearly the same as that of a sphere of equal volume. I see no way, however, of evaluating the above integral so as to make a similar comparison, either as regards volume or surface.

The capacity of an anchor-ring without an aperture, the radius of whose axis is a, may be found from Green's Function by means of the series in § 50, viz., it is

$$\frac{8a^2}{\pi} \int_0^\infty \frac{\mathrm{K}_0(j\lambda a)}{\mathrm{J}_0(j\lambda a)} \, d\lambda.$$

Expression of any Ellipsoidal Harmonic in terms of the Conjugate System.

53. Let $u = 1/r$, where $r^2 = (x-f)^2 + (y-g)^2 + (z-h)^2$, and let $v = \mathrm{V}_n(x-p, y-q, z-r)$ be a potential arising from an ellipsoid with its centre at p, q, r and its axes parallel to the coordinate axes. We are to consider the invariant operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^n$$

upon uv; or, if we take the suffix 1 to be put to differential operators upon u only, and the suffix 2 to those on v only, the operation in question is

$$2^n \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^n uv.$$

Now this may be represented by a series of terms of the form

$$\mathrm{A}_n{}^\sigma \mathrm{H}_n{}^\sigma\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right) \mathrm{H}_n{}^\sigma\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right) uv$$

where

$$2^n n!\ \mathrm{A}_n{}^\sigma = \mathrm{H}_n{}^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \mathrm{H}_n{}^\sigma(x, y, z).$$

If we suppose x, y, z put equal to zero after the operations are performed, this result is clearly the same as

$$\mathrm{A}_n{}^\sigma \mathrm{H}_n{}^\sigma\left(\frac{\partial}{\partial f}, \frac{\partial}{\partial g}, \frac{\partial}{\partial h}\right) \mathrm{H}_n{}^\sigma\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}, \frac{\partial}{\partial r}\right) \frac{\mathrm{V}_n(p, q, r)}{\sqrt{f^2+g^2+h^2}}$$
$$= \mathrm{A}_n{}^\sigma (2n)!\, \frac{\mathrm{H}_n{}^\sigma(f, g, h)}{d^{2n+1}}\, \mathrm{G}_n{}^\sigma(p, q, r), \quad \S\S\ 7 \text{ and } 20,$$

where $d^2 = f^2 + g^2 + h^2$.

Now let the axes of coordinates be transformed in such a way that the new axis of z is in the direction of the radius vector to fgh; the above operator, which is an invariant, can then be thrown into the form

$$\frac{(2n)!}{n!\ n!}\frac{\partial^n}{\partial z_1}\frac{\partial^n}{\partial z_2} + \ldots + \frac{2^{2\sigma}(2n)!}{(n+\sigma)!\ (n-\sigma)!}\left(\frac{\partial^\sigma}{\partial \xi_1}\frac{\partial^\sigma}{\partial \eta_2} + \frac{\partial^\sigma}{\partial \xi_2}\frac{\partial^\sigma}{\partial \eta_1}\right)\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^{n-\sigma} + \ldots$$

The new coordinates of fgh are 0, 0, d; let those of p, q, r be p', q', r', and let the transformed values of u, v be u', v'. We may omit all but the first term of the operator, inasmuch as the others give vanishing values, and the result is

$$\frac{(2n)!}{n!\ n!}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^n u'v'$$
$$= \frac{(-1)^n}{d^{n+1}} \cdot \frac{(2n)!}{n!}\left(\frac{\partial}{\partial z_2}\right)^n v',$$

or, returning to the old axes,

$$(-1)^n \cdot \frac{(2n)!}{n!}\frac{1}{d^{n+1}}\left(\frac{\partial}{\partial h}\right)^n \mathrm{V}_n(x-p,\ y-q,\ z-r),$$

where

$$\frac{\partial}{\partial h} = \frac{f}{d}\frac{\partial}{\partial x} + \frac{g}{d}\frac{\partial}{\partial y} + \frac{h}{d}\frac{\partial}{\partial z}.$$

We have, finally, putting $x, y, z = 0$,

$$\left(f\frac{\partial}{\partial p} + g\frac{\partial}{\partial q} + h\frac{\partial}{\partial r}\right)^n \mathrm{V}_n(p, q, r) = 2^n\, n!\ n!\ \Sigma\ \frac{\mathrm{H}_n^{\sigma}(f, g, h)\ \mathfrak{S}_n^{\sigma}(p, q, r)}{\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^{\sigma}(x, y, z)}$$

By equating the coefficients of the various powers and products of f, g, h we may obtain any assigned differential coefficient of V_n.

Addition Theorems for Spherical Harmonics of the Second Kind and for Bessel's *Functions.*

54. The method of the preceding article may be used for establishing addition theorems for harmonics of the second kind, but before applying it for this purpose it will be convenient to make a few preliminary observations for the purpose of bringing those harmonics under a similar system of definitions as those for harmonics of the first kind. I begin with a geometrical interpretation of

$$\int_0^{\infty} \frac{du}{\mu + \sqrt{\mu^2 - 1}\cosh u},$$

where μ lies between 1 and -1.

Let P be any point whose projection on the axis of z is N, so that $\mathrm{ON} = z$, $\mathrm{PN} = \rho$, $\mathrm{OP} = r$, $\cos \mathrm{NOP} = \mu$; then the potential at P due to a uniform distribution of matter on ON is

$$\int_0^z \frac{d\zeta}{\sqrt{(z-\zeta)^2 + \rho^2}} = \tfrac{1}{2}\log\frac{r+z}{r-z} = \tfrac{1}{2}\log\frac{1+\mu}{1-\mu}.$$

In like manner, if we draw PM at right angles to OP to meet the axis of z in M, we shall find

$$\frac{Q_0}{r} = \int_{OM}^{\infty} \frac{d\zeta}{\zeta\sqrt{(\zeta - z)^2 + \rho^2}} = \frac{1}{r} \cdot \tfrac{1}{2} \log \frac{1+\mu}{1-\mu}$$

$$= \frac{1}{r}\int_0^{\mu} \frac{dv}{1 - v^2}$$

$$= \frac{1}{r}\int_0^{\infty} \frac{\mu\, du}{\mu^2 + (1-\mu^2)\cosh^2 u}, \text{ where } v = \mu \tanh u$$

$$= \frac{1}{2r}\int_0^{\infty} \frac{du}{\mu + j\nu\cosh u} + \frac{1}{2r}\int_0^{\infty} \frac{du}{\mu - j\nu\cosh u}.$$

Also

$$-j\tfrac{1}{2}\pi\frac{P_0}{r} = \frac{1}{2r}\int_0^{\infty} \frac{du}{\mu + j\nu\cosh u} - \frac{1}{2r}\int_0^{\infty} \frac{du}{\mu - j\nu\cosh u},$$

therefore

$$\frac{Q_0 - j\tfrac{1}{2}\pi P_0}{r} = \frac{1}{r}\int_0^{\infty} \frac{du}{\mu + j\nu\cosh u}.$$

We shall write R_n for $Q_n - j\tfrac{1}{2}\pi P_n$, so that

$$\frac{R_0}{r} = \int_0^{\infty} \frac{du}{z + j\rho\cosh u}.$$

Following now the corresponding definitions for harmonics of the first kind, we may write

$$n!\, Q_n = (-1)^n r^{n+1}\left(\frac{\partial}{\partial z}\right)^n \frac{Q_0}{r}$$

$$n!\, Q_n^{\sigma} = (-1)^n r^{n+1}\left(\frac{\partial}{\partial z}\right)^{n-\sigma}\left(\frac{\partial^{\sigma}}{\partial \xi} + \frac{\partial^{\sigma}}{\partial \eta}\right)\frac{Q_0}{r}$$

$$n!\, Q'^{\sigma}_n = (-1)^n r^{n+1} j\left(\frac{\partial}{\partial z}\right)^{n-\sigma}\left(\frac{\partial^{\sigma}}{\partial \xi} - \frac{\partial^{\sigma}}{\partial \eta}\right)\frac{Q_0}{r}.$$

It is obvious that similar definitions also apply to the R-functions.

In carrying out the differential operations indicated we may observe that, where the function to be operated upon does not contain the longitudinal angle ϕ, the following results will be useful, viz.,

$$2^{\sigma}\frac{\partial^{\sigma}}{\partial \xi} = e^{-\sigma\phi j}\left(\frac{\partial}{\partial\rho} - \frac{\sigma-1}{\rho}\right)\left(\frac{\partial}{\partial\rho} - \frac{\sigma-2}{\rho}\right)\cdots\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$= e^{-\sigma\phi j} f_{\sigma}\left(\frac{\partial}{\partial\rho}\right) \text{ suppose,}$$

and

$$2^{\sigma}\frac{\partial^{\sigma}}{\partial \eta} = e^{\sigma\phi j} f_{\sigma}\left(\frac{\partial}{\partial\rho}\right).$$

We then have, readily,

$$P_n^{\sigma} = j^{\sigma} 2^{-\sigma} \,.\, 2\cos\sigma\phi \,.\, p_n^{\sigma}$$
$$R_n^{\sigma} = j^{\sigma} 2^{-\sigma} \,.\, 2\cos\sigma\phi \,.\, r_n^{\sigma},$$

where

$$p_n^{\sigma} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos\sigma\theta}{(\mu + j\nu\cos\theta)^{n+1}}\, d\theta \qquad r_n^{\sigma} = \int_0^{\infty} \frac{\cosh\sigma u\, du}{(\mu + j\nu\cosh u)^{n+1}}.$$

When $\sigma = 0$, these general results must be divided by 2. There are also corresponding functions with $\sin\sigma\phi$ substituted for $\cos\sigma\phi$.

55. [*March* 4, 1891.—The expressions contained in § 54 may also be readily obtained if we notice that R_0/r is equal to the integral

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{z + jx\cosh u + y\sinh u},$$

or,

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{z + \frac{1}{2}j\,(\xi e^{-u} + \eta e^{u})}.$$

Let us find the result, in a series, of the operation denoted by

$$\frac{(-1)^n}{n!}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right)^n \frac{R_0}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A).$$

One method is given in § 53, but inasmuch as the expression (A) is a spherical harmonic in f, g, h we may at once put it equal to

$$\Sigma\, A_{\sigma} H_n^{\sigma}(f, g, h)\, H_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{R_0}{r},$$

and determine A_{σ} as in §§ 10, 49. The result is

$$\Sigma\, 2^{2\sigma-1}\frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\, H_n^{\sigma}(f, g, h)\,\frac{(-1)^n}{n!}\left(\frac{\partial^{\sigma}}{\partial\xi^{\sigma}} + \frac{\partial^{\sigma}}{\partial\eta^{\sigma}}\right)\frac{\partial^{n-\sigma}}{\partial z^{n-\sigma}}\,\frac{R_0}{r}.\Big]$$

If now $d_1\mu_1\phi_1$ and $d_2\mu_2\phi_2$ be the polar coordinates of fgh and xyz respectively, the series first found, when the expressions in the preceding article are entered in it, will become

$$\frac{d_1^n}{d_2^{n+1}}\left\{P_n R_n + \ldots + \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\, p_n^{\sigma} r_n^{\sigma}\, 2\cos\sigma(\phi_1 - \phi_2) + \ldots\right\}.$$

But the operation (A) also gives

$$\frac{d_1^n}{d_2^{n+1}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{(h + j \cosh u\, f + \sinh u\, g)^n}{(z + j \cosh u\, x + \sinh u\, y)^{n+1}}\, du$$

Hence

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\{\mu_1 + j\nu_1 \cosh (u + j\phi_1)\}^n}{\{\mu_2 + j\nu_2 \cosh (u + j\phi_2)\}^{n+1}}\, du$$

$$= \mathrm{P}_n(\mu_1)\, \mathrm{R}_n(\mu_1) + \ldots + (-1)^\sigma \frac{n!\, n!}{(n+\sigma)!\, (n-\sigma)!} p_n^\sigma(\mu_1)\, r_n^\sigma(\mu_2)\, 2 \cos \sigma (\phi_1 - \phi_2) + \ldots$$

We may denote the left-hand side of this identity by $\mathfrak{R}_n$ and put $\mathfrak{R}_n = \mathfrak{Q}_n - j\frac{1}{2}\pi \mathfrak{P}_n$. Then, since

$$\mathfrak{P}_n = \mathrm{P}_n(\mu_1)\, \mathrm{P}_n(\mu_2) + \ldots + (-1)^\sigma \frac{n!\, n!}{(n+\sigma)!\, (n-\sigma)!} p_n^\sigma(\mu_1)\, p_n^\sigma(\mu_2)\, 2 \cos \sigma (\phi_1 - \phi_2) + \ldots$$

We must also have

$$\mathfrak{Q}_n = \mathrm{P}_n(\mu_1)\, \mathrm{Q}_n(\mu_2) + \ldots + (-1)^\sigma \frac{n!\, n!}{(n+\sigma)!\, (n-\sigma)!} p_n^\sigma(\mu_1)\, q_n^\sigma(\mu_2)\, 2 \cos \sigma (\phi_1 - \phi_2) + \ldots$$

56. The symbol $\mathfrak{Q}_n$ has been used to denote the real part of the result of the differential operation

$$\frac{(-1)^n}{n!} r^{n+1} \left(\frac{\partial}{\partial h}\right)^n \int_0^\infty \frac{du}{z + j\rho \cosh u}.$$

To make it clear what this coefficient is and how related to the corresponding function $\mathfrak{P}_n$ it may be noticed that the former, as found in § 53, is at once recognisable as the coefficient of d_1^n/d_2^{n+1} in the expansion of

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{d_2(\mu_2 + j\nu_2 \cosh u) - d_1(\mu_1 + j\nu_1 \cosh u)},$$

or, of

$$\frac{1}{r_{12}} \left\{ \log \frac{r_{12} + d_2\mu_2 - d_1\mu_1}{r_{12} - d_2\mu_2 + d_1\mu_1} - j\,\tfrac{1}{2}\pi \right\}$$

where

$$r_{12}^2 = (x - f)^2 + (y - g)^2 + (z - h)^2.$$

If we strike out the second term within the brackets we obtain the generating function of $\mathfrak{Q}_n$.

The expansion in the preceding article has been proved in the case when μ_2 is not

greater than 1, that case admitting of a geometrical interpretation, but a similar relation, from its character, must also hold when μ_2 is greater than 1.

Reduction to BESSEL'S *Functions.*

57. If we make n infinite and write $n\nu_1 = \lambda_1$ and $n\nu_2 = \lambda_2$, where λ_1, λ_2 are finite, the left-hand side of the integral in § 55, expressing Q_n, will become

$$\begin{aligned} &\tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\lambda_1 \cosh(u+j\phi_1) - j\lambda_2 \cosh(u+j\phi_2)}\, du, \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\Lambda \cosh(u+j\Phi)}\, du, \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\Lambda \cosh u}\, du, \end{aligned}$$

where

$$\begin{aligned} \Lambda \cos\Phi &= \lambda_1 \cos\phi_1 - \lambda_2 \cos\phi_2, \\ \Lambda \sin\Phi &= \lambda_1 \sin\phi_1 - \lambda_2 \sin\phi_2, \end{aligned}$$

and, therefore,

$$\Lambda^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos(\phi_1 - \phi_2).$$

Following HEINE, to whose work I am indebted for the integrals expressing R_0 in §§ 54 and 55, I shall denote the external BESSEL'S function by K, that is to say, the limiting form of Q in the same way as J is used as the limiting form of P. The above integral is therefore equal to $K_0(\Lambda) - j\tfrac{1}{2}\pi J_0(\Lambda)$. The series in § 55 then readily give

$$J_0(\Lambda) = J_0(\lambda_1) J_0(\lambda_2) + \ldots + 2J_\sigma(\lambda_1) J_\sigma(\lambda_2) \cos\sigma(\phi_1 - \phi_2) + \ldots$$

$$K_0(\Lambda) = J_0(\lambda_1) K_0(\lambda_2) + \ldots + 2J_\sigma(\lambda_1) K_\sigma(\lambda_2) \cos\sigma(\phi_1 - \phi_2) + \ldots$$

VII. *The Conditions of Chemical Change between Nitric Acid and certain Metals.*

By V. H. Veley, *M.A., the University Museum, Oxford.*

Communicated by Professor Odling, *F.R.S.*

Received October 23,—Read November 27, 1890.

Table of Contents.

Page.

Introduction 280
Millon's Experiments 281
H. Ste.-Claire Deville's Experiments 281
Russell's Experiments 281
Armstrong and Acworth's Experiments 282
Divers' Experiments 283
Apparatus and Method of Investigation 284
Purification of Nitric Acid 287
The Metals used 288
Method of Estimation of Nitrous Acid 288
Copper and Nitric Acid 290
(1) The Reaction between Copper and Nitric Acid depends upon the Presence of Nitrous Acid 290
(2) The Amount of Chemical Change between Copper and Nitric Acid depends upon the Amount of Nitrous Acid 294
Effect of Nitric Acid upon the Chemical Change between Copper and Nitrous Acid . . 298
Mercury and Nitric Acid 300
(1) The Reaction between Mercury and Nitric Acid depends upon the Presence of Nitrous Acid 300
(2) The Amount of Chemical Change between Mercury and Nitric Acid depends upon the Amount of Nitrous Acid 303
Effect of Nitric Acid upon the Chemical Change between Mercury and Nitrous Acid . . 305
Bismuth and Nitric Acid 305
(1) The Reaction between Bismuth and Nitric Acid depends upon the Presence of Nitrous Acid 305
(2) The Amount of Chemical Change between Bismuth and Nitric Acid depends upon the Amount of Nitrous Acid 306
Effect of Nitric Acid upon the Chemical Change between Bismuth and Nitrous Acid . . 311
The Interpretation of the Results 312
Reactions between Hydrogen and Metallic Nitrates in Presence of Nitric Acid 313
Conclusions 316

29.5.91.

Introduction.

The science of chemistry presents few problems at once of such technical importance and such almost infinite complexity as the transformations of nitric acid; few problems, therefore, have been studied with so much diligence by a number of investigators. The preparation of certain metallic nitrates, the chemical changes and correlated electromotive forces of certain forms of batteries, require but to be mentioned, while the preparation of nitric oxide gas from metallic copper and moderately concentrated nitric acid forms both an exercise set before the veriest tyro in chemistry and the subject of several extensive memoirs.

Within the last ten years no investigations have attracted so much attention as those which have proved beyond doubt that chemical changes hitherto regarded as almost fundamental, and as occurring between two substances, whether elementary or compound, do not in reality take place except in presence of some third substance. Thus, Dixon* has shown that carbonic oxide will not combine with oxygen; H. B. Baker,† that sulphur, phosphorus, and carbon do not burn in oxygen; Wanklyn,‡ as also Cowper,§ that certain metals will not burn in chlorine; and other results might also be enumerated. In all these cases the necessary *tertium quid* is water.

The main object of the present communication is to endeavour to demonstrate that some metals will not enter into chemical change with dilute nitric acid save in the presence of nitrous acid. It is also hoped that the method of investigation herein applied for the elucidation of the problem presents some degree of novelty, more particularly in that hitherto it has been customary to examine the products of the chemical changes when both metal and acid were at rest, whereas in the experiments to be described both were in continuous and regular motion.

On previous occasions‖ I have ventured to point out the disadvantages inherent to the former method; it is not, therefore, considered necessary to discuss this point. Further, an especial endeavour has been made to explain the various chemical changes investigated without recourse to the convenient, and it is to be feared, often misused hypothesis of "nascent hydrogen." For it appears improbable that in many cases hydrogen plays any part whatever, while to connote certain properties of a kind of matter of which we can have no cognition, is to carry the powers of imagination beyond their legitimate bounds. The results obtained by the analytical or so-called chemical methods and given in detail in this paper, receive some confirmation by those obtained by electrical methods and described in a succeeding paper.

* 'Phil. Trans.,' 1884, pp. 617–684.

† 'Phil. Trans.,' 1888, A., pp. 571–591; also 'Chem. Soc. Journ.,' 1885 (*Trans.*), pp. 349–352.

‡ 'Chem. News,' vol. 20, p. 271.

§ 'Chem. Soc. Journ.,' 1883 (*Trans.*), pp. 153–155.

‖ 'Chem. News,' vol. 55, pp. 221, 222; and 'Chem. Soc. Journ.,' 1889 (*Trans.*), p. 362.

Millon's Experiments.*

Some years ago Millon published an interesting memoir, curiously overlooked by writers of text-books and dictionaries, upon the promotion on the one hand of the chemical change between nitric acid and metals by the presence of nitrous acid, produced either by passing in a few bubbles of nitric oxide gas or adding a few drops of a solution of a metallic nitrite, and its prevention on the other by the presence of some substance such as ferrous sulphate, which removes the nitrous acid as fast as it might be formed. His investigations were conducted principally with copper and nitric acid of sp. gr. 1·07 (10 per cent. concentration) heated to 20° C., but were, to judge from the paper, purely qualitative; it will be seen that the experiments to be described, most of which were conducted in ignorance of Millon's work, differ from his only in this respect, that acids varying from 25 to 35 per cent. concentration, and temperatures from 25° to 35° were used. Millon remarks that the oxidation of copper by nitric acid is dependent upon the following conditions, (i.) the concentration of the acid, (ii.) the temperature, (iii.) the presence of nitric oxide, and (iv.) the solubility of the products formed in the acid mixture. Similar results were also obtained with silver, mercury, bismuth, and tin. Millon explains the whole course of the change in the following words :—

"La marche générale des oxydations s'explique sans peine; l'acide nitreux forme des nitrites de cuivre, de mercure, d'argent qui sont détruits par l'acide nitrique à mesure qu'ils se forment; cette destruction donne naissance au deutoxide d'azote qui, retrouvant de l'acide nitrique refait de l'acide nitreux; d'où résulte une nouvelle attaque et une nouvelle destruction." My experiments tend to show that in the case of the metals copper, mercury, and bismuth, as also of silver from the experiments of Russell (*vide infra*) this is the conclusion of the whole matter.

H. Sainte-Claire Deville's† Experiments.

Deville examined the products formed when zinc dissolves in nitric acid with and without addition of sulphuric or hydrochloric acid; his main purpose was to show that the changes taking place can be explained without recourse to the hypothesis of nascent hydrogen. As however in this paper the changes induced by metallic copper are only incidentally discussed, it is necessary only to refer to these results.

Russell's Experiments.‡

Russell showed that silver does not enter into chemical change with dilute nitric acid, because when hydrogen gas is passed through silver nitrate metallic silver

* 'Comptes Rendus,' vol. 14, 1842, pp. 904–912.

† 'Comptes Rendus,' vol. 70, 1870, pp. 20–26, and 550–557.

‡ 'Chem. Soc. Journ.,' 1874, pp. 3–12.

separates with consequent liberation of nitric acid. The reaction, however, does not merely consist in the replacement of silver by hydrogen, since the amount of the latter absorbed is in considerable excess over the amount of the former equivalent to it; nitrous acid is simultaneously produced in which the silver re-dissolves (when the passage of the hydrogen is stopped) with formation of silver nitrite. The explanation of the chemical changes between silver and nitrous acid in presence of nitric acid is practically the same as that given by MILLON in the case of copper, and will subsequently be alluded to. In further experiments RUSSELL states that a solution of copper nitrate by long standing in contact with hydrogen, judging only from the change of colour, is converted into the nitrite, mercuric nitrate is changed into basic nitrate, while bismuth nitrate is unaltered under these conditions. In another paragraph RUSSELL showed that when silver dissolves in nitric acid it dissolves initially as silver nitrite.

ARMSTRONG AND ACWORTH'S EXPERIMENTS.*

These experimenters published an elaborate investigation mainly upon the nature and composition of the gas evolved when various metals dissolve in nitric acid of different degrees of concentration. These gases are undoubtedly the final outcome of complicated changes taking place both between nitric acid and its reduction products and between these products *inter se*. ARMSTRONG and ACWORTH regard the first change as consisting of the replacement of the hydrogen of the acid by the metal, and the formation of the metallic nitrate; the displaced hydrogen is, however, not evolved as such, but reduces the nitric acid to nitrous acid, hyponitrous acid, hydroxylamine and ammonia according to the equations

$$\text{(i.)}\quad HNO_3 + H_2 = HNO_2 + H_2O,$$
$$\text{(ii.)}\quad HNO_3 + 2H_2 = HNO + 2H_2O,$$
$$\text{(iii.)}\quad HNO_3 + 3H_2 = NH_2OH + 2H_2O,$$
$$\text{(iv.)}\quad HNO_3 + 4H_2 = NH_3 + 3H_2O.$$

As however most of these products can exist neither in the presence of nitric acid, nor in the presence of one another, further changes take place resulting in the formation of nitric and nitrous oxides and of nitrogen, thus :—

$$\text{(v.)}\quad 3HNO_2 = HNO_3 + 2NO + H_2O,$$
$$\text{(vi.)}\quad NH_2OH + HNO_2 = N_2O + 2H_2O,$$
$$\text{(vii.)}\quad HNO_2 + NH_3 = N_2 + 2H_2O.$$

Though it is true that all the compounds given on the right-hand side of the equations, with the possible exception of hyponitrous acid, are formed when certain metals dissolve in nitric acid, yet the question seems to arise whether the products given in equations

* 'Chem. Soc. Journ.,' 1877 (ii.), p. 54.

Nos. i.–iv. result from the successive reduction of nitric acid by so-called nascent hydrogen. Again, it is possible that the formation of certain gases is due not only to the interaction of the products of reduction *inter se*, but also to such an interaction between a portion of the unaltered nitric acid, the reduction products, and the metallic nitrates, in all their various permutations. Indeed Armstrong and Acworth themselves account for the formation of nitrous oxide, when copper dissolves in nitric acid, not to an interaction between hydroxylamine and nitrous acid, but to the reduction of the nitric acid by the cuprous nitrate. This view was further confirmed by the non-production of this gas when silver, a metal analogous in its behaviour to copper, dissolves under the same conditions in nitric acid; for the silver nitrate formed is capable of no further reduction. According to these authors the extent to which reduction takes place when a given metal is dissolved in nitric acid "is a function of the energy developed in the equation $R_n + nHNO_3 \pm aq = nRNO_3 \pm aq + nH$, *i.e.*, in the displacement of the hydrogen of the acid by the metal, including that developed subsequently by the action of the water present on the salt which is produced." This view they regard merely as a "working hypothesis."

Divers' Experiments.*

In the course of this paper, dealing mainly with the conditions of formation of hydroxylamine the author divides metals into two classes as regards their behaviour with nitric acid, viz., "those which yield nitrite and nitrate with water, but exert no further action, forming neither ammonia nor hydroxylamine." In this class are placed silver, mercury, copper, and bismuth. Secondly, those "which form ammonia and generally also hydroxylamine, but do not yield nitrite or nitrous acid, with free nitric acid; on the other hand they readily form nitrite by acting on their own nitrate. To this class belong tin, zinc, cadmium, magnesium, aluminium, lead, iron, and the alkali metals." Again, in another paper, Divers† states his belief "that nitrous acid has only a doubtful existence, and may not have one at all even in the presence of nitric acid." In a foot-note to the former paper Armstrong put forward the view, supported by some experiments, as yet unpublished, "that the primary product of the reduction of nitric acid is nitrous acid, whatever the nature of the reducing agent, be it hydrogen or metal, or arsenious acid for example." Experiments to be detailed in the sequel show that those metals belonging to the first class of Divers, namely, mercury, copper, and bismuth, yield nitrous acid when the conditions are such that the metals enter into chemical change; and an experiment made with a sample of pure assay silver, also pointed to the same conclusion. But other observations made in connection with this enquiry tend to show that those metals which belong to the second class of Divers, though they may form hydroxylamine and ammonia as more ultimate products, yet

* 'Chem. Soc. Journ.,' 1883, pp. 443-466.

† 'Chem. Soc. Journ.,' 1885, p. 231.

undoubtedly they yield nitrous acid as the primary product of reduction, and that, too, in the presence of a considerable excess of nitric acid. For example, some experiments, reserved for a future communication, were conducted with the purest metallic lead procurable; these proved that under certain conditions nitrous acid was formed immediately after a reaction had set in, as shown by the meta-phenylene diamine test, while not a trace of ammonia could be detected by NESSLER'S reagent. In order to afford a further corroboration of the correctness of ARMSTRONG'S conclusion a series of experiments were conducted with various metals, as follows :—about half-a-gram of the metal was placed in 25 c.c. nitric acid of 29 per cent. concentration, and purified from all traces of nitrous acid.

The metal was allowed to dissolve in the acid for five minutes only, and at the same time another similar volume of the same nitric acid was kept under identical conditions. The two quantities of acid were then poured simultaneously into two separate volumes of 90 c.c. of water, and the same volume of meta-phenylene diamine hydrochloride solution added to each mixture. The following metals gave indication of the formation of considerable quantities of nitrous acid, when the reaction with the nitric acid was conducted at ordinary temperatures :—magnesium (in the form of ribbon), steel (pianoforte wire), cobalt, zinc* (three times fractionally distilled, and of such a degree of purity that a portion of the same sample was quite unattacked by dilute sulphuric acid), tin (freshly granulated), cadmium, and arsenic.

The following metals produced only insignificant traces of nitrous acid at ordinary temperature, but an abundance at a temperature of 30° :—nickel, aluminium, and antimony.

In no case was the slightest indication of nitrous acid produced in the blank experiment with nitric acid by itself, nor was any coloration obtained when a small quantity of hydroxylamine hydrochloride was added to the nitric acid, and the mixture kept for the same interval of five minutes at a temperature of 30°.

Hence it would appear that there can be no doubt that when the conditions are such that the metals enter into reaction with nitric acid, nitrous acid is found immediately after chemical change, and that, too, in the presence of a very considerable excess of nitric acid. Thus, the second point of the differentiation of the zinc-tin from the copper-mercury class of metals, as set forth by DIVERS, is not valid. However, it is probable that the *amount* of reaction between the metal and the nitric acid is dependent upon the proportion of nitrous acid in the case of members of the latter, but not of those of the former class.

APPARATUS AND METHOD OF INVESTIGATION.

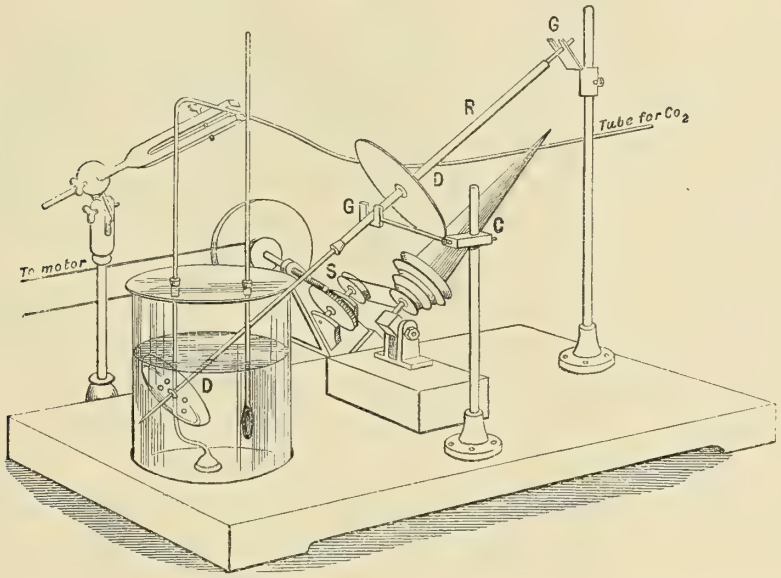
It will be convenient for the elucidation of the text of the present memoir to briefly describe again the apparatus and methods adopted, though a full account has

* I am indebted to Mr. F. PULLINGER for this sample of metallic zinc, in the purification of which considerable labour had been bestowed.

been given in a former paper of mine, entitled, "A Method of Investigating the Dissolution of Metals in Acids" ('Chem. Soc. Journ.' 1889 (*Trans.*), p. 365).

The apparatus consisted in outline of a wooden cone C (Fig. 1)* driven by a small

Fig 1.



water-motor; a brass disc D worked by friction on this cone, and the former could be moved along a brass rod R provided with a scale of 200 mm. and set parallel to the cone; the rod rested above and below upon guides G, G, screwed to a wooden base board. The lower extremity of the rod ended in a hollow ferule, into which was inserted a rubber cork; through this passed a piece of glass tubing, bearing at a suitable distance from the end of the rod a small glass dish D about 70 mm. in diameter, perforated with holes and with slightly upturned edges. The tube and dish were set at about an angle of 45° within a beaker of 1300 c.c. capacity, which contained the nitric acid. The glass dish was set eccentrically on the tube to prevent the metallic sphere, which rested upon the rim, from always revolving on the same axis and thus becoming spheroidal in shape.

To reduce the velocity of rotation of the water-motor, the driving wheel of which revolved about 1560 times per minute, a tangent screw was interposed between it and the cone; in all comparative experiments the rotation of the cone was kept constant.

The acid-containing beaker was covered with a circular glass plate provided with two holes; through one of them passed a piece of quill tubing ending in a thistle funnel, the flow of gas through which served to keep the acid in agitation; through the other hole passed a thermometer of long range.

* I am indebted to the Chemical Society for the use of this illustration, which appeared in their Journal.

In the wooden base board a circular hole was cut of dimensions slightly larger than the beaker, and the whole arrangement placed on a wooden box covered with an iron lid. A small gas flame was placed within the box, the side of which nearest to the observer could be opened by means of a wooden flap; thus, the flame could be moved nearer to or further from the beaker according as the temperature was too low or too high.

In order to compare together the amount dissolved off from the metallic spheres of different areas in the various results, these spheres were weighed before and after each experiment, as also their diameters measured; from the latter the mean area was calculated. Then M, the loss in weight, divided by A, the mean area, is taken as the factor for comparison; this value M/A is for convenience expressed in terms of decimilligrams dissolved off per one square millimetre surface.

In the paper alluded to above, wherein the variation of the amount of metallic copper dissolved off by chromic acid under various conditions was investigated, it was proved that within experimental error concordant values for M/A were obtained, not only when the same copper sphere was dissolved in the same acid, but also when two different spheres were dissolved under the same conditions.

As a preliminary part of this investigation, this same fundamental point was proved in the case of nitric acid; it will be necessary to give only one such series of experiments, to show once for all that the apparatus and method used yields concordant results.

In the following table are given (i.) the weights of a copper sphere at the commencement of each experiment; (ii.) its weight at the conclusion; (iii.) the difference between these two values on the loss in weight, M; (iv.) the mean area A in square mm.; and (v.) the value of M/A. The first result obtained is omitted for a reason which will be discussed in the sequel.

Series I.—Conditions of Experiment:—Sp. gr. $\frac{12}{12}$ of nitric acid = 1·171. Percentage of acid = 28. Volume of acid = 1000 c.c. Temperature 25°.

Weight before in Grams.	Weight after in Grams.	Loss in Grams.	Mean Area in Sq. Mms.	Value of M/A.
4·3465	4·0463	·3002	289·93	10·35
4·0463	3·7673	·279	276·4	10·1
3·7673	3·5035	·2638	263·4	10·0
3·5035	3·2458	·2577	250·63	10·28
3·2458	3·0045	·2413	237·61	10·15
3·0045	2·7713	·2332	225·15	10·33
2·7713	2·5511	·2202	213·34	10·32
2·5511	2·3453	·2058	201·76	10·2
2·3453	2·1558	·1895	189·9	10·0
2·1558	1·9693	·1865	179·73	10·38
1·9693	1·8	·1693	169·35	10·0
1·8	1·6391	·1612	159·25	10·13
			Mean Value	10·19

The mean value of these results is 10·19, with a probable error of ± ·0317; this is within the limits of experimental error if the great variation produced by a very slight alteration in temperature be taken into consideration. In the course of this series of experiments the area of the bullet is reduced by nearly one-half; the amount of acid used up by the copper (assuming, by way of argument, the correctness of the equation $3Cu + 8HNO_3 = 3Cu(NO_3)_2 + 2NO + 4H_2O$) was 9·086 grms., or 2·77 per cent. of the total amount present at the commencement. Here and in all experiments described in the present paper, the amount of nitric acid taken was, unless otherwise stated, far in excess of that required to dissolve the whole of the metallic spheres.

Purification of Nitric Acid.

Nitric acid of sp. gr. 1·41, labelled by the manufacturers as "Pure Special," was purified from nitrous acid, with which it was invariably contaminated, by the following method: a rapid stream of air was forced from a gasholder through (i.) a wash-bottle containing water, and (ii.) a piece of glass tubing about 40 cms. long and 2 cms. diameter, which was filled rather tightly with cotton wool, which served to retain any particles of dust which might be carried over accidentally. The air then passed by means of a thistle funnel into the acid contained in a beaker, which was heated to a temperature not exceeding 35° C., and protected from direct sunlight. The current of air was continued until a small test portion, about 4–5 c.c., when removed and poured into a dilute solution of potassium iodide and starch gave no immediate blue coloration. As stated in my preliminary paper on the reaction between copper and

nitric acid* if the temperature rose to 40° C., or the acid were not properly sheltered, nitrous acid was formed by the decomposition of the nitric acid at a rate faster than that at which it was removed by the current of air. This process of purification is, certainly at first and probably throughout, merely mechanical, the nitrous acid being simply blown off. In some experiments made for another purpose, and conducted in a different manner, it appeared that the last traces of nitrous acid present in a sample of nitric acid could be removed as effectually by a current of hydrogen as by air. The acid, thus purified, was poured as quickly as possible into such a volume of water as would make the mixture of the dilution required.

In successful operations the amount of nitrous acid present in the diluted acid was less than one part in three, but more than one part in five, millions of nitric acid; this quantity was determined by the meta-phenylene diamine reaction (*vide infra*). The specific gravity of each sample of acid was determined by a delicate pyknometer of the form devised by SPRENGEL, and the amount of free acid determined in a portion of the same sample by the usual process of acidimetry.

The Metals used.

Unless otherwise stated the metallic spheres used in this investigation were of the highest degree of purity supplied by Messrs. JOHNSON and MATTHEY. Each set of spheres was made from the same rod or block of metal to ensure so far as possible their homogeneity, and each sphere was about 5 grms. in weight, 10 mm. in diameter, and turned to 0·013 mm.

Before experiments were conducted with any one sphere it was revolved for some time in nitric acid in order to remove any superficial coating of metallic oxide, as also any polishing material, as previous experience had shown that this precaution was necessary.

Herein it is proposed to investigate the conditions of chemical change between nitric acid and the following metals: copper, mercury, and bismuth.

In the several sections a description will be given of the results for each metal, while their interpretation will be discussed in a general summary at the conclusion.

Method of Estimation of Nitrous Acid.

The detection and estimation of nitrous acid† forms an important factor of the results to be detailed in the various sections of the present memoir. In a selection

* 'Roy. Soc. Proc.,' vol. 46, pp. 216–222.

† As DIVERS ('Journ. Chem. Soc.,' 1885, 231) is inclined to doubt the existence of nitrous acid, "even in the presence of nitric acid," the term *nitrous acid* is applied here, and throughout the paper, to imply that kind of matter which liberates iodine from an acidulated solution of potassium iodide, and which gives certain colour reactions with organic bases, such as meta-phenylene diamine hydrochloride, sulphanilic acid, and naphthylamine hydrochloride. The question whether this kind of matter is to be

of the various methods proposed consideration must be paid to the presence of free nitric acid and of metallic nitrate in addition to the nitrous acid, as also to the minute quantities to be estimated. For the latter reason the determination by means of potassium permanganate solution was not available; for the former, that by sulphanilic acid, as requiring a slightly alkaline solution, was inadvisable. The phenylene diamine hydrochloride method, proposed originally by P. Griess, and modified by Preusse and Tiemann, was adopted, though it possesses the disadvantages inherent to all colorimetric methods, and secondly, the full depth of tint is not reached until about half-an-hour after the nitrous acid has been added. As a matter of experience it was found that after this interval the tint was of constant depth for some time, but if the solution were allowed to stand for an hour or more the tint gradually deepened, probably owing to some secondary changes between the free nitric acid and the organic base. It is not, therefore, pretended that the determinations are of absolute accuracy, though the relative results obtained can be satisfactorily interpreted. In the earlier experiments quantities of the nitric acid solution varying from 1 to 5 c.c. were added to 100 c.c. of distilled water contained in a glass cylinder, and the tint given by the meta-phenylene diamine compared with that produced on addition of a solution of sodium nitrite of known strength, acidulated with a volume of dilute sulphuric acid the same as that of the nitric acid used, and made up with water to the same volume of 100 c.c. In the later experiments a more expeditious, and doubtless more accurate process was adopted, which was based upon a method proposed by Otto Hehner* for "Nesslerising." The apparatus to be described was constructed and placed at my disposal by Mr. H. G. Madan, of Queen's College, Oxford, to whom I would express my best thanks.

The apparatus consisted essentially of two plane mirrors, the lower one of which was inclined at an angle of 30° to the horizon, and the upper set at an angle of 180° to the lower, between them were placed two glass tubes 320 mm. in length, enclosed within a wooden box with blackened sides, open at both ends, and which could also be opened at the side facing the observer. One of the tubes was provided at its lower extremity with a T-piece connected by means of flexible tubing with a reservoir which could be raised or lowered by a ring working on a metallic rod. Between these tubes was a wooden projection on which a paper scale, divided into ninety divisions, was gummed. A small plate of ground glass was placed on the upper part of the box arrangement. Light reflected from the lower mirror passed through the tubes filled with the solutions to be examined, and thus two coloured discs appeared on the upper mirror. The solution to be estimated was placed in the plain tube until the level of the liquid was at the division marked 90, and a standard solution in the reservoir, which was raised or lowered until the tints of the discs on the upper mirror

represented by the symbolic expression HNO_2, commonly assigned to it, or is merely nitrogen peroxide, N_2O_4, in presence of water, is beside the subject of the present inquiry.

* 'Chem. News,' vol. 33, p. 184.

were accurately matched; the level of liquid in the second tube was then read off, say x divisions; then $x/90$ gives the depth of the solution to be examined as compared with the standard solution. The standards were made up by dissolving Bismarck brown (the dye-stuff formed in the reaction between nitrous acid and meta-phenylene diamine) in dilute alcohol and adding it to such a volume of acidulated* water that the tint produced was the same as that of a solution of acidulated sodium nitrite of known strength contained in the same volume of water. Three such standards were made up, of tints corresponding to those produced by solutions containing ·15, ·3, and ·6 milligrams of nitrous acid in 1 c.c.; each of these solutions was also standardised one against another, and from time to time again compared with a nitrous acid solution as before. It was then found that the tints of the solutions, when not exposed to sunlight more than was necessary, were permanent for several weeks.

Copper and Nitric Acid.

(1.) *The Reaction between Copper and Nitric Acid depends upon the presence of Nitrous Acid.*

In the preliminary communication to the Royal Society on the subject (*vide supra*), attention was drawn to the observation that when a sphere of copper was revolved in dilute nitric acid no bubbles of gas were evolved from the metallic surface for some few minutes. But if the same sphere were introduced into the same acid, containing in small quantities the products of the reaction between the metal and the acid, then the evolution of gas at once commenced. Tables of figures were also given to show that, other conditions remaining the same, the amount of copper dissolved off per unit surface in a given interval of time, viz., one hour, was less in the former than in the latter case.

Further, it was shown that the addition of a small proportion of urea was sufficient to prevent any change between the metal and the acid from taking place. Since the above communication it has been found that substances other than urea are equally effective. The results of the experiments are given in detail below.

I. *Hydrogen Peroxide.*—About ·75 grm. of potassium peroxide was added in small portions at a time to a few c.c. of water, and the solution, while still effervescing with oxygen gas, quickly introduced into one litre of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}1813$, percentage of acid = 29·86); the mixture was then heated to 28°·5–29° C. and a rapid current of carbonic acid was passed in. A copper sphere, of weight 4·9188 grms., and exposing a surface of 321·87 sq. mm., was placed upon the little glass dish, which was set in rotation as quickly as possible. For 78 minutes no gas was evolved from the surface, no blue colour imparted to the solution, and the loss in weight was only ·0015 grm.

* If free acid, either sulphuric or hydrochloric, is not added, the tints are of too red a hue, and one standard solution is not thus directly comparable with another.

If then it be assumed that the potassium peroxide had been decomposed without loss in accordance with the equation

$$K_2O_4 + 2HNO_3 = 2KNO_3 + H_2O_2 + O_2,$$

then the quantity of hydrogen peroxide present in 352·5 grms. nitric acid was ·1647 grm., or one part of the peroxide in 2140 parts acid; yet this small quantity suffices to prevent the chemical change.

The acid was then heated to 32°·1; no gas was evolved for 28 minutes, then the reaction started, and the loss of weight of the copper sphere observed was ·0085 grm. As all the hydrogen peroxide had been destroyed, a fresh quantity of 1·48 grm. of the potassium peroxide was added as before, and the sphere again introduced.

No gas was evolved for 60 minutes, and the loss in weight observed was ·003 grm. It will be evident from these results that hydrogen peroxide is a very effective material for the purpose; the small proportion added suffices merely to destroy the small trace of nitrous acid not removed by the process of purification. Then, as nitrous acid is neither present initially nor is allowed to be formed, no reaction can take place.

I have been accustomed to show before large audiences the difference of behaviour of nitric acid, with and without hydrogen peroxide, towards metallic copper by means of the following experiment:—A small glass trough, divided into three sections, is made of two pieces of plate glass and four strips of glass cemented together by marine glue; into each of the divisions is introduced the same quantity of purified nitric acid of 25–28 per cent. concentration; to one of the portions are added a few drops of a solution of sodium nitrite, to the second, about the same quantity of water, and to the third, the same quantity either of a solution of hydrogen peroxide in nitric acid, or of a thin paste of barium peroxide with water. The glass trough is then placed in the slide of a lantern provided with an inverted prism, and projected upon a screen. Small strips of electrotype copper are then placed in each of the divisions, when a very rapid stream of gas bubbles is seen to arise at once from the strip of copper immersed in the nitric acid containing the nitrite; after a few minutes, gas bubbles will arise from the copper in the purified nitric acid, while no change will take place in the nitric acid containing the peroxide during the interval of time required for a lecture experiment.

II. *Potassium Chlorate.**—To a litre of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}1772$, percentage of acid = 29·2) were added 2 grms. of potassium chlorate, dissolved in 5 c.c. hot water; the acid was then heated to 28°, and a copper sphere, weighing 5·0648 grms. and exposing a surface of 328·2 sq. mm., was introduced. As in the previous experiments, the sphere was kept in rapid rotation, and the liquid in violent agitation. For 60 minutes no gas was evolved, and the loss in weight experienced was ·0006 grm.; the

* I am indebted to my colleague, Mr. W. W. Fisher, for suggesting to me the case of this salt as a preventive of the formation of nitrous acid.

sphere was again introduced and kept rolled about for 100 minutes, when there was a further loss in weight of ·0069 grm., or a total loss after the 160 minutes insertion, of ·0075 grm., or ·23 decimilligrams per 1 square millimetre surface. Into the same acid containing the potassium chlorate and heated to the same temperature, the same copper sphere was again introduced, the acid was kept at rest, and the sphere not rotated but, from time to time, gently rolled in order to expose as far as possible all portions of the sphere equally to the action of the acid. The evolution of gas commenced 2 minutes after the insertion of the sphere; for the same time of 160 minutes the loss was 1·644 grms., or 555·2 decimilligrams per 1 square millimetre surface. The amounts of copper dissolved in the two cases are practically in the ratio of 1 to 2300. In the latter experiment, a trace of nitrous acid is formed in the immediate vicinity of the sphere; this, after the chlorate is destroyed, continually increases, while, as a consequence, the rate at which the copper dissolves also continually increases.*

III. *Urea.*—In the former communication it was shown that the addition of a small proportion of urea completely stopped the reaction between copper and nitric acid; two experiments were cited in illustration. As a similar example, Cross and Bevan† have shown that the conversion of ligno-cellulose (jute) to cellulose by means of nitric acid, a change dependent upon the presence of nitrous acid, is modified or even prevented, other conditions remaining the same, by the addition to the acid of urea nitrate in increasing quantities.

IV. *Air.*—In an experiment conducted in a manner similar to those described above, a current of air was substituted for that of carbonic acid. A copper sphere was introduced into a sample of acid (sp. gr. = 1·1723, percentage of acid = 28·2) heated to 27°. When gas bubbles were seen to arise from the metallic surface, the sphere was quickly taken out, washed with water, dried, weighed, and again introduced. The following results were obtained :—

Experiment I.	No gas evolved for	14	minutes.	Loss in weight =	·0016	grm.
,, II.	,,	8	,,	,, =	·0012	,,
,, III.	,,	30	,,	,, =	·002	,,
	Total time	52	,,	Total loss	·0048	,,

The same sphere was then placed in the same acid and a current of carbonic acid used as before. The loss in weight during the same interval of time was ·31 grm. The amounts of copper dissolved in the two experiments are in the proportion of 1 : 64·6.

V. *Potassium Permanganate.*—Two grams of crystalline potassium permanganate was dissolved in 700 c.c. nitric acid (sp. gr. $\frac{19}{19}$ = 1·1765, percentage of acid = 29), and

* A similar experiment is more fully discussed in the section devoted to the chemical changes between mercury and nitric acid.

† 'Chemical News,' vol. 60, pp. 13-14.

the mixture heated to 31°·2. A copper sphere was introduced and kept in rotation for 30 minutes, a current of carbonic acid being passed in as before. At the end of the operation, there was a considerable deposit of manganese peroxide on the beaker and glass dish, produced by the reduction of the permanganate either by the copper or by the lower oxides of nitrogen, and possibly also, to a slight extent, by the mere process of heating. The loss in weight observed was ·0136 grm. The experiment was then repeated under precisely the same conditions, but without the addition of the potassium permanganate; the loss in weight observed, after the same interval of time, was ·417 grm.

The results of the above experiments, and of two others conducted in a similar manner, are set forth in the following table.

Series II.

Weight of permanganate introduced.	Weight of sphere at commencement.	Weight of sphere at conclusion.	Loss.	Mean area.	Value of M/A.
grms.	grms.	grms.	grms.		
0	4·8708	4·4538	·416	311·09	13·40
2	4·8944	4·8708	·0136	319·88	·42
3	3·631	3·5855	·0455	259·29	1·75
1·5	3·5173	3·49	·0273	257·36	1·06

It will be evident from the above results that the amount of copper dissolved is considerably diminished by the addition of the permanganate. It is probable that the loss in weight is due rather to a reaction between the copper and permanganic acid than to one between the copper and the nitric acid.

VI. *Ferrous Sulphate.*—By some qualitative experiments the observations of Millon (*loc. cit.*) were confirmed, that ferrous sulphate added in comparatively small quantities to nitric acid, of about 20 per cent. concentration, prevented at ordinary temperature the change between the acid and the metallic copper. But this substance was not found to be effective with a more concentrated acid, heated to a higher temperature, when added in such small proportions as those used in the experiments described above.

It may then be legitimately concluded from these results that copper by itself will not dissolve in nitric acid by itself, even in the presence of highly oxygenated substances. For these latter serve not to oxidise the copper, but rather any nitrous acid which might be present initially or might be produced either from impurities in the metal or from accidental causes.

(2.) *The Amount of Chemical Change between Copper and Nitric Acid depends upon the Amount of Nitrous Acid.*

In order to establish this point, the copper was allowed to dissolve in the acid; the sphere was rotated slowly and as uniformly as possible, while a slow and uniform current of carbonic acid (about forty bubbles per minute) was passed in. In all the experiments described below, the same quantity, one litre, of acid was used and the reaction allowed to continue for the same interval of time, one hour, unless otherwise stated. From time to time a quantity of acid varying from 5 to 1 c.c. was removed, and the amount of nitrous acid contained therein was determined as quickly as possible by the method described above.

In these dilute solutions the blue colour of the copper solution did not interfere with the orange tint of the reaction.

In the earlier experiments it was observed that the amount of nitrous acid very rapidly increased after the reaction had commenced, until it reached a maximum and a constant point. The following results serve to illustrate this point.

Series III.—Concentration of Nitric Acid = 28·2 per cent. Temperature 27°–27°·2.

Time in minutes after reaction had commenced.	Amount of nitrous acid in milligrams contained in 1 c.c. of the nitric acid.
First hour.	
0	·00074
2	·00141
11	·0124
30	·093
60	·105
Second hour.	
15	·099
30	·105
45	·099
60	·105

At the end of the first hour the sphere was taken out, dried, weighed, its axes determined: it was subsequently introduced into the same acid mixture.

SERIES IV.—Concentration of Acid = 29·86 per cent. Temperature 27°–27°·2.

Time in minutes after reaction had commenced.	Amount of nitrous acid in milligrams per 1 c.c.
First hour.	
0	·0042
2	·015
11	·064
30	·111
60	·124
Value of M/A = 17·75.	
Second hour.	
15	·124
30	·166
45	·195
60	·195
Value of M/A = 18·88.	
Third hour.	
15	·173
45	·195
60	·195
Value of M/A = 18·46.	
Fourth hour.	
30	·173
45	·185
60	·185
Value of M/A = 18·62.	

The extreme difference in the last nine observations falls within the errors of estimation. In both the above series of experiments the amount of copper dissolved off in the first hour was less than in those succeeding; in the preliminary paper attention was called to this observation.

Secondly, it was found that those variations of the conditions, such as greater concentration of the acid or higher temperature, which increase the amount of copper dissolved off per unit time, are equally those conditions which increase the maximum and constant proportion of nitrous acid, that is to say, the amount of copper dissolved and the amount of nitrous acid present are concomitant variables.

This point is illustrated by the results set forth in the following Table, which are selected from a number of experiments :—

Series V.

Concentration of nitric acid per cent.	Temperature.	Amount of constant proportion of nitrous acid in milligrams per 1 c.c.	Value of M/A.
24·91	27°–27°·2	·036	1·52
24·91	27°–27°·2	·036	1·61
24·91	27°–27°·2	·036	1·42
29·86	27°–27°·2	·18	18·88
29·86	27°–27°·2	·18	18·46
29·86	27°–27°·2	·18	18·62
29·86	35°–35°·2	·526	36·65

It will be evident from these data that if, temperature remaining the same, the concentration be increased, then the amount of nitrous acid present, as also the amount of copper dissolved per unit time, is increased ; the same result ensues if the concentration of the acid remains the same, but the temperature is increased. As a further confirmation of the above results, as also of the simple qualitative experiments of Millon, it was shown that if nitrous acid be added to the nitric acid by the introduction therein of sodium nitrite, the amount of copper dissolved off per unit time, other conditions remaining the same, is also increased. This ready-made nitrous acid was not, however, so efficacious a material as nitrous acid resulting from the chemical change between the copper and the nitric acid. Further, the amount of nitrous acid present at the conclusion of the operation was less than that at the commencement—a result partly due to some mechanical blowing-off of the acid by the current of the carbonic acid, and possibly also to the fact that the destruction of the nitrous by the nitric acid was more rapid than its supply by the reaction between the copper and the nitric acid.

In the following Table the results are compared of the amount of copper dissolved off per unit time, with and without the addition of nitrous acid ; in the latter case, the amount of nitrous acid present both at the commencement and at the conclusion of the experiment is also given.

SERIES VI.

Concentration of nitric acid per cent.	Temperature.	Amount of nitrous acid in milligrams per 1 c.c.	Value of M/A.
Without addition of nitrous acid.			
24·91	27°–27°·2	·036	1·52
With addition of nitrous acid.			
24·91	27°–27°·2	1·61 1·19	6·63
24·91	27°–27°·2	1·262 0·891	6·20

SERIES VII.

Concentration of nitric acid per cent.	Temperature.	Amount of nitrous acid in milligrams per 1 c.c.	Value of M/A.
Without addition of nitrous acid.			
26·77	27°–27°·2	0·133	7·16
26·77	27°–27°·2	0·133	6·65
With addition of nitrous acid.			
26·77	27°–27°·2	1·92 1·36	15·97
26·77	27°–27°·2	1·73 1·18	12·8

Thus, in the first case, the amount of copper dissolved off per unit time is increased from 1·52 to 6·4, and, in the second, from 6·9 to 14·38; but under these conditions a great increase of nitrous acid produces a comparatively small result.

The experiments of LOTHAR MEYER, BABINET, and others have shown that the chemical change between metallic zinc and sulphuric acid is completely arrested by the pressure of the evolved hydrogen when the acid and metal are sealed up in a glass tube. If, then, the reaction between copper and nitric acid is dependent upon the formation and presence of nitrous acid, it appeared *à priori* probable that, under the same condition, the chemical change, far from being diminished, would be increased.

For not only would the nitrous acid formed as a product of the change be retained in solution, but also the nitric oxide gas, which would otherwise be evolved, would reduce the nitric acid to form a further quantity of nitrous acid. To ascertain the correctness of this supposition, two copper spheres, of practically the same weight and area (viz., 5·1163 and 5·115 grms. and 330·72 sq. mm. respectively), were introduced simultaneously into the same volume, 25 c.c., of the same sample of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}177$, concentration 29·2 p.c.), contained in two pieces of hard glass tubing. One of these was sealed up as quickly as possible, and the extremity of the other bent at a right angle. These tubes were then placed side by side on a small wooden board, which was continually rocked up and down by a crank connected with an eccentric; thus both the acid and spheres were kept in continual motion. This comparative experiment was continued for four hours, when the former tube was unsealed, and the copper spheres were taken out, dried, and weighed. The loss in weight of the sphere in the sealed tube was 2·5985 grms.; that of the sphere in the open tube was 1·9875 grms. The ratio of the former to the latter is practically as four to three.

A similar experiment was repeated under the same conditions for an interval of six hours, when the sealed tube was burst by the internal pressure of the gas. The losses in weight of the two spheres were 2·7865 and 3·154 grms. respectively. The ratio is less than in the preceding experiment, but it is obvious that after a certain time more acid would be used up, and a greater proportion of copper nitrate would be formed in the sealed tube; both of these conditions would tend to retard the solution of the metal, and thus equalise the amount dissolved in the sealed and open tubes. These results obtained are quite confirmatory of those described above, and point most markedly to the essential difference between the two cases of chemical change, zinc and sulphuric acid on the one hand and copper and nitric acid on the other.

Effect of Nitric Acid upon the Chemical Change between Copper and Nitrous Acid.

In all the experiments detailed above, the effects produced by comparatively small quantities of nitrous acid present in the nitric acid have been the object of enquiry; it seemed desirable to study the converse proposition, namely, the effect of nitric acid upon the chemical change between copper and nitrous acid.

In order to obtain the latter, silver nitrite was prepared by precipitating the nitrate with a slight excess of sodium nitrite; the precipitate was purified by washing and re-crystallisation from hot water. A weighed quantity of the salt was decomposed by slowly adding dilute hydrochloric acid in slight deficiency of that required for the complete conversion of the nitrite into the chloride; the process was conducted in a vessel surrounded by a freezing mixture and filled with an atmosphere of carbonic anhydride. The acid solution was poured off as quickly as possible from

the chloride and ice crystals, but owing to its rapid decomposition, a solution of only about 1 per cent. concentration could be obtained.

In a preliminary experiment a copper sphere, 3·0633 grms. in weight, was placed in 15 c.c. of the nitrous acid solution at a temperature of 10° C.; an immediate and violent evolution of gas ensued. After 15 minutes' insertion the loss in weight was ·0635 grm.; on repeating the experiment for the same interval of time the loss in weight was ·06 grm. To the nitrous acid solution was added one-tenth of its volume of nitric acid of 40 per cent. concentration; the losses in weight in two experiments, each of 15 minutes' duration, were ·0445 and ·041 grm. respectively. Thus the former pair of results are fairly concordant among themselves, as also are the latter pair; but the value of the former is greater than that of the latter, though the only variations in the successive second and third experiments were, the slight increase of dilution due to the addition of nitric acid, and the slight diminution in the proportion of nitrous acid.

As this result seemed to point to the remarkable conclusion that the presence of nitric acid not only does not increase, but even retards the reaction between copper and nitrous acid, a more strictly comparative experiment was conducted as follows :—

Into two test-tubes were placed 10 c.c. of the same solution of nitrous acid, containing ·0942 grm. of the acid;* to one, 5 c.c. of water, to the other, 5 c.c. of nitric acid, containing 1·7228 grm. of the acid, were added; into the former mixture was placed a copper sphere 2·824 grms. in weight, and exposing a surface of 214·0 sq. mm.; into the latter, a sphere 2·5051 grms. in weight, and exposing a surface of 208·8 sq. mm.; the two tubes were placed side by side in a beaker of cold water. After half-an-hour the loss in weight of the sphere in the water-nitrous acid mixture was ·1398 grm., while that in the nitric-nitrous acid mixture was ·0472. The difference of these two values is far in excess of that to be attributed to the slight difference of areas exposed, while a further experiment showed that the amounts dissolved off these two spheres by the same quantities of nitric acid, under identical conditions, were practically the same, thus eliminating the variable of the difference in material of which the two spheres were composed.

It appeared, however, probable that the result of the above experiments was due rather to the greater stability of nitrous acid in the presence of nitric acid, than to any preventive action of the latter. To test this supposition, two mixtures were made, one consisting of 10 c.c., containing ·0985 grm. nitrous acid, and 5 c.c. water; the other of 10 c.c. of the same nitrous acid and 5 c.c., containing 1·7678 grm. sulphuric acid. Into each of these mixtures was introduced a copper sphere, one 5·1065, the other 5·115 grms. weight, both exposing the same surface of 330·84 sq. mm. At the end of half-an-hour, the loss in weight of the sphere in the nitrous acid-water

* The amount of nitrous acid was determined in another portion of the same sample by a standard solution of potassium permanganate immediately after the experiment with the copper sphere had been commenced.

mixture was ·068 grm. (value of M/A = 2·21), while that in the nitrous-sulphuric acid mixture was ·0745 grm. (value of M/A = 2·4). Thus the numbers expressing the amount of copper dissolved off per unit area are in the ratio of 1 : 1·09. The addition, therefore, of the sulphuric acid produces but little effect, and this too in a direction opposite to that caused by the addition of nitric acid.

Mercury and Nitric Acid.

(1.) *The Reaction between Mercury and Nitric Acid depends upon the presence of Nitrous Acid.*

For the sample of mercury used in these experiments I am indebted to the kindness of Mr. F. J. Smith; the metal was purified in the usual manner by dilute nitric acid, then heated for some time nearly to the boiling-point, and finally re-distilled twice in a vacuum.

The method of experiment was varied slightly in a manner suitable for dealing with a liquid instead of a solid. The revolving glass dish was not provided with holes; on it was placed a small globule of mercury of about 5 c.c. in volume; two small glass marbles were also added, which served to tilt over continually the edges of the globule, and thus to expose successively each portion of it to the acid. At the conclusion of an experiment the dish and its contents were quickly plunged into a large volume of distilled water; the mercury was then washed several times with water, dried over sulphuric acid, and weighed.

A blank experiment was made in which water heated to 35° was substituted for nitric acid; after all the necessary manipulations had been performed, it was found that the weight of the globule of mercury at the conclusion of the experiment, lasting one hour, was within a few milligrams identical with that at its commencement. The loss of metal by vaporisation was therefore quite inappreciable.

Into 700 c.c. of nitric acid (specific gravity = 1·181, concentration per cent. 29·2, and containing one part in 1,000,000 of nitrous acid), heated to 30°–30°·5, was introduced a globule of mercury, of weight 7·8432 grms.; for one hour no bubbles of gas were seen to rise from the surface of the metal, and its weight at the conclusion of the experiment was 7·8417 grms.; there was then a loss of only ·0015 grm., a quantity quite accounted for by the difficulties of manipulation.

The *same* globule of mercury was then introduced into the *same* acid heated to the *same* temperature; but neither the acid nor the mercury were kept in motion. In a few minutes bubbles of gas were seen to rise from the metallic surface, and the globule after insertion in the acid for one hour weighed 5·7227 grms., representing a loss of 2·119 grms., or 28·32 per cent.

The effect produced by the variation of one condition only, that of motion, is most remarkable, the result to all appearances paradoxical. For all investigations on

chemical change have taught us that its amount is increased by the removal of the products of change from the immediate environment of the reacting substances. Herein an example of the reverse is presented. The explanation lies in the fact that by this chemical change the solvent is produced of one of the substances entering into the change, and this, remaining within the sphere of action, increases the amount of the chemical change, and is itself simultaneously increased. In another experiment a globule of mercury, 7·5885 grms. in weight, was introduced into 750 c.c. of nitric acid (specific gravity 1·2106, concentration = 34·12 per cent.), heated to 26° ; at the end of one hour the globule weighed 7·5745 grms. ; there was thus a loss of ·014 grm., equal to ·18 per cent. Into the same acid, heated to the same temperature, a globule, of weight 6·535 grms., was introduced, while the acid and mercury were kept at rest. At the end of one hour the globule weighed 3·815 grms., or a loss of 2·72 grms., equal to 41·62 per cent. Thus the losses in weight in the two cases are in the ratio of 1 : 307·2.

Into the acid used in the experiment immediately preceding, and which then contained ·43 milligram nitrous acid in 1 c.c., a globule of mercury, of weight 6·3902 grms., was introduced ; both the acid and the metal were kept in motion, other conditions being unaltered. The evolution of gas commenced immediately, and the loss in weight was 2·022 grms., or 31·61 per cent. Again, to another portion of the same acid, heated to the same temperature, sodium nitrite solution, in such a quantity that the liquid contained ·825 milligram nitrous acid in 1 c.c., and a globule, 5·2351 grms. in weight, were introduced ; the immediate evolution of gas was observed, and the loss in weight at the end of one hour was 2·2847 grms., equivalent to 43·64 per cent. These results show that, while mercury does not dissolve in nitric acid containing only insignificant traces of nitrous acid, yet it readily dissolves when the nitrous acid is allowed to form and to accumulate, or when the nitric acid contains nitrous acid, either resulting from the solution of the mercury in the acid, or produced by the addition of sodium nitrite solution. Several attempts were made to prevent the chemical change between the mercury and the nitric acid from taking place at higher temperatures, and with more concentrated acid ; but these proved unsuccessful, notwithstanding slight variations of manipulation, which might have been supposed to contribute to success. The effect, therefore, produced by the addition of various substances which react with nitrous acid was investigated.

Urea.—An aqueous solution of urea (4 grms. in 5 c.c.) was added to 750 c.c. nitric acid (sp. gr. $\frac{19}{19}$ = 1·2041, percentage of free acid = 33·3, proportion of nitrous acid present = 1 in 1,000,000 parts) ; the mixture was heated to 31°·2–31°·4. A globule of mercury, 8·7242 grms. in weight, was introduced, and the metal, as also the acid, kept in agitation ; the globule, after immersion for 70 minutes, weighed 8·701 grms., or a loss of ·0232 grm., equivalent to a mean loss of ·27 per cent. During the time, however, required for the removal of the metal from the acid, the chemical change was seen to commence. The same globule was introduced into the same acid under the

same conditions, except that of agitation of metal and acid ; the evolution of gas from the surface of the mercury commenced immediately, and the loss in weight after 70 minutes' immersion was 2·2365 grms., equivalent to a loss of 25·7 per cent. Thus the variation of one condition, that of motion, reduces the amount dissolved in the same interval of time in the ratio of 100 : 1·05.

Again, the same globule, 6·4645 grms. in weight, was introduced into another sample of the same acid, heated to the same temperature, and both the mercury and acid kept in agitation, but no urea was added. No reaction occurred apparently for 20 minutes, and the loss in weight after 70 minutes' immersion was 1·956 grms., equivalent to a loss of 30·26 per cent. Thus the addition of 1·3 parts of urea to 100 parts of nitric acid reduces the amount dissolved, other conditions remaining the same, in the ratio of 100 : 9. It is evident from these experiments that neither the presence of urea, nor the agitation both of metal and acid, are of themselves sufficient to prevent the mercury from entering into chemical change with the nitrous acid, either present initially, or formed locally.

Potassium Chlorate.—Three grams potassium chlorate were dissolved in 750 c.c. of nitric acid from the same sample, and the mixture heated to 33°–33°·2. A globule, of weight 7·1675 grms., after immersion in the acid for 65 minutes, both the acid and metal being kept in motion, lost only ·001 grm., a quite inappreciable quantity. On repetition of the experiment under the same conditions, except that of motion, there was a loss of 1·7076 grms., equivalent to 23·83 per cent. ; under these conditions the evolution of gas commenced 1 minute 30 seconds after introduction of the globule. Thus the alteration of the one condition, viz., motion, completely prevents the chemical change under these conditions.

Hydrogen Peroxide.—Four grams of barium peroxide were dissolved in 750 c.c. of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}2081$, percentage of free acid = 33·83), and the mixture heated to 33°·2–33°·4.

A globule, of weight 7·037 grms., after immersion in the mixture for 60 minutes, lost ·0233 grm., corresponding to a loss of 3 per cent. when the metal and acid were kept in motion ; but when this condition was not fulfilled there was a loss in weight of 4·0332 grms., corresponding to a loss of 57·51 per cent. Thus the condition of motion reduces the amount dissolved in the ratio of 100 to ·52.

In the following table the more important results of the experiments described in the foregoing paragraphs are set forth, whereby the remarkable differences, produced by the variation of one condition, that of motion or rest, in each pair of experiments is rendered the more manifest.

SERIES VIII

Condition.	Temperature.	Concentration of nitric acid.	Substance added.	Loss in weight per cent.
Motion	30°–30°·5	29·2	..	Nil
Rest	,,	,,	..	28·32
Motion	26°	34·12	..	·18
Rest	,,	,,	..	41·62
Motion	31°·3	33·3	Urea	·27
Rest	,,	,,	,,	25·7
Motion	33°·1	,,	Potassium chlorate	Nil
Rest	,,	,,	,,	23·83
Motion	33°·3	33·83	Hydrogen peroxide	·3
Rest	,,	,,	,,	57·51

It may be concluded from these experiments that *metallic mercury does not dissolve in diluted nitric acid,* containing 34 per cent. of the free acid, provided that only infinitesimal quantities of nitrous acid are present initially, and that this same acid does not accumulate in the immediate environment of the metal.

(2.) *The Amount of Chemical Change between Mercury and Nitric Acid depends upon the Amount of Nitrous Acid.*

By experiments conducted in a manner similar to those described in the section referring to copper it was shown (i.) that the amount of nitrous acid present, when mercury is allowed to react with nitric acid, gradually increases until a maximum and constant proportion is present, when concomitantly the rate at which the mercury dissolves also becomes constant; and (ii.) that those variations of conditions, be they greater concentration of acid or higher temperature, which increase the amount of mercury dissolved, increase also this maximum and constant proportion of nitrous acid. In these comparative experiments the mercury and nitric acid were kept at rest for 5 minutes to ensure the starting of the reaction; they were then kept in uniform motion for a succeeding period of 55 minutes. To avoid the precipitation of mercurous chloride in the determination of nitrous acid by means of meta-phenylene diamine hydrochloride the latter was converted into the corresponding sulphate by precipitation with silver sulphate.

(i.) The amount of nitrous acid increases up to a maximum and constant proportion.

SERIES IX.—Concentration of Nitric Acid = 35·35. Temperature 21°.

Time in minutes after reaction had commenced.	Amount of nitrous acid in milligrams contained in 1 c.c. of the nitric acid.
First hour.	
20	·235
40	·583
60	·643
Loss in weight of mercury per cent. = 36·75.	
Second hour.	
20	·582
40	·582
60	·608
Loss in weight of mercury per cent. = 60·35.	
Third hour.	
30	·608
60	·560
Loss in weight of mercury per cent. = 63·12.	

Thus the amount of mercury dissolved in the two latter experiments is approximately constant, and considerably greater than that dissolved in the first hour, during which the amount of nitrous acid was increasing up to its maximum and constant point of ·593 milligram in 1 c.c., the mean of the last seven determinations.

(ii.) Those conditions which increase the amount of mercury dissolved per unit time are those conditions which increase the constant proportion of nitrous acid.

This point is illustrated by the results embodied in the following table.

SERIES X.

Concentration of nitric acid per cent.	Temperature.	Amount of constant proportion of nitrous acid in milligrams per 1 c.c.	Percentage amount of mercury dissolved.
32·9	21°	·26	13·35
32·9	21°	·272	11·39
32·9	21°	·223	9·80
32·9	21°	·247	9·85
32·9	26°	·371	23·55
32·9	26°	·359	17·27
32·9	26°	·396	18·27
35·35	21°	·591	60·35
35·35	21°	·584	63·12

These results, though not so concordant as those obtained in the case of copper, yet are sufficient to show that whether the concentration be the same and the temperature be increased, or the temperature the same and the concentration increased, the amount of mercury dissolved and the constant proportion of nitrous acid present vary *pari passu*.

Thus the results obtained with mercury and nitric acid confirm to the fullest degree those obtained with copper and nitric acid.

Effect of Nitric Acid upon the Chemical Change between Mercury and Nitrous Acid.

In order to determine the effect produced on the chemical change between mercury and nitrous acid by the addition of a small proportion of nitric acid, an experiment was conducted in a manner similar to that described in the section relating to copper. Two globules of mercury, of weight 8·049 and 8·3255 grms. respectively, were placed at the bottom of two small basins of similar shape and dimensions and kept under identical conditions.

In one basin was placed a mixture of

10 c.c. nitrous acid containing ·1357 grm. of the acid,
5 c.c. water;

in the other,

10 c.c. of the same nitrous acid,
5 c.c. of nitric acid containing 1·7228 grm. of the acid.

The globule of mercury in the former mixture completely lost its mobility, while bubbles of gas rose continually from its surface; the liquid also remained perfectly clear. In the latter mixture, however, there was formed a bulky white deposit, probably consisting of basic mercuric nitrate. At the end of half an hour the loss in weight in the former case was ·2415 grm., in the latter ·0365, numbers which are in the ratio of 6·6 : 1. It is thus evident that the chemical change proceeds with greater rapidity in the nitrous acid water, than in the nitrous-nitric acid mixture, a result in accordance with that obtained in the case of copper; it is not, however, so conclusive, inasmuch as the metallic surface in the latter case may have been coated by the basic nitrate and thus preserved from the action of the acid.

Bismuth and Nitric Acid.

(1.) *The Reaction between Bismuth and Nitric Acid depends upon the Presence of Nitrous Acid.*

For these experiments spheres of the best commercial bismuth were obtained from Messrs. Johnson and Matthey; these were of similar dimensions and weight as those of metallic copper. The method of procedure was precisely similar for the two metals. Since a preliminary experiment showed that, as in the case of copper, the reaction commenced in purified nitric acid of about 28 per cent. concentration, small quantities of various substances were introduced to remove the nitrous acid.

I. *Potassium Chlorate.*—Three grams of potassium chlorate were dissolved in 750 c.c.

of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}1743$, percentage of acid = 28·3); and the mixture heated to 31°·7. A bismuth sphere, of weight 4·3148 grms., and exposing a surface of 267·76 sq. mm., was rolled about in the acid; after 90 minutes the loss in weight was ·0228 grm., value of M/A = ·85. No evolution of gas was perceptible and the loss was probably occasioned in part by the mere rubbing off of a loosely adhering black deposit, which invariably formed when the metal was introduced into the acid. It is probable that this deposit consisted of the dioxide, Bi_2O_2, but it was not further examined.

The experiment was repeated under the same conditions except that no potassium chlorate was dissolved in the acid; the loss in weight after the same interval of time was ·565 grm., value of M/A = 22·03. Hence the presence of ·34 part of potassium chlorate in 100 parts of diluted nitric acid reduces the amount dissolved per unit area in the ratio of 100 : 3·68.

As a further confirmation of this result, 3 grms. of potassium chlorate were dissolved in 750 c.c. of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}186$, percentage of acid = 30·66), heated to 31°·8. A bismuth sphere, 5·0173 grms. in weight, and exposing a surface of 298·35 sq. mm., was rolled about in the acid; after 100 minutes' insertion the loss in weight was ·02 grm.; value of M/A = ·67. The experiment was repeated with the same mixture under the same conditions, except that the sphere was kept at rest. The loss in weight after the same interval of time was 2·3949 grms.; value of M/A = 96·47; therefore the alteration of one condition, that of motion, reduces the amount dissolved in the ratio of 100 : ·62.

II. *Hydrogen Peroxide.*—Two grams of barium peroxide were added in small portions at a time to 900 c.c. of nitric acid (sp. gr. $\frac{19}{19} = 1{\cdot}1745$, percentage of acid = 28·3), heated to 29°·5; a bismuth sphere, 306·79 sq. mm. in surface, was rolled about in the mixture for 30 minutes. At first there was a slight evolution of gas; the loss in weight was ·017 grm.; value of M/A = ·58.

The experiment was repeated with the same mixture under the same conditions, except that both the metal and acid were kept at rest; the loss in weight was ·3665 grm.; value of M/A = 124·5. Therefore the alteration of one condition, that of motion, reduces the amount dissolved in the ratio of 100 : 4·64. Experiments made with a view of preventing the reaction by the addition of urea to nitric acid of concentration similar to that used in the experiments described above proved unsuccessful. It may, however, be concluded from the results obtained with potassium chlorate and hydrogen peroxide, that the reaction between metallic bismuth and diluted nitric acid is inappreciable, provided that nitrous acid is not initially present, nor allowed to accumulate.

(2.) *The Amount of Chemical Change between Bismuth and Nitric Acid depends upon the Amount of Nitrous Acid.*

The results of several series of experiments showed that, as in the case of copper, the amount dissolved off per unit area from a sphere of metallic bismuth was less in amount when it was introduced into pure nitric acid than when introduced into the

same acid under the same conditions, but containing in small quantities the products of the change between the metal and the acid.

Further, the amount of nitrous acid present gradually increased until it reached a maximum and constant proportion, when the amount of metal dissolved also became constant. This is made evident by the results given below.

SERIES XI.—Concentration of Acid = 28·2. Temperature, 25°.

Time in minutes after reaction had commenced.	Amount of nitrous acid in milligrams per 1 c.c.
First hour.	
0	Nil
5	·011
15	·051
30	·065
45	·097
60	·15
Value of M/A = 18·95.	
Second hour.	
15	·179
30	·21
45	·22
60	·217
Value of M/A = 22·07.	
Third hour.	
30	·223
45	·236
60	·236
Value of M/A = 22·48.	
Fourth hour.	
15	·233
30	·24
45	·236
60	·24
Value of M/A = 21·74.	
Fifth hour.	
30	·22
45	·223
60	·23
Value of M/A = 22·7.	

The values obtained for the amount of nitrous acid in the last thirteen observations are within experimental errors of estimation, while the amount dissolved per unit area in the last four intervals of one hour are concordant and greater than that in the first interval.

One more series of experiments may be quoted as a further illustration of these points of analogy between bismuth and copper.

SERIES XII.—Concentration of Acid = 28·2. Temperature, 30°

Time in minutes after introduction of metal.	Amount of nitrous acid in milligrams per 1 c.c.
First hour.	
15	·083
30	·147
45	·233
60	·246
Value of M/A = 44·29.	
Second hour.	
30	·24
45	·28
60	·286
Value of M/A = 45·7.	
Third hour.	
30	·283
45	·283
60	·283
Value of M/A = 45·32.	
Fourth hour.	
20	·240
40	·276
60	·283
Value of M/A = 45·4.	

The values obtained for the amount of nitrous acid are practically constant during the three latter periods of one hour, and the amounts dissolved off are equally constant.

Again, as with copper and mercury, so also with bismuth, those conditions which increase the amounts of metal dissolved per unit time are equally those conditions

which increase the amounts of this constant proportion of nitrous acid. This point is illustrated by the results given below.

Series XIII

Concentration of nitric acid per cent.	Temperature.	Amount of constant proportion of nitrous acid in milligrams per 1 c.c.	Value of M/A.
26·77	25°	·015	1·33
26·77	25°	·015	1·39
26·77	25°	·015	1·23
28·2	25°	·232	22·07
28·2	25°	·232	22·48
28·2	25°	·232	21·74
28·2	25°	·232	22·7
28·2	30°	·283	45·7
28·2	30°	·283	45·32
28·2	30°	·283	44·88
30·66	30°	·292	49·06
30·66	30°	·292	49·44

These results show that whether the temperature be kept constant and the concentration increased, or conversely, the concentration kept constant and the temperature increased, the values for the amount of metal dissolved off per unit surface in unit time, and the amount of maximum proportion of nitrous acid concomitantly vary.

Again, as in the case of copper, the addition of sodium nitrite to nitric acid, of such a degree of dilution as to be without action upon the metal bismuth, starts at once the chemical change, though this ready-made nitrous acid is not so efficacious as nitrous acid in process of formation and decomposition.

Thus, a bismuth sphere, 5·1738 grms. in weight, and exposing a surface of 309·02 sq. mm., was placed in a sample of nitric acid of 25·19 per cent. concentration, and heated to 20°. The loss in weight after immersion for one hour was ·003 grm., a quantity quite insignificant; on repetition of the same experiment under the same conditions, there was a loss in weight of ·0235 grm., value for M/A = ·73. A few c.c. of a concentrated solution of sodium nitrite were then placed in the nitric acid, and the experiments again repeated. A stream of gas bubbles was seen to rise at once from the metallic surface, and, as shown in the results given below, the amount of metal dissolved under otherwise the same conditions was considerably increased.

The amount of nitrous acid was determined immediately before and after the experiment.

Series XIV.

Weight before.	Weight after.	Nitrous acid in milligrams in 1 c.c.	Loss.	Mean area.	Value of M/A.
5·1473	4·5968	{32 / 21}	·5508	297·89	18·48
4·5968	4·1406	{19·6 / 19·3}	·4562	275·7	17·75

On a comparison of the above results with those given in Series XI., it will be evident that the efficacy of ready-made nitrous acid is about 1 per cent. of the nitrous acid produced by the solution of the metal, for, in the experiments of Series XI., the amount of nitrous acid present was about ·23 milligram, and in those of Series XIV., about 22·0 milligrams, yet the amount dissolved off per unit surface was rather higher in the former than in the latter case.

As a further illustration, results are given of the amounts dissolved off per unit area without and with addition of nitrous acid, all other conditions being kept invariable.

Series XV.

Concentration of nitric acid per cent.	Temperature.	Amount of nitrous acid in milligrams in 1 c.c.	Value of M/A.
Without addition of nitrous acid.			
26·77	25°	·015	1·32
With addition of nitrous acid.			
26·77	25°	{16·0 / 7·65}	14·28
26·77	25°	{15·65 / 9·0}	17·03

Further comparative experiments, similar to those described in the section relating to the metal copper (*vide* p. 298) were conducted to determine the amount of bismuth dissolved by the same quantity of nitric acid when contained in an open and sealed tube. Thus, two spheres of metallic bismuth of practically the same weight and area (4·8633 and 4·775 grms., of 294·98 and 291·83 sq. mm. respectively) were placed simultaneously in the same quantity of nitric acid (29·2 per cent. concentration). The loss in weight of the sphere in the open tube was ·8685 grm., that in the sealed tube 2·09 grms., numbers which are in the ratio of 1 to 2·5. The temperature of the room was 17° C., and the duration of the experiment, four and a half hours.

Another experiment was conducted in a precisely similar manner for six hours, but at a lower temperature, viz., 10° C. The weight of the spheres introduced were 5·1693 and 5·1525 grms., both exposing a surface of 310·02 sq. mm. : the loss in weight of the sphere in the open tube was ·5445 grm., that in the sealed tube ·957, numbers which are in the ratio of 1 : 1·8. These results are quite confirmatory of those obtained with copper, proving that if the conditions are such that the nitrous acid is allowed to accumulate within the nitric acid, the amount of metal dissolved is thereby increased.

Effect of Nitric Acid upon the Chemical Change between Bismuth and Nitrous Acid.

In order to determine whether nitric acid hinders the reaction between bismuth and nitrous acid, and to afford a further confirmation of the results described in the case of the metal copper, a similar comparative experiment was conducted as follows :— Two mixtures were made, the one consisting of 10 c.c. of a solution containing ·108 grm. nitrous acid with 5 c.c. of water, the other of 10 c.c. of the same nitrous acid, with 5 c.c. of a solution containing 1·722 grm. nitric acid. Into the former was introduced a bismuth sphere, 4·1945 grms. in weight, and exposing a surface 273·24 sq. mm. ; into the latter a sphere 3·993 grms. in weight, and exposing a surface of 261·87 sq. mm. The two experiments were conducted under identical conditions. After 30 minutes the loss in weight of the bismuth in nitrous acid-water mixture was ·1865 grm. (value of M/A = 6·94), while of that in the nitrous-nitric acid mixture was ·1035 grm. (value of M/A = 3·99) ; these numbers are in the ratio of 9 : 5. Further, the difference in the rate of evolution of gas bubbles from the metallic spheres in the two cases was very evident. But in order to prove that this result was not due to any difference of homogeneity of the two spheres, they were introduced simultaneously into the two portions of nitric acid of the same volume (concentration = 29·2 per cent.), under the same conditions. The experiment was continued for 30 minutes and the following results obtained :—

Weight before.	Weight after.	Loss.	Mean area.	Value of M/A.
4·008	3·807	·191	258·4	7·56
3·8895	3·692	·1975	253·66	7·76

These values for the amounts dissolved per unit area are concordant to within 3 per cent.

Therefore, under the conditions described above, the presence of nitric acid actually impedes the solution of the bismuth ; the results perfectly confirm those obtained with metallic copper.

Another experiment was conducted in a similar manner with two other spheres, of

weight 3·7847 and 3·651 grms., and exposing surfaces of 254·36 and 24·539 sq. mm. respectively, placed, the former into a mixture of 10 c.c. containing ·112 nitrous acid mixed with 5 c.c. of water, the latter into a mixture of the same volume of the same nitrous acid mixed with 5 c.c. of a solution containing 1·722 grm. nitric acid. At the end of half-an-hour the loss in weight of the former was ·2198 grm. (value of M/A = 8·85); that of the latter, ·1465 grm. (value of M/A = 6·05), or numbers nearly in the ratio of 3 : 2. Temperature 17° C.

The Interpretation of the Results.

The experiments detailed above for the metals copper, mercury, and bismuth, to which, according to Russell's experiments, silver must also be added, have established the following facts :—(i.) the primary change is that between the metals and nitrous acid ; (ii.) no gas is evolved at first from the surface of the metal ; (iii.) the amount of nitrous acid increases up to a constant and maximum proportion ; and (iv.) those conditions which increase the amount of metal dissolved per unit time, are equally those conditions which increase this constant proportion of nitrous acid. If then a trace of nitrous acid becomes once formed, and if, also, the quantity of nitric acid is present in very considerable excess, it would appear that these results may be explained as follows :—(i.) the metal dissolves in nitrous acid to form the metallic nitrite and nitric oxide ; (ii.) the nitrite formed is decomposed by the excess of nitric acid to reproduce the nitrous acid ; and (iii.) the nitric oxide formed in (i.) is not evolved as such, but reduces the nitric acid or the nitrate to produce a further quantity of nitrous acid. These changes may be represented thus, taking copper for example :

$$\text{(i.)}\quad Cu + 4HNO_2 = Cu(NO_2)_2 + 2H_2O + 2NO,$$
$$\text{(ii.)}\quad Cu(NO_2)_2 + 2HNO_3 = Cu(NO_3)_2 + 2HNO_2,$$
$$\text{(iii.)}\quad 2NO + HNO_3 + H_2O = 3HNO_2.$$

The same equations apply to the other metals, *mutatis mutandis*. The condition of the increase in amount of the nitrous acid is satisfied, for at the start four molecular proportions take part, while five are ultimately produced ; the non-evolution of nitric oxide in the earlier stage of the change is also accounted for. But when the amount of nitrous acid has reached a certain point, the reverse of equation (iii.) takes place, and the nitric oxide is evolved from the liquid.

$$\text{(iv.)}\quad 3HNO_2 = HNO_3 + 2NO + H_2O.*$$

At this point the nitrous acid is decomposed as fast as it is formed, the amounts of

* [Since the text was written Clemente Montemartini has proved that the decomposition of nitrous acid takes place precisely in accordance with the equation given above, the rate of change being expressible by the formula $aC/dT = kC$ (C = concentration, T = time, and k a constant). 'Accad. Lincei Rendiconti,' 1890, *Semestre* 2, p. 264.—April 24, 1891.]

the reactions (iii.) and (iv.) per unit time being doubtless dependent, as in other similar cases, upon the relative masses of the nitrous and nitric acids, upon the temperature and other conditions of the experiment.

This explanation is, however, only valid provided that the mass of nitric acid be in very large excess over that of the nitrous acid, for it is evident from the experiments described above, that if there is no very great difference between the masses of the two acids the nitric acid serves to impede rather than to promote the chemical change, a result which is probably due to the greater stability of nitrous acid in the presence of a small quantity of nitric acid.

There yet remains one further point; if these metals by themselves do not dissolve in dilute nitric acid by itself, by what manner of means is the necessary nitrous acid initially formed? It is, of course, possible that the metal will enter into chemical change with an amount of nitrous acid less than that detected by the meta-phenylene diamine; it is also possible that some metallic impurity forms with the metal a couple which electrolyzes the nitric acid, thereby producing nitrous acid. As regards this latter point, it is worthy of remark that of the metals experimented with, mercury was less susceptible of chemical change than copper, and copper in its turn than bismuth. This order is equally that of their probable degree of purity.

A few experiments were accordingly made to determine the amount of a foreign metal which could be added to pure mercury without promoting its reaction with nitric acid. These showed that one part in a thousand of pure copper and one part in eight hundred of pure silver added to mercury did not render the latter susceptible of chemical change, even with nitric acid of 30·5 per cent. concentration, and heated to 28°·5, provided that both metal and acid were continually stirred. If both were at rest, then, as in the previous experiments, the change commenced immediately.

Reactions between Hydrogen and Metallic Nitrates in presence of Nitric Acid.

The observations of Russell on the reactions between hydrogen and metallic nitrates have been alluded to above, and their importance as regards the chemical changes between metals and nitric acid also discussed.

Hydrogen and Cupric Nitrate.

Russell inferred from the change in colour only that cupric nitrate was reduced to the corresponding nitrite by prolonged contact with hydrogen. This point was more fully investigated in the course of some experiments made with a view of reducing, if possible, cupric nitrate to metallic copper. Hydrogen prepared by electrolysis of acidulated water, according to Bunsen's method, was purified by bubbling through (i.) a wash-bottle containing a saturated solution of chromic acid, (ii.) a similar bottle containing an aqueous solution of potassium iodide with starch, (iii.) a test-tube con-

taining the same solution. It then passed into a set of GEISSLER'S bulbs partly filled with dilute nitric acid, wherein was dissolved the metallic nitrate, and thence into a set of LIEBIG'S bulbs containing a solution of potassium iodide with starch, similar to that used in the anterior portion of the apparatus.

Thus, the hydrogen, after passage through the nitric acid, did not come in contact with anything of the nature of cork and caoutchouc, except the small piece of tubing used to join the GEISSLER'S bulbs with the LIEBIG'S bulbs. In a blank experiment, 15 c.c. of nitric acid, of 24·9 per cent. concentration and containing 1·5 part per million of nitrous acid, were placed in the GEISSLER'S bulb and hydrogen passed through for ten hours, at the temperature of the laboratory. No blue colouration was produced either in LIEBIG'S bulb or in any part of the apparatus. Then ·25 grm. of copper carbonate was dissolved in the nitric acid, whereby the proportion of nitrous acid was not increased, and the experiment repeated. After the hydrogen had passed for 10 minutes a faint blue tinge was apparent in the LIEBIG'S bulbs, which gradually deepened in colour, while the colour of the iodide of potassium solution in the anterior portion of the apparatus was in no wise altered. These experiments show that whereas ready-made hydrogen by itself will not reduce nitric to nitrous acid, yet this change takes place in the presence of cupric nitrate. It is probable that, at first, the cupric nitrate is reduced to the nitrite by the hydrogen, and the nitrite is subsequently decomposed by the free nitric acid to form cupric nitrate and nitrous acid; this latter is partly blown off by the current of hydrogen. The chemical changes may be represented as follows:—

$$\text{(i.)}\quad Cu(NO_3)_2 + 2H_2 = Cu(NO_2)_2 + 2H_2O,$$
$$\text{(ii.)}\quad Cu(NO_2)_2 + 2HNO_3 = Cu(NO_3)_2 + 2HNO_2.$$

After the hydrogen had passed through the cupric nitrate solution for six hours, during which time nitrous acid was continually being evolved, the blue colour of the nitrate had changed into the green colour of the nitrite; the solution now contained 300 parts instead of 1·5 part per million of nitrous acid originally present. The proportion of this acid is therefore increased in the ratio of 1 : 200, without taking into account that lost by the passage of the hydrogen bubbles. This reaction, however, takes place much less readily, if at all, when only a small quantity of free nitric acid is present.

An attempt was made to reduce a nearly neutral solution of cupric nitrate to metallic copper in accordance with RUSSELL'S observations with silver nitrate; it was thought that the addition of a considerable quantity of urea to the solution might assist the formation of the metallic copper, as also prevent its re-solution by any nitrous acid which might be formed. The experiment conducted in this manner proved, however, unsuccessful; nitrous acid was formed in preference to the metal and this reacted with the urea to form nitrogen and carbonic anhydride; the

formation of the latter was evidenced by a turbidity in an aqueous solution of barium hydrate placed in the LIEBIG'S bulbs.

Hydrogen and Mercuric Nitrate.

Some crystalline mercuric oxide was purified by repeatedly washing with potash solution; it was then dissolved in glacial acetic acid, the mercuric acetate crystallised out, re-dissolved in water and precipitated with sodium carbonate; the precipitate was washed several times, at first with a dilute aqueous solution of sodium carbonate and then with water; it was finally dried at 100°. Mercuric oxide was thus obtained free from all traces of nitrous acid, with which the original material was contaminated. A weight of ·167 grm. of this oxide was dissolved in 10 c.c. of nitric acid, from the same sample as that used in the cupric nitrate experiments, the solution introduced into the GEISSLER'S bulb, and the hydrogen passed, as described above. After some time a faint blue tinge could be observed in the LIEBIG'S bulbs, though the production of nitrous acid in this case was quite insignificant as compared with that of the cupric nitrate. It was further found that after passage of the hydrogen for six hours the amount of nitrous acid present in the nitric acid was not materially increased. But on addition of dilute hydrochloric acid to the solution, an abundant precipitate of mercurous chloride was produced, whereas a similar solution of mercuric nitrate in nitric acid, through which hydrogen had not been passed, gave only a very slight turbidity with dilute hydrochloric acid. Thus, under these conditions mercuric is reduced to mercurous nitrate, according to the equation

$$2Hg(NO_3)_2 + H_2 = Hg_2(NO_3)_2 + 2HNO_3.$$

In order to confirm this observation by quantitative results the following experiment was conducted:—5 grms. of the purified mercuric oxide were dissolved in 50 c.c. of the same sample of nitric acid as that used previously. Through 10 c.c. of this solution a current of hydrogen was passed for 24 hours; at the end of this time the mercurous chloride was precipitated by excess of dilute hydrochloric acid. The same quantity of the same solution of mercuric nitrate in nitric acid, through which no hydrogen had been passed, was simultaneously mixed with the same excess of hydrochloric acid. The mercurous chloride obtained in the two cases was collected on a tared filter, dried at 100°, and weighed. From the solution through which the hydrogen had passed ·0272 grm. mercurous chloride was obtained, but, from the solution through which the hydrogen had not passed, ·001 grm. only was obtained. The ·1 grm. mercuric oxide dissolved in the portion of 10 c.c. nitric acid would yield ·1067 grm. mercurous chloride; therefore, by the current of hydrogen, 25·5 per cent. of the mercuric was reduced to mercurous nitrate, while, as before, the quantity of nitrous acid produced was inappreciable.

Hydrogen and Lead Nitrate.

·25 grm. of purified lead carbonate was dissolved in 15 c.c. of nitric acid from the same sample, and the experiment with the current of hydrogen conducted as before ; in about 20 minutes a blue colouration was produced in the potassium iodide solution posterior to the nitric acid solution. After the hydrogen had passed through for six hours the quantity of nitrous present in the nitric acid was also considerably increased. Thus nitric is reduced to nitrous acid through the intervention of lead, as of cupric nitrate.

Other Metallic Nitrates.

Experiments were also conducted with solutions of purified bismuth and zinc carbonates in nitric acid, but no trace of nitrous acid was apparent even after passage of the hydrogen for several hours.

Conclusions.

The main points contained in the paper may briefly be summed up as follows :—

I. The metals copper, mercury, and bismuth do not dissolve in nitric acid of about 30 per cent. concentration, and heated to a temperature of about 30° C., provided that nitrous acid is neither present initially nor formed subsequently. To prevent these contingencies it is necessary to add a small quantity of some oxidising substance such as hydrogen peroxide or potassium chlorate, or as less efficacious, potassium permanganate, or to pass a current of air, or lastly such a substance as urea, which destroys the nitrous acid by its interaction.

II. If the conditions are such that these metals dissolve, then the amount of metal dissolved and the amount of nitrous acid present are concomitant variables, provided that the nitric acid is in considerable excess. Change of conditions, such as concentration of acid and variation of temperature, which increase the former increase also the latter. Again, if the oxides of nitrogen and nitrous acids formed are kept within the liquid by enclosing it within a sealed tube, then the amount of metal dissolved in unit time is also increased.

III. If the conditions are such that these metals dissolve, it would appear that the metallic nitrite is at first formed together with nitric oxide ; the former is decomposed by the excess of nitric acid to liberate nitrous acid, while the latter reduces the nitric acid to form a further quantity of nitrous acid.

Eventually the net result is the product of two reverse chemical changes represented by the equations

$$\text{(i.)}\quad 2NO + HNO_3 + H_2O = 3HNO_2,$$
$$\text{(ii.)}\quad 3HNO_2 = 2NO + HNO_3 + H_2O.$$

The nitrous acid is thus destroyed as fast as it is generated.

IV. If the conditions are such that metals dissolve in nitric acid then nitrous acid is invariably the initial product of reduction.

V. The metals copper, mercury, and bismuth dissolve very readily in a 1 per cent. solution of nitrous acid (prepared by decomposing silver nitrite with hydrochloric acid); under these conditions nitric acid present in slight excess interferes with, rather than promotes the chemical change. This result is probably due to the greater stability of nitrous in the presence of nitric acid.

VI. Hydrogen gas reduces nitric to nitrous acid in presence of cupric or lead nitrate ; it also converts mercuric into mercurous nitrate, but does not produce any change in solutions of bismuth and zinc nitrates when dissolved in nitric acid.

Finally, I would wish again to thank those of my fellow-workers, whose names have been mentioned in the course of this paper, for their valuable advice and suggestions, and also to express my obligations to the Chemical Department of this Museum for supplying me with some of the materials and instruments necessary for this present inquiry.

VIII. *The Variations of Electromotive Force of Cells consisting of certain Metals, Platinum, and Nitric Acid.*

By G. J. Burch, *B.A., and* V. H. Veley, *M.A., the University Museum, Oxford.*

Communicated by Professor Odling, *F.R.S.*

Received October 23,—Read November 27, 1890.

Introduction.

The experiments described by one of us in another paper have established that the necessary condition for the chemical reactions between nitric acid and the metals copper, mercury, and bismuth, is the presence and continuous formation of nitrous acid. These results confirm the previous observations of Russell in the case of the metal silver. Further, it has been shown that when these metals are introduced into purified nitric acid, no apparent change takes place at first, but after some time gas bubbles are seen to rise from the surface of the metal, and the liquid then contains an amount of nitrous acid which can be detected by the more delicate reagents. The amount of this acid increases up to a maximum and constant point, while concomitantly the amount of metal dissolved per unit of time equally increases to a maximum and constant value. It seemed, therefore, desirable to ascertain if these phenomena determined by the balance are correlated with any alteration of electromotive force. To avoid any misunderstanding, however, it would be advisable at the outset to state that this paper is separated from that of one of us "On the Condition of Chemical Change between certain Metals and Nitric Acid," rather on account of division of labour than of difference of subject. The idea of measuring the electromotive force of cells consisting of platinum, certain metals, and nitric acid occurred to the latter of us, while the apparatus and method of measurement finally adopted are due solely to the former.

Previous Investigations.

Many years ago Faraday* pointed out that neither nitrogen peroxide nor nitric acid are conductors when pure, but that the addition of water to the former, or of nitrous acid to the latter, produces the property of conductivity "in a very high degree among electrolytes." Thus the presence of nitrous acid is the necessary

* 'Experimental Researches,' series xvi., 1815–1818.

condition that nitric acid should become an electrolyte. Again, FARADAY* proved that the electric position of certain pairs of metals in nitric acid is varied according to the conditions of concentration and temperature; in fact, experiments on this point were considered by him to be of especial importance in confirming the chemical and disproving the contact theory of the voltaic cell.

POGGENDORFF gives one determination of the E.M.F. of a cell consisting of copper, nitric acid (sp. gr. = 1·222, with nine times its volume of water), and platinum; the value found, taking the E.M.F. of a Daniell's cell at 1070 millivolts, is calculated out as 660 millivolts.

THOMSEN† alludes to the various reactions occurring in the combinations zinc / H_2SO_4, 100 H_2O / fuming nitric acid / carbon, as also zinc / H_2SO_4, 100 H_2O / HNO_3, 7 H_2O / carbon; the chemical changes taking place in these cells are shown to vary with the concentration of the acid and the chemical energy produced. In the latter cell there is a reduction of the hydrate of nitric acid to nitrogen peroxide, which dissolves without evolution of gas, but with more dilute acid nitric oxide is formed and evolved. The E.M.F. observed of these combinations is substantially in accord with that calculated from the chemical energy developed, expressed in heat units, an agreement which, as ARMSTRONG has pointed out, is the more remarkable considering the gradual accumulation of the products of decomposition of the nitric acid within it, and the effects liable to be produced thereby.

OSTWALD,‡ in the course of his investigations on contact electricity, measured the difference of potential between metals and aqueous solutions of various electrolytes by the dropping electrode method; amongst other results he gives values for the E.M.F. of certain metals in nitric acid of three different degrees of concentration. He regards, however, his determinations as uncertain, as the metals were strongly attacked by the acid liquid.

But in all these investigations no especial care seems to have been taken to purify the nitric acid from the nitrous acid, which is almost invariably present, neither are details given as to any variation of E.M.F., conditions of temperature and concentration remaining the same. These variations form the principal subject of the present enquiry.

Preliminary Experiments.

Certain preliminary experiments made with the aid of a quadrant electrometer showed that the E.M.F. of a copper-platinum couple dipping in nitric acid of 28 per cent. concentration gradually increased from the moment of contact of the metal with the acid for about two minutes, when it reached its maximum value. But if a small trace of a nitrite were added to the pure nitric acid this maximum E.M.F. was reached

* 'Experimental Researches,' series xvii.

† 'WIEDEMANN, Annalen,' vol. 11, 1880, p. 263; also 'Thermochem. Untersuch.,' vol. 3, p. 437.

‡ 'Zeits. Phys. Chem.,' vol. 1, 1887, p. 602.

at the moment of immersion of the metals; while, on the other hand, the addition of a small trace of urea caused this phenomenon of the increase of E.M.F. to be delayed for a greater interval of time.

This method, however, presented the disadvantage, that, owing to the long period of the vibrations of the needle of the quadrant electrometer, the E.M.F. could not conveniently be determined at the very moment of immersion of the metals. It was, therefore, thought desirable to construct some form of apparatus, which should permit of immediate measurement, and should also ensure that no current passed in the galvanic cell, whereby the proportion of nitrous acid and, consequently, the E.M.F. might be increased.

It was decided to use the capillary electrometer, as the quickest in its action of all instruments for measuring differences of potential; secondly, to balance the E.M.F. of the platinum-copper couple from first to last against a difference of potential, which could be varied at will, and its amount at any given moment automatically recorded; and, finally, to interpolate in the circuit a resistance sufficiently great to make the movements of the mercury of the electrometer sluggish to some degree, so as to minimise the quantity of electricity which would pass while the operator was balancing the E.M.F. of the cell, and at the same time to preserve the instrument from damage.

The Apparatus.

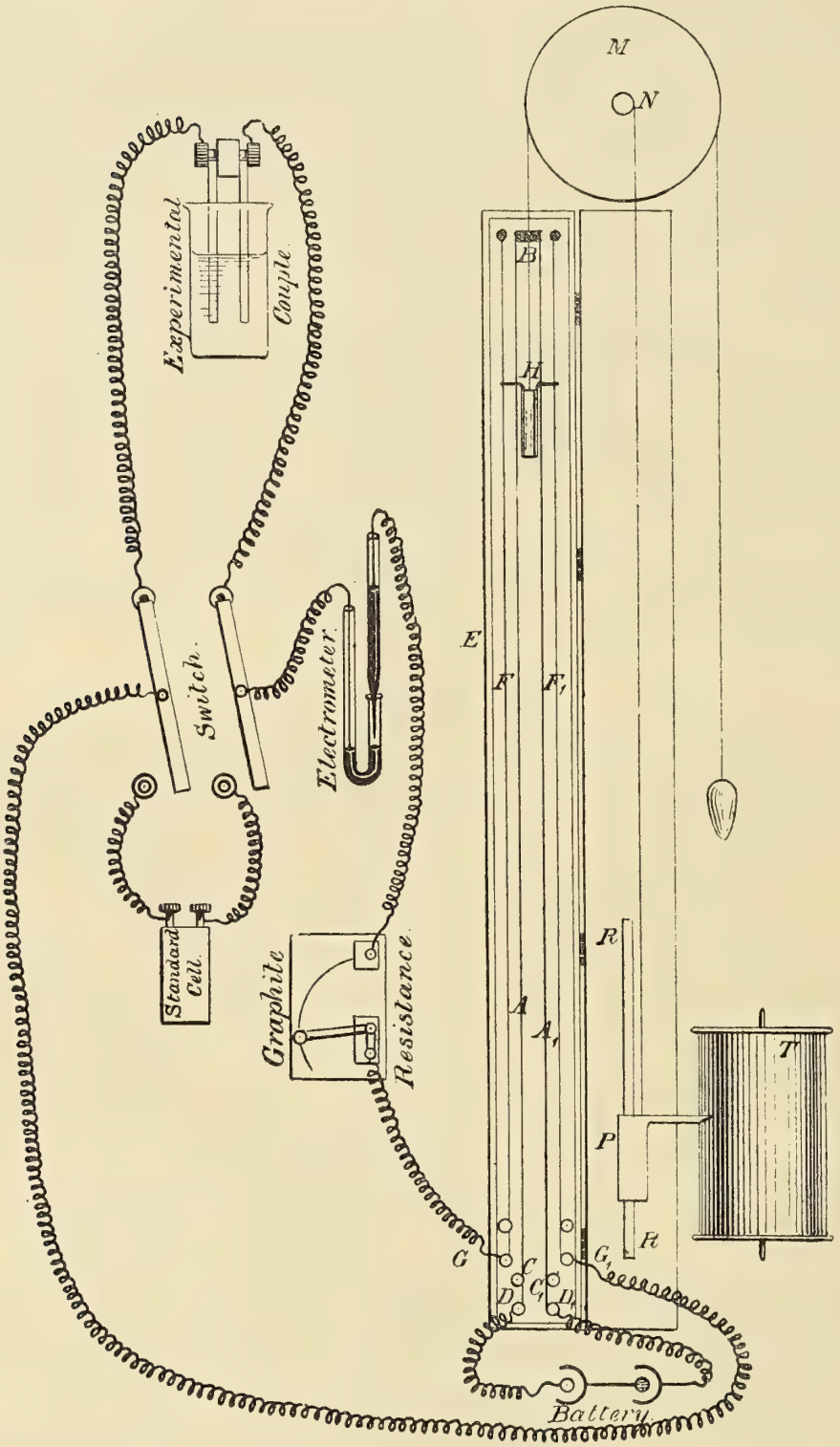
In balancing the E.M.F. of the experimental cell a rheochord (fig. 1), designed by one of us, was employed. A, A′ is a German silver wire 4·5 metres long, and of 20 ohms resistance, which is soldered by the middle to the brass yoke B, the resistance of which is practically zero. The two ends of the wire are attached to springs C, C′, which keep it stretched, and through these to the terminals D, D′.

The yokes, springs, and terminals are cemented to varnished glass pegs in the wooden trough E, which is at the same time the baseboard and the box of the instrument. Two stouter wires, F, F′, insulated from each other and parallel to A, A′, are attached in a similar manner to the terminals G, G′. A heavy rider, H, carries two pairs of spring jaws, made also of German silver, which pinch the wires so as to make electrical connection between A and F and between A′ and F′. Two Daniell's cells are connected with the terminals D and D′, and the terminals G and G′ of the derived circuit are connected with the platinum-copper couple through the electrometer.

The difference of potential thus introduced to balance the E.M.F. of the experimental cell is proportional (errors of calibration excepted) to the distance of the rider H from the yoke-piece B.

The edge of the trough E is graduated in millimetres, so that the position of the rider can be read off. The movement of the rider during an experiment is automatically recorded as follows:—

Fig. 1.



The rheochord.

On the end of the trough E, which for this purpose is fixed in a vertical position, is a flanged wheel, M, 7·5 inches in diameter, on which is wound a thin gut line attached to the rider. Another line, with a counterpoise, is also wound on the same wheel so that the rider may be kept, by the slight friction of its jaws on the rheochord wire, in any position in which it may be placed. On the axle of the wheel is a flat-bottomed groove, N, ·75 inch in diameter, on which also a gut line is wound in such a manner that the weight, P, attached to it may rise when the rider is pushed downward and *vice versâ*, the movement of the weight, P, being only one-tenth of that of the rider. The weight works in a slide, R, attached to the lid of the box or trough, which is held open when in use by the clamps by which the instrument is fastened to the bench.

The weight, P, carries a light arm, on the end of which is a style for the purpose of writing on the smoked paper upon the recording-cylinder, T, which is driven by clockwork in the usual manner. Thus the position of the rider throughout each experiment is recorded on a diminished scale.

The Standard of Electromotive Force.

The standard of E.M.F. to which all the measurements were ultimately referred was a Clark's cell, made up according to the directions given by Lord Rayleigh.* The form adopted was a ∪-shape, the limb containing the mercury being of a greater diameter than that of the limb containing the zinc pole; in fact, the cell was made of a disused pipette, one end of which was cut off and the delivery tube bent round. It was supported by passing the limbs through a cork fitted with a wide-mouthed bottle, and this was filled with water in which a thermometer was immersed. It was thought that the temperature of the cell could thus be more easily ascertained and would be less liable to variation.

Before and after each experiment the Daniell's cells were balanced against the Clark's cell and the position of the rider noted. It has been shown by one of us that, with a capillary electrometer arranged in this way, the quantity of electricity allowed to pass in the process of balancing need never be more than sufficient to charge ·25 microfarad to ·005 volt.

The Capillary Electrometer.

The capillary electrometer was of the form introduced by one of us, and gave a measurable excursion with a difference of potential of ·001 volt.† The working pressure was about 6·5 cms. of mercury, and the meniscus was very rapid in its movements. A micrometer scale was used in the eye-piece of the observing microscope to ensure accuracy in balancing the E.M.F., and connection was made with the circuit by a double-make key which was closed only at the moment of the observation.

* 'Phil. Trans.,' 1885, p. 781.

† A more sensitive instrument would have been unmanageable.

The Interpolated Resistance.

The resistance interpolated in the circuit to prevent damage to the electrometer was a streak of graphite upon ground glass. By a simple arrangement the effective length of the streak could be adjusted, varying the resistance from zero to about a million ohms. In practice, 750,000 ohms was found to be ample. It may be noted that even this amount of resistance does not affect the sensitiveness of the electrometer if the insulation is good, but simply makes the movement of the mercury slower.

The Materials Used.

The purest electrotype copper procurable, and metallic silver of the degree of purity known as test metal, were supplied to us by Messrs. JOHNSON and MATTHEY; for the mercury, which had been distilled twice *in vacuo*, we are indebted to Mr. F. J. SMITH.

The nitric acid was purified according to the method described by one of us; it was sufficiently free from nitrous acid to give no colouration with metaphenylene diamine; the proportion of this impurity was, therefore, probably less than one part in two or three millions. For each separate experiment the metallic strips were thoroughly washed and cleansed, the success of the experiments depending upon the care bestowed on this point.

The Experimental Cell.

The copper and the platinum strips were fixed vertically in insulated brass screw-clamps, the lower end of each strip passing through a glass tube and projecting about a centimetre beyond it, the purpose of this arrangement being to prevent the possibility of the strips coming in contact with one another, or of the acid with the clamps. An ebonite bar which carries the clamps slides freely on a rod, and is prevented from twisting on it by a guide rod, at the top of which is a catch to fix the bar while everything is being prepared. On the rod is an adjustable stop to regulate the depth to which the copper and platinum strips can be immersed. The acid is contained in a beaker enclosed in a jacket supplied with water, the temperature of which remained constant in the interval of time required for an experiment.

The Circuit.

The general arrangement of the apparatus is shown in fig. 2 (from a photograph).

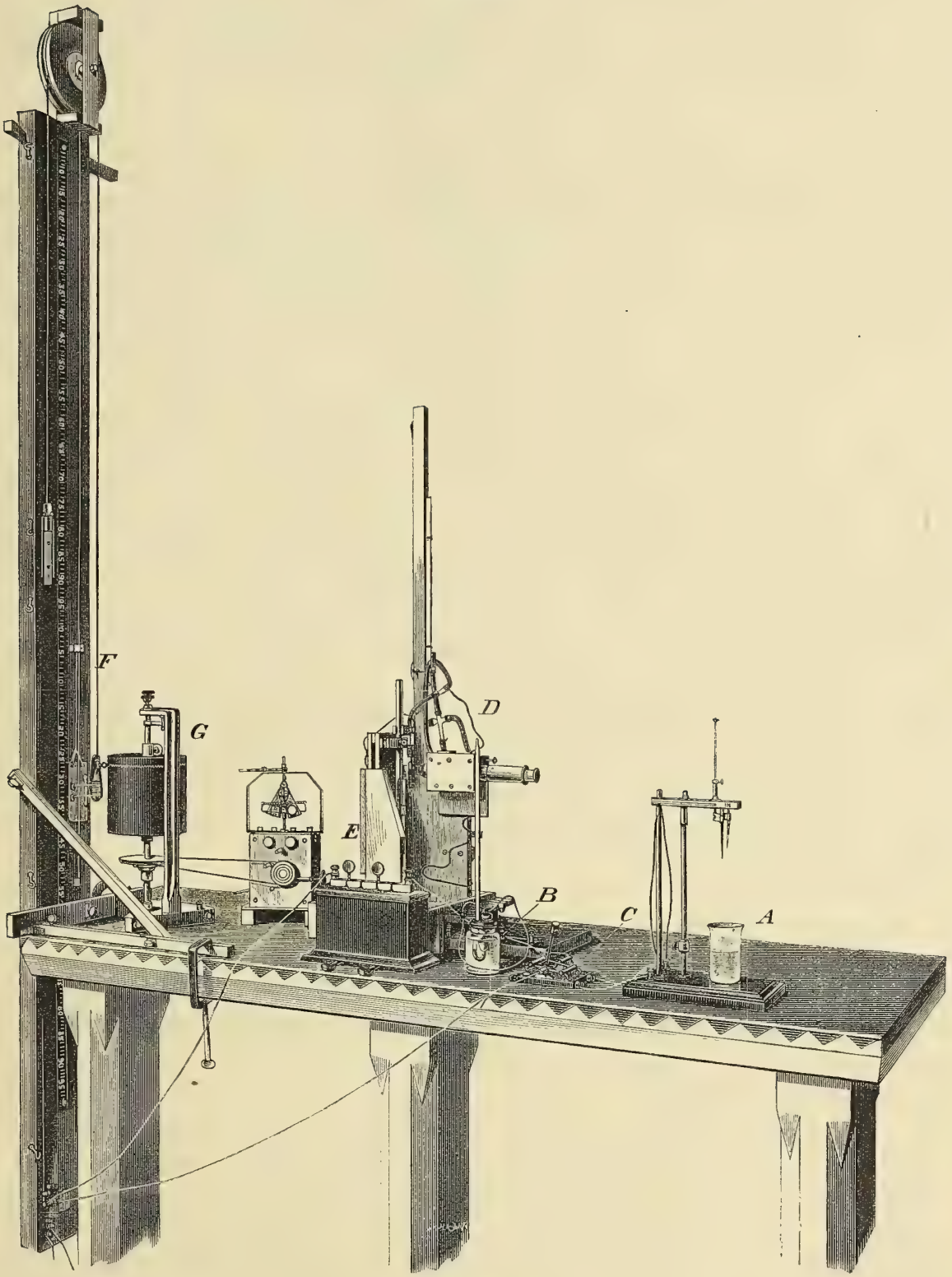
A = the experimental cell.

B = the standard cell.

C = the switch by means of which either the experimental cell or the rheochord can be put into the circuit.

D = the electrometer.

Fig. 2.



The apparatus.

E = the graphite resistance (represented by a resistance box).

F = the rheochord.

G = the recording cylinder, driven by clockwork (the Daniell's cells are omitted).

A diagram of the circuit showing the connections is given in fig. 1.

The Method of Experiment.

The method of conducting an experiment was as follows :—The copper and platinum strips were fixed in the clamps, and the beaker containing the acid placed beneath them. The rider of the rheochord was then set as nearly as possible in the position in which the E.M.F. of the derived circuit would balance the initial E.M.F. of the experimental cell, this having been determined by a preliminary experiment; the driving-clock was set going at such a rate as would bring the expected duration of the more important part of the experiment well within one revolution of the recording cylinder. At a given signal, one operator released the catch, and the bar carrying the clamps was dropped, suddenly immersing the metal strips in the acid. At the same instant the other observer pressed the contact key of the electrometer, which had up to that time been short-circuited and disconnected. If there was any movement of the mercury, he brought it back to zero as quickly as possible by shifting the rider of the rheochord up or down as might be required.

If, however, the position of the rider had been well chosen, there was no movement of the meniscus of the mercury, which remained at zero for a longer or shorter time, till the E.M.F. of the experimental cell began to rise. When this took place, the difference of potential derived from the rheochord being no longer sufficient to balance it, the meniscus commenced to move slowly down. The movement was checked by drawing the rider gently down, and thus increasing the balancing E.M.F.; and this was continued so long as the electrometer showed any change, the level of the mercury being kept constantly at zero.

When the E.M.F. of the experimental cell had become steady it was disconnected, and the standard cell put into the circuit instead by means of the switch. The rider was shifted until the E.M.F. of the standard cell was exactly balanced by that of the rheochord circuit, and its position was noted.* A line was next marked round the cylinder corresponding to some known position of the rider, generally at 1000 mm., and a series of short marks, corresponding to intervals of 100 mm., was added to serve as a scale of reference.

In addition to the record on the drum, the position of the rider was read off on the scale attached to the rheochord at the beginning and end of each experiment.

The curves were measured by a sliding T ruler, the vertical portion of which was graduated in millimetres and ended in a point, the zero of the scale, while the horizontal portion was provided with a vernier graduated into fifths of a millimetre.

* If the rider is set at 1438 mms., and the standard cell balanced by altering the resistance of the battery circuit, the rheochord readings are in decimals of a volt.

Thus the height of the curve above any arbitrary base line could be determined for each second of time, and by means of the recorded data the scale-readings could be directly converted into millivolts, the value of the Clark's cell being taken as 1·438 volt.

Copper / Nitric Acid / Platinum.

In the earlier experiments the platinum and copper strips were heated *in vacuo*, to eliminate traces of occluded gases; but in the later experiments, as the required apparatus was not available, this precaution was omitted.

In the succeeding Table the values are given for each interval of five seconds (unless there was no alteration) from the moment of immersion of the copper-platinum strips in the acid.

TABLE I.

Concentration of acid = 29·2 per cent. Temperature 15°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	54	50	214	105	448
5	93	55	234	110	450
10	114	60	266	115	468
15	133	65	297	125	470
20	155	70	341	130	480
25	167	75	374	145	482
30	177	80	413	175	490
35	189	90	422	185	494
40	198	95	434	205	512*
45	200	100	441		

In Table II. the results are given of another experiment conducted under precisely similar conditions.

TABLE II.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	47	40	272	120	524
5	92	45	389	130	527
10	117	50	475	155	533
15	122	55	491	190	535
20	134	75	503	225	539
25	148	90	507	265	543
30	175	95	513		
35	215	105	518		

* It is probable that the final values given in Tables I. and II. do not represent the maximum attainable, as our subsequent experience showed that this is only reached after the lapse of a number of minutes.

It will be seen, not only from these results, but more clearly from the recorded curve, that the initial E.M.F. of the cell Pt / HNO_3 dil. / Cu, under these conditions is at first very small, and that it rises at first slowly, then more or less rapidly, and finally very slowly until it becomes nearly constant. This last stage is necessarily very protracted as the curve more nearly approaches its asymptote; thus, in the results given in Table II. the alteration of E.M.F. during the last two minutes was only 19 millivolts. It is also worthy of note in these and succeeding experiments that although the general shape of the curve representing the rise of E.M.F. is fairly regular, yet indications are not wanting that there are minor irregularities in the rise, which may be described as secondary oscillations. The explanation of this almost unique case of an increase of E.M.F. is doubtless as follows: the E.M.F. at the start is that (or nearly so) of the combination platinum / nitric acid / copper, while that at the conclusion is that of the combination platinum / nitrous acid / copper. The rate of rise is due, other conditions remaining the same, to the number of molecules of nitric converted into nitrous acid. If the containing vessel was not perfectly clean, if the strip of copper contained a trace of oxide, or even was touched with the finger, or if the strip of platinum was moistened with a drop of liquid used in a previous experiment, then the rise of E.M.F. was almost instantaneous. The initial value depends upon the degree upon which these conditions of purity have been fulfilled, and also upon the success attending the somewhat difficult process of the purification of the nitric acid.

If, then, this alteration of E.M.F. is dependent upon the conversion of nitric to nitrous acid, the presence of a substance, such as urea, which would destroy the nitrous acid, would cause the phenomenon to be protracted. Accordingly an experiment was conducted under precisely the same conditions as before, except that ·25 grm. of urea was added to the 25 c.c. of nitric acid.

TABLE III.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	85	420	287	610	425
5	105	430	301	630	429
Constant	for 87″	450	314	640	437
95	150	460	327	660	448
105	160	470	345	670	450
120	166	480	356	680	460
Constant	for 45″	490	358	690	472
165*	171	510	368	710	483
330	235	520	371	750	494
340	243	540	379	780	515
360	249	560	382	790	519
390	252	570	393	810	527
400	266	580	412	895	542
410	272	590	414	915	550

* At this point there was a failure in the recording of the style.

On comparing the results given in Tables II. and III., it will be evident that in the latter case a much greater interval of time is required before the maximum value is reached, the numbers being 265 and 915 seconds respectively. This last interval is probably the measure of time required for the complete decomposition of the urea added, by the nitrous acid formed. These results were also confirmed by another pair of observations.

TABLE IV.—Conditions as before.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	109	55	370	115	542
5	297	60	404	135	548
10	233	65	437	155	555
15	—	70	453	160	568
20	257	75	476	200	575
25	270	80	492	240	581
30	288	85	503	280	584
35	302	90	510	320	586
40	313	95	516	360	592
45	332	100	523		
50	350	110	534		

The experiment was then repeated with introduction of ·25 grm. urea as in the case of Table III.

TABLE V.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	255	25	348	85	368
5	307	30	350	105	372
10	329	45	354	271	372
15	332	55	360		
20	341	65	364		

The difference in the rate of rise of the E.M.F. in the two cases is very evident, and confirms, in every respect, the previous observations. Other series of experiments were conducted, in order to determine the effects produced by alterations of conditions, such as concentration of acid, temperature, and addition of nitrous acid.

Effect of Concentration.

A sample of nitric acid was purified as before, and made up to a concentration of 15 per cent., *i.e.*, nearly one-half the strength of that used in the preceding experiments.

TABLE VI.—Concentration of Acid = 15 per cent. Temperature = 16°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	369	135	498	868	746
5	386	255	516	898	789
15	411	315	531	958	795
35	421	Interval	of 213″	988	800
45	428	528	588	1018	816
55	435	558	603	1048	820
65	446	588	622	1078	828
75	448	618	644	1108	832
85	462	668	648	1168	839
95	465	718	674	1198	846
105	477	778	679	1228	858
125	496	808	—		

If these numbers be compared with those of Table IV., it will be seen that the effect produced by decreasing the concentration is to lengthen out to a considerable degree the phenomenon of the increase of E.M.F., for to reach the same point, viz., 592 millivolts, 360″ were required in the one and about 540″ in the other case.

The higher initial value is probably due to a greater proportion of nitrous acid present initially in the nitric acid, as our experience has shown us that a difference of a very few parts per million makes a considerable alteration not only in the rate of chemical change, but also in the variations of electromotive force; it is also probably conditioned by the amount of occluded gases in the metallic strips.

Decrease of concentration also seems to increase to a slight extent the final value of the E.M.F., which is doubtless influenced to a greater degree by the proportion of nitrous acid, but to a less degree by the quantity of metallic salt dissolved per unit volume of the acid.

The experiment was then repeated under the same conditions, except that a small quantity of sodium nitrite was added to the nitric acid; a maximum value of 711 millivolts in one case and 727 in another was reached *at once*. This result shows that the increase of E.M.F. observed is due for the most part to the *production of nitrous acid*. The slightly lower final value of the E.M.F., when the nitrous acid has been added as such, is in accordance with the observations of one of us, that nitrous acid ready formed is not so active a material as nitrous acid in the alternative process of formation and decomposition.

As a confirmation of the above values the results are given below of one preliminary set of determinations made by a different method and by means of a Thomson's quadrant electrometer; the values were compared with another Clark's cell of different construction to that used by us.

TABLE VIA.—Concentration of Acid = 26 per cent. Temperature = 16°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	648	100	738	200	808
10	648	110	748	220	814
20	642	120	760	240	822
30	656	130	768	280	828
40	668	140	774	340	828
50	685	150	783	370	858
60	689	160	783	430	853
70	705	170	794	550	863
80	724	180	801		
90	732	190	808		

On comparison of these results with those given in Table VI., it will be seen, as before, that the time required to reach the maximum value of the E.M.F. is less with more concentrated acid. The experiment was then repeated with addition of 1 per cent. of sodium nitrite; a value of 755 millivolts was reached at once, and there was no further increase. Other preliminary experiments gave nearly identical results, but it is not thought necessary to quote them.

Effect of Temperature.

To determine the effect produced by variation of temperature, two experiments were conducted with the same acid as that used above, but heated in a jacket of water.

TABLE VII.—Concentration of Acid = 15·0 per cent. Temperature = 36°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	600	40	779	120	860
5	636	45	807	140	866
10	648	50	815	150	898
15	700	70	821	After 2′	862
20	712	80	834	„ 5′	848
25	742	90	845	„ 7′	846
30	768	100	859		

TABLE VIII.—Concentration of Acid = 15·0 per cent. Temperature = 50°.

Time.	E.M.F.	Time.	E.M F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	634	6	858	14	884
2	755	8	870	Constant at	864
4	811	10	874		

It will be evident that an increase of temperature causes a more rapid rise of the E.M.F., a result due to the more rapid solution of the copper, and consequent production of nitrous acid; it is probable from other experiments that, if the conditions could be kept absolutely fixed, the rise of E.M.F. per unit of time would be found to be a logarithmic function of the temperature. As a comparison, the times required for the rise of E.M.F. between the same limits, viz., 634 and 858 millivolts, are 840, 75, and 6 seconds respectively.

Another point is worthy of observation, namely, that at the higher temperatures the E.M.F. reaches a maximum value, and then slowly, but somewhat irregularly, decreases until it reaches a constant value; this may be due to the vapour of water either oxidising the surface of the metal above the acid, or carrying up bubbles of oxides of nitrogen which attack the metal, and thus produce a temporary increase.

In conclusion, the value of the E.M.F. in water for the copper-platinum strips used in the above experiments was determined. It was found to be 410 millivolts (and after 3 hours 441), which is practically the initial value given in Table IV., thus showing that the E.M.F. of the combination copper / water / platinum is not very different from that of copper / dilute nitric acid / platinum, while it is even probable that the still lower value obtained with more concentrated acid is due to an immediate formation of oxide.

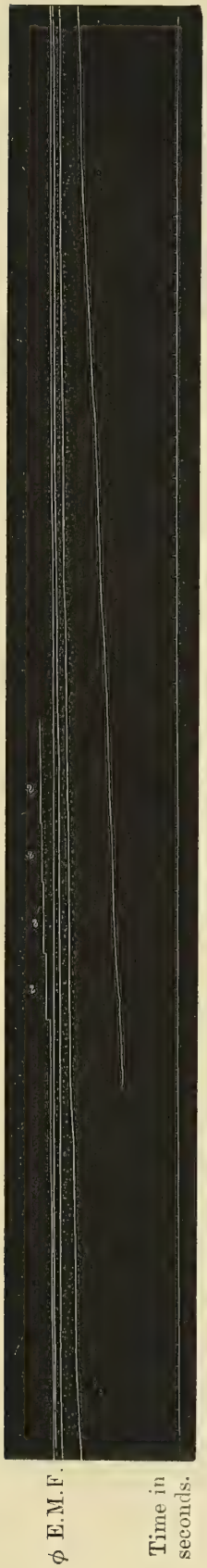
A characteristic curve is given of the rise of E.M.F. in one experiment, in which the abscissæ represent the time in seconds from the start, while the ordinates are reduced rheochord readings which are functions of the E.M.F. The same remarks apply in the case of the curves for other metals.

Silver / Nitric Acid / Platinum.

Precisely similar experiments to those described above were conducted with metallic silver, which showed the same general results of rise of E.M.F.; this will be evident from the results given in the following Tables:—

TABLE IX.—Concentration of Acid = 15·0 per cent. Temperature = 15°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	319	90	414	240	560
10	325	110	442	250	572
20	350	120	446	280	577
30	355	130	465	300	600
40	364	140	474	370	619
50	377	150	484	440	629
60	395	180	491	560	650
70	400	190	506	680	674
80	409	200	525	800	681



Characteristic curve of Copper / Nitric Acid / Platinum. Temp. 16°. Acid, 15 per cent.

Characteristic curves of Silver / Nitric Acid / Platinum. *a* at temp. 29°; *β* at temp. 16°.

The curve representing the variation of E.M.F. rises at first with uniform regularity, but in the *later stages*, in an irregular or staircase form ; this is not due to instrumental error, but is the actual record of a periodic change in the experimental cell.*

There can be but little doubt, considering the results obtained from a chemical standpoint by Russell, that in the case of this metal also the initial E.M.F. is that of silver / nitric acid / platinum, while the final is that of silver / nitrous acid / platinum. But to meet the possible objection that the rise of E.M.F. is the effect of that cause to which we attribute it, and not to some gradual formation of oxide or, it may be, of metallic nitrate, on the surface of the metal above the level of the acid liquid, the experiment above was repeated under precisely the same conditions, except that the metallic surface above the liquid was coated over with vaseline to protect it from acid fumes and oxides of nitrogen.

Table X.—Conditions as in Table IX.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	267	100	349	320	450
10	303	110	356	390	456
20	305	120	367	400	487
30	319	130	374	410	499
40	322	150	399	480	508
60	332	220	402	490	520
80	338	300	426	610	552
90	345	310	435	1570	680

On a comparison of the results given in Tables IX. and X., the values for the initial and final E.M.F.'s are seen to be within the limits of experimental error ; the form of the curves in the two cases is the same, and in many parts of their course they are superposable. This shows, therefore, that the rise of E.M.F. is not due to any alteration of the metallic surface above the acid liquid.

Effect of Temperature.

The experiments were repeated, the one condition of temperature being varied ; the results obtained are given below.

* It was noticed that if the graphite resistance was cut out and the rider allowed to remain in one position for a few seconds, the mercury of the electrometer did not always rise steadily, but frequently moved with a series of rapid oscillations, which apparently coincided with the evolution of bubbles larger than usual. This was seen also in the case of copper.

TABLE XI.—Concentration of acid = 15·0 per cent. Temperature 29°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	260	60	485	250	547
10	314	70	503	390	569
20	388	80	509	410	579
30	427	100	519	531	594
40	455	110	524	Probably not complete.	
50	479	180	538		

TABLE XII.—Concentration of acid = 15·0 per cent. Temperature 39°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	360	40	636	90	664
10	522	50	643	120	668
20	563	60	651	130	680
30	597	70	660		

A fourth experiment was conducted at 41°, but the rise of E.M.F. was too rapid for correct observation.

The results obtained with silver confirm in every particular those obtained with copper, viz.: that increase of temperature produces a more rapid rise, but does not appreciably alter the final value of the electromotive force.

Bismuth / Nitric Acid / Platinum.

The metal used for these determinations was cast into a small cylinder, 6 inches long and $\frac{1}{6}$ inch diameter; between each operation the coating of black oxide, formed by the immersion of the metal in the acid, was carefully removed.

TABLE XIII.—Concentration of acid = 15 per cent. Temperature 16°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	685	60	946	190	995
10	760	70	—	220	1024
20	810	80	956	310	1055
30	850	90	971	490	1058
40	875	100	982		
50	896	120	986		

On repetition of the above experiment, under precisely the same conditions, the following values were obtained.

TABLE XIV.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	737	40	913	80	1006
10	809	50	940	90	1014
20	872	60	944	160	1040
30	902	70	985	Constant.	

Thus the behaviour of a circuit consisting of bismuth, platinum, diluted nitric acid is precisely similar to those containing the metals copper and silver. The net result obtained was to be expected from the observations of one of us, that no chemical reaction takes place between bismuth and nitric acid, save in the presence of nitrous acid. As in the previous example, the initial value of the E.M.F. is approximately that of bismuth / nitric acid / platinum, while the final value is that of bismuth / nitrous acid / platinum.

Effect of Temperature.

The above experiments were repeated with alteration of the one condition of temperature. In the two following tables the results are given of experiments conducted at 34° and 66° respectively.

TABLE XV.—Temperature 34°.

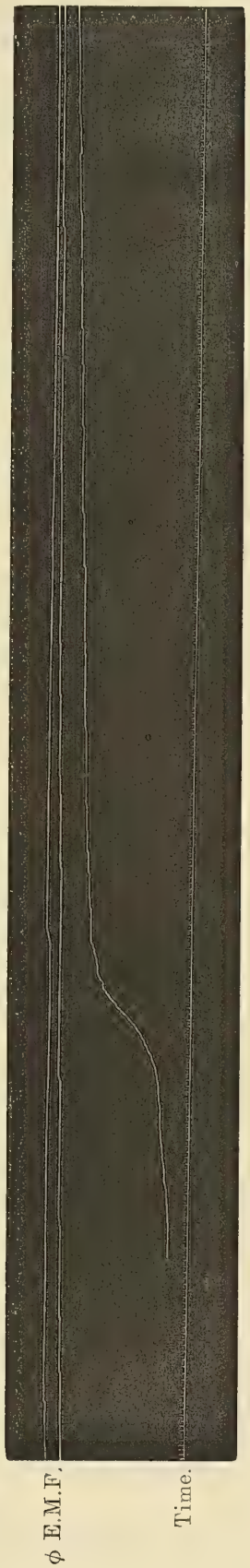
Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	884	40	996	120	1033
10	896	50	1011	Constant.	
20	950	80	1058		
30	950	110	1045		

TABLE XVI.—Temperature 66°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	682	15	1128	30	1091
5	999	20	1123	55	1061
10	1020	25	1112	Constant.	

As in the case of copper and silver, so also in that of bismuth, increase of temperature produces a more rapid rise of E.M.F.; and, in fact, at a temperature of 66°

Characteristic curve of Bismuth / Nitric Acid / Platinum. Temp. 16°. Acid, 15 per cent.



Characteristic curve of Mercury / Nitric Acid / Platinum. Temp. 15°. Acid, 29·1 per cent.

it was almost too sudden for correct observation; the final value, however, is not affected. It will be seen that at the higher temperatures the E.M.F. rises to a maximum and then falls slightly to the value at which it remained constant for several minutes. The possible reasons for this phenomenon have been discussed above.

Mercury / Nitric Acid / Platinum.

The mercury was introduced into the acid by means of a small U-tube having limbs of unequal length, in the shorter one of which, the acid came in contact with the metal, while down the longer, a platinum wire passed serving as a connection with the rest of the apparatus. In the use of mercury there is, however, this disadvantage, that there is a difference of potential due to capillarity, which produces an accidental but permanent charge in the instrument, and diminishes or increases, as the case may be, the readings by a constant amount. In the particular experiment quoted, the readings were diminished; and the values given below are intended merely to illustrate the similarity of the behaviour of mercury to that of the other metals discussed above.

TABLE XVII.—Concentration of acid = 29·2 per cent. Temperature 16°.

Time.	E.M.F.	Time.	E.M.F.	Time.	E.M.F.
seconds.	millivolts.	seconds.	millivolts.	seconds.	millivolts.
0	$0 + x$	90	$260 + x$	195	$305 + x$
10	13 „	100	262 „	215	307 „
20	29 „	105	266 „	235	315 „
30	41 „	115	271 „	255	317 „
40	103 „	125	279 „	295	331 „
50	225 „	145	281 „	335	335 „
60	239 „	155	285 „	355	339 „
70	248 „	165	291 „	420	361 „
80	255 „	175	301 „		

Other experiments conducted in the same manner gave precisely similar results with different errors due to capillarity, so that in this case also it is probable that the final E.M.F. is that of the combination mercury / nitrous acid / platinum.

Conclusion.

I. The experiments detailed above show that when the metals, copper, silver, bismuth, and mercury are introduced into purified nitric acid of varying degrees of concentration, and a couple made with platinum, the electromotive force of such a cell increases considerably until it reaches a constant and (in most cases) a maximum value.

This rise of E.M.F. is attributed to the production of nitrous acid by the decomposi-

tion of the nitric acid, and the final value is considered to be due to the former acid only, while the initial value is due for the most part to the latter acid, though it is affected to a remarkable degree by the amount of impurity of nitrous acid either initially present or produced by minute and unavoidable uncleanliness of the metallic strips and the containing vessel.

II. If nitrous acid has been previously added to the nitric acid then the maximum E.M.F. is reached at once.

III. If the conditions, namely, increase of temperature, of impurity, and of concentration of acid, are such as could favour a more rapid production of nitrous acid, then the rise of E.M.F. is concomitantly more rapid.

IV. Conversely, if the conditions are unfavourable to the production of nitrous acid the rise of E.M.F. is less rapid.

V. If any substance, such as urea, be added, which would tend to destroy the nitrous acid as fast as it may be formed, then the rise of E.M.F. is extremely slow, being dependent upon the number of molecular impacts of nitrous acid upon the surface of the metal. Thus the results obtained by the electrometer and of the chemical balance are in every way confirmatory, the one of the other. These results open out the further question as to whether the E.M.F. of batteries, in which concentrated nitric acid forms an ingredient, is dependent not upon the nitric acid *per se*, as hitherto supposed, but upon the nitrous acid present in the fuming nitric acid, and formed also by the reduction of the acid by the hydrogen, which would otherwise be given off from the zinc and sulphuric acid.

Also since recent investigations have tended to show that the products of electrolysis of sulphuric acid are instrumental in effecting the solution of zinc, it is probable that the E.M.F. of a combination zinc / sulphuric acid / platinum may exhibit variations similar to those of combinations, wherein nitrous acid, one of the products of electrolysis of nitric acid, is continuously being formed.

To these problems we hope at no distant date to direct our attention.

Finally, we would express our thanks to Mr. F. J. Smith, of Trinity College, for kind assistance and advice; to Mr. H. G. Madan, of Queen's College, for taking photographs of our apparatus; and especially to Professor Burdon Sanderson for kindly placing at our disposal the apparatus from his department.

The experiments were conducted partly in the Chemical Department of the Museum, and partly in a private laboratory.

IX. *The Influence of Stress and Strain on the Physical Properties of Matter.*—Part III. *Magnetic Induction* (continued).—*The Internal Friction of Iron, Nickel, and Cobalt, studied by means of Magnetic Cycles of very minute range.**

By Herbert Tomlinson, *B.A., F.R.S.*

Received August 22,—Read December 5, 1889.
Revised May 19, 1891.

Table of Contents.

	PAGE.
Purpose of the investigation	341
Description of apparatus	342
Mode of annealing	344
Relation between change of induction and change of force in any cycle, $y = Ax + Bx^2$	345
The values of A and B the same in cycles of different ranges of force	348
Mathematical formulæ	349
The limit of force within which A and B are constant	350
The effect of altering the magnitude of the steps by which the force is increased or diminished	351
Molecular "accommodation"	351
"Fatigue of elasticity," or upset of "accommodation"	355
Effect of annealing on the values of A and B	358
The temporary effect of change of temperature on the values of A and B	361
The effect of sudden cooling on the values of A and B	362
The effects of permanent extension, hammering, and torsion on the values of A and B	362
The temporary effect of loading on the values of A and B	363
Relations between the dissipations of energy in torsional and magnetic cycles	366
Summary	368

Purpose of the Investigation.

Three years ago the author had the honour of presenting to the Royal Society two memoirs relating to the internal friction of metals, as studied by the logarithmic

* An abstract of this paper has been published in 'Roy. Soc. Proc.,' vol. 47, p. 13.

In prosecuting the research, the author has received assistance from the "Elizabeth Thompson Science Fund," U.S.A., and from the English "Government Grant." For the aid rendered from these sources he begs to return his most grateful thanks.

decrement of arc of torsionally oscillating wires;* these were followed by a third, dealing with the effect of magnetisation on the internal friction of iron.† The results recorded in these memoirs seemed to favour the view adopted by G. Wiedemann, that the main part of the dissipation of energy met with in a torsional cycle arises from *the permanent rotation of the molecules about their axes*, and served as an encouragement to attempt to ascertain how far the dissipation of energy which accompanies *cyclic changes of magnetisation* would be amenable to the laws which regulate the dissipation of energy in torsional cycles. The cyclic changes of magnetisation which attend on cyclic changes of magnetising force have been studied by Ewing and others,‡ but mainly for comparatively great magnetising forces—for forces, in fact, far exceeding those for which there seemed any likelihood of any *simple* laws holding good.

Lord Rayleigh has shown, however,§ that when the magnetising force is sufficiently minute the ratio of induction to force (permeability) in iron, tends rapidly to a *finite constant value* as the force diminishes, and that the dissipation of energy in any cycle can be very simply expressed. The author proposed, therefore, to make use of very feeble magnetising forces, and to render their effects sufficiently evident by very sensitive arrangements. Since the same experiments would suffice to determine both the *constant* permeability mentioned above, and the dissipation of energy in any cycle, the two have been studied side by side. The metals which have been examined are iron, nickel, and cobalt; they have been tested not only at the ordinary temperature of the room but also at 100° C. Finally, the effects of mechanical stress on the *constant* permeability and on the dissipation of energy have received somewhat careful attention. The "ballistic" method of observation has been adopted in these researches.

Description of Apparatus.

Induction Coils.

Three induction coils were used at different times, the first of which was constructed as follows: A brass tube, slit throughout its entire length, was coated with paper stuck on with shellac varnish; the length of the tube was 360 cm., and the internal and external diameters were 5 and 5·6 cm. respectively. On the paper-covered tube, to within about 1 cm. of each end, was wound uniformly, and in a single layer, a silk-covered copper wire, there being 8·500 turns per cm. of length;‖ this served as

* 'Phil. Trans.,' vol. 177 (1886); 'Roy. Soc. Proc.,' No. 244, 1886 (vol. 40, p. 343).

† 'Phil. Trans.,' vol. 179 (1888), A.

‡ 'Phil. Trans.,' 1885.

§ 'Phil. Mag.,' vol. 23, March, 1887.

‖ The winding of the coils and the counting of the turns were in all cases carefully performed by the author himself.

the primary coil. The primary coil was then coated with paper, and on it, to within 20 cm. of each end, and, in a single layer, were wrapped no less than 2234 convolutions of silk-covered copper wire; this served as the secondary coil.

The second of the induction coils had its primary wire wrapped round the outer of two coaxial copper tubes, each 120 cm. in length, connected at the ends, and enclosing between them an annular air-space 0·6 cm. thick; there were 8·00 turns per cm. of length. Inside the inner of the two coaxial tubes was placed the secondary coil. This consisted of 2494 convolutions, wrapped in three layers, on a thin brass tube, slit from end to end, whose length was 100 cm., and inner diameter, 1·2 cm. This second induction coil was used both for experiments at the ordinary temperature of the room and at 100° C. In the latter case steam was made to pass through a tube soldered into one end of the annular air-space mentioned above, and out again through another tube at the other end, where it was condensed in cold water.

The third induction coil was constructed mainly with the view of testing the metal cobalt, which could not be procured in the same lengths as the iron and nickel. The primary coil was the same as in the last induction coil, but the secondary consisted of a thin brass tube slit throughout its length, round which were wrapped in six layers 5752 convolutions of silk-covered copper wire. The length of the coil was 48 cm., and its inner diameter, 0·9 cm.

The induction coils, when in use, were placed horizontally with their axes at right angles to the magnetic meridian, so that there might be no appreciable effect produced by the earth's magnetic force. The lengths of the primary coils were such that in all cases the metal cores may be regarded as endless.

The Galvanometer.

The magnetic induction was measured by the throw of the needles of a very sensitive Thomson's reflecting galvanometer, whose scale was placed 2 metres from the mirror. For reasons which will be mentioned presently, the time of vibration was made very short,* whilst the damping was rather considerable; but this, as will be seen, did not interfere with the accuracy of the observations.

The Battery.

For most of the experiments, a single Daniell's cell, charged with a saturated solution of sulphate of copper and a semi-saturated solution of sulphate of zinc, was used. The E.M.F. of the cell was compared from time to time with a Latimer Clark's standard cell.

* The needles took about $\frac{3}{4}$th second in swinging out from their position of equilibrium

The Earth Coil.

This was employed for determining in absolute measure the magnetic induction. It consisted of a wooden circular plate carefully turned, and capable of revolving about a horizontal axis. Round the rim of the plate were wound in a single layer seven complete turns of silk-covered copper wire, whose ends were soldered to amalgamated copper discs dipping in mercury cups, and revolving with the circular wooden plate. The plane of the coil was kept in a horizontal position, and at the end of each experiment was quickly turned through 180 degrees; at each turning the throw of the galvanometer was observed. This earth coil reading was the throw corresponding to

$$2 \times 7 \times 607 \times 0{\cdot}4371$$

lines of induction; where 7 is the number of turns, 607 the area of the earth coil in sq. cm., and 0·4371 the local value in C.G.S. units of the vertical component of the earth's magnetic force.*

Arrangement for Demagnetising by the Method of Reversals.

This was the same as that adopted by Mr. Bidwell in his experiments described in the 'Philosophical Transactions,' vol. 179 (1888), A., p. 207.

Mode of Annealing.

Those of the rods or wires which did not exceed 4 ft. in length were placed inside an iron tube of 1·5 cm. internal diameter; the tube was then put with its axis horizontal, and in a direction at right angles to the magnetic meridian in one of Fletcher's new tube gas furnaces. The furnace was 4 ft. in length, and was provided with 40 burners, which were capable of heating the tube and its contents to a temperature of about 1000° C. This high temperature was maintained for several hours, and afterwards the tube and its contents were allowed to cool slowly in the furnace. The longer rods were annealed by Mr. Thomas Andrews, of the Wortley Iron Works, Sheffield, who was good enough to supply three rods of iron (Wortley best scrap iron) and three of steel (Firth's soft cast steel), each 360 cm. in length and 1 cm. in diameter. The rods were enclosed in a wrought iron tube, and placed in a furnace at a temperature of 566° C., and allowed to cool gradually.

* In order to ascertain whether this value might be affected by masses of iron in and about the building, the earth coil readings inside the room were compared with those taken when the coil was turned in a garden some distance away from the house; the two sets of readings did not materially differ.

Relation between Change of Induction and Change of Magnetising Force in any Cycle.

Lord Rayleigh has shown* that provided the magnetising force does not exceed a certain limit, the relation between change of induction and change of force in any cycle is very simple.

If x be any *change* of force, and y the corresponding *change* of induction as the force varies from its maximum value to zero, if the force be always applied in the same direction, or from its maximum value in one direction to an equal maximum in the opposite, if the force be alternately applied in opposite directions, then

$$y = Ax + Bx^2$$

where A and B are constants. Lord Rayleigh experimented with *hard* iron, using the "*magnetometric*" method. When examined by this method, *annealed* iron was found to be a much less satisfactory subject, because the settling down of the iron into a new magnetic state proved to be far from instantaneous. As the main object of the enquiry was to compare the dissipations of energy in torsional and magnetic cycles, and as the laws of the former were deduced from experiments on wires freely vibrating with comparatively short vibration-periods,† it was decided to use the "*ballistic*" method with a galvanometer of *very short* vibration-period, so as to avoid as much as possible the effects of time-lag in the annealed metals. It has been already mentioned that the time taken by the galvanometer needles to swing out from their position of equilibrium was in most of the experiments only $\frac{3}{4}$th second, so that the complete vibration-period was only 3 seconds; but preliminary experiments *very carefully made with complete vibration-periods of* 2, 3, *and* 4 *seconds gave precisely the same results even with most thoroughly annealed iron.* It was accordingly assumed that with a sufficiently short vibration-period the effects of time-lag do not make themselves materially felt in the "ballistic" method.‡ Nevertheless it was thought advisable to make other preliminary trials, for, at first sight, the galvanometer from its very short vibration-period, and the rather considerable damping of its needles seemed unsuitable for observations by the ballistic method, in which it is usual to employ a galvanometer of long vibration-period, and with a small amount of damping. All that is required, however, in order to interpret the readings correctly in absolute measure, is to *compare* the throw of the needles produced by suddenly altering the magnetising force with that produced by suddenly turning the earth coil through 180 degrees; and provided the throw is proportional to the transient current, the reduction to absolute measure will be accurate.

* 'Phil. Mag.,' vol. 23, March, 1887, p. 236.

† Within the limits of the experiments the dissipation of energy was found to be independent of the vibration-period.

‡ It need hardly be said that as a consequence of this the present investigation is entirely concerned with the *instantaneous* changes which are produced by the application or removal of the magnetising force.

This point was closely examined in different ways, and it was found that the deflection of the galvanometer was strictly proportional to the transient current within the limits of the scale.*

The following experiments show that the simple relation between change of magnetic induction and change of force which Lord RAYLEIGH proved to hold good for *hard* iron when tested by the *magnetometric* method will equally hold for *well-annealed* iron, nickel, and cobalt when only the *instantaneous* changes of induction such as are registered in the ballistic method by a galvanometer of short vibration-period are taken into consideration.

Experiment I.

A well-annealed iron rod, 366 cm. in length and 1·116 cm. in diameter, was, as a preliminary,† put through several magnetic cycles whose maximum magnetising forces were $\pm$ 0·07441 C.G.S. units. The magnetising force was changed by steps, which could be accomplished by suddenly altering, by means of a single plug, the resistance in a resistance box connected with the primary circuit. The observations recorded in the next Table were then made.

TABLE I.

Sum of the throws obtained with the iron in the induction coil, as the magnetising force was changed by steps from its greatest value in one direction to its greatest value in the opposite direction. T.	Ditto with the iron out of the coil. C.	T − C.	Total *change* of magnetising force in C.G.S. units, reckoned from its highest value in one direction to an equal value in the opposite direction. x.	Total *change* of induction in C.G.S. units, reckoned from its highest value in one direction to an equal value in the opposite direction. y.	Ditto calculated from the formula $y = 196{\cdot}3x + 88{\cdot}8x^2$.
255·75	15·43	240·32	0·01750	3·454	3·461
528·06	31·60	496·46	0·03570	7·111	7·121
817·63	48·45	769·18	0·05467	10·998	10·996
1126·79	66·25	1060·54	0·07441	15·098	15·098
1438·53	84·05	1354·48	0·09415	19·240	19·286
1741·90	100·90	1641·00	0·11311	23·319	23·340
2036·25	117·08	1919·17	0·13132	27·284	27·307
2322·22	132·50	2189·72	0·14880	31·175	31·175

The numbers given in Table I. are the means of ten sets of observations. The observed changes of magnetic induction agree very closely with those calculated from the formula

$$y = Ax + Bx^2.$$

* Of course if a flat scale be used allowance must be made for the flatness of the scale. Even when the scale is 2 metres from the mirror the correction amounts to more than 1 per cent. when the deflection reaches 300 mm.

† Unless the contrary is stated, it must be understood that in all cases the metals were subjected to *preliminary* cycles of the same range of force as in the actual experiment.

The values of A and B, in the present instance, are 196·3 and 88·8 respectively; they were obtained by substituting in the formula the two values of x, 0·07441 and 0·14880, and the corresponding values of y. Of course, a still closer agreement between calculation and observation could have been obtained by adopting the method of least squares.

Experiment II.

Four pieces of well-annealed nickel wire, each 120 cm. in length and 0·1525 cm. in diameter, were tied together in a bundle and tested in the same manner as the iron with magnetising forces whose extreme ranges were $\pm$ 0·4043 C.G.S. units. The wire contained 97·5 per cent. of nickel, and 0·67 per cent. of iron.

TABLE II.—Nickel Wire.

Total change of force reckoned from its greatest value in one direction to an equal value in the opposite direction. x.	Total change of induction ditto. y.	Ditto calculated from the formula $y = 63{\cdot}31x + 4{\cdot}211x^2$.
0·1060	6·74	6·76
0·2085	13·39	13·38
0·3080	19·88	19·90
0·4043	25·60	25·60
0·5006	32·78	32·76
0·6001	39·56	39·51
0·7026	46·54	46·56
0·8086	53·96	53·96

Experiment III.

A strip of well-annealed cobalt, 56 cm. in length, 0·08547 cm. thick, and 0·7285 cm. wide. The strip contained 98 per cent. of cobalt, no trace of nickel, and barely a trace of iron.* The extreme ranges of magnetising force were $\pm$ 1·359 C.G.S. units.

* For the analyses of the metals nickel and cobalt, the author is indebted to Mr. J. S. JOHNSON and Professor J. M. THOMSON, of King's College, Strand.

TABLE III.—Cobalt Strip.

Total change of force reckoned from its greatest value in one direction to an equal value in the opposite direction. x.	Total change of induction ditto. y.	Ditto calculated from the formula $y = 65{\cdot}14x + 1{\cdot}26x^2$.
0·2620	17·23	17·16
0·5080	33·51	33·42
0·7395	48·88	48·85
0·9579	63·55	63·55
1·1640	77·54	77·53
1·3590	90·78	90·84
1·5540	104·28	104·28
1·7601	118·58	118·54
1·9785	133·91	133·78
2·2100	150·28	150·12
2·4560	167·69	167·58
2·7180	186·21	186·35

From Tables II. and III. it appears that the formula $y = Ax + Bx^2$ holds good for nickel and cobalt.

We may now regard it as established that not only for *hard iron*, but also for *well-annealed iron, nickel, and cobalt*, the *instantaneous* changes of induction follow the above-mentioned simple law. Lord RAYLEIGH has, moreover, proved that with *hard iron* A and B will have the *same* values for cycles of *different* ranges of force. The next experiments show that this is the case with *annealed iron, nickel, and cobalt* when tested as above.

The values of A and B the same for different ranges of force.

Experiment IV.

Ten pieces of well-annealed iron wire, each 120 cm. in length and 0·1 cm. in diameter, were tied in a bundle and tested with cycles of different ranges of force. As a preliminary the wires were subjected to a series of cycles whose ranges were gradually diminished from ± 0·2144 to zero.

TABLE IV.—Bundle of Iron Wires.

Maximum ± force used in the cycle. C.G.S. units.	A. C.G.S. units.	B. C.G.S. units.
0·01192	129·2	40·6
0·07300	129·3	41·9
0·13932	129·5	40·5
0·21440	129·4	42·0

Experiment V.

The same nickel wire as before, and treated like the iron to preliminary cycles of range gradually decreasing from the highest force to zero.

TABLE V.—Bundle of Nickel Wires.

Maximum ± force used in the cycle. C.G.S. units.	A. C.G.S. units.	B. C.G.S. units.
0·1043	57·58	4·758
0·1924	57·56	4·643
0·3902	57·66	4·835

Experiment VI.

The same cobalt strip as before, and treated like the iron and nickel to preliminary cycles.

TABLE VI.—Cobalt Strip.

Maximum ± force used in the cycle. C.G.S. units.	A. C.G.S. units.	B. C.G.S. units.
0·6195	65·090	1·261
0·8510	65·269	1·273
1·1097	65·086	1·263
1·3590	65·094	1·256

MATHEMATICAL FORMULÆ.

Having thus established that, within the limits of the forces used, the *change* of induction (y) can be expressed in terms of the *change* of force (x), as the force varies by any amount, in one direction going from + X to − X, or from − X to + X, where X represents the maximum force in any cycle by the equation

$$y = Ax + Bx^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and having also shown that A and B are not only constant in any particular cycle, but have the same values for cycles of different ranges of force, it can be easily shown that

$$\mu_0 = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where μ_0 is Lord RAYLEIGH'S *constant* permeability, *i.e.*, the value to which the permeability tends as the force becomes infinitesimally small.

Again, if ΔE denote the dissipation of energy per cubic cm. in any cycle of range of force $\pm X$

$$\Delta E = \frac{2B}{3\pi} X^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If the force be always applied in the *same* direction, and have a maximum value X the value of ΔE is, of course, $\frac{1}{8}$th of that in (3).*

Again, each time in a $\pm$ cycle the force is taken from 0 to $\pm X$, the change of induction exceeds the change of induction as the force is taken from $\pm X$ to 0 by a certain amount (P). If 2Y be the whole change of induction as the force varies from $\pm X$ to $\mp X$

$$A = \frac{Y - P}{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$B = \frac{P}{2X^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

These last are convenient equations for determining the values of A and B.

Also, we have

$$\Delta E = \frac{PX}{3\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as a convenient expression for determining the dissipation of energy per cubic cm.

THE LIMIT OF MAGNETISING FORCE WITHIN WHICH A AND B ARE CONSTANT.

It has been shown that with annealed iron, nickel, and cobalt, the values of A and B are the same for cycles of different ranges of force *provided the force does not exceed a certain limit.* It is impossible to define this limit at all exactly for any metal, but the following table will give a rough notion of the value of the limit for *well-annealed* metals.

TABLE VII.

Metal.	Limit of force within which A and B have the same values in cycles of different ranges. C.G.S. units.
Iron	0·1 to 0·4†
Nickel	0·39
Cobalt	1·36

* This was actually proved by experiment.

† Depending upon the amount of "accommodation." See article on "Molecular Accommodation."

The Effect of Altering the Magnitude of the Steps by which the Force is Increased or Diminished.

So long as the force is confined within the limits given in the last table, the *magnitude of the steps by which it is altered is of no moment*,* but as soon as these limits are passed the magnitude of the steps becomes material.

Experiment VII.

A bundle of well-annealed iron wires was put through cycles of ranges of $\pm$ 0·6 C.G.S. units of force. At first this force was applied and removed in six steps† of equal magnitude; the number of steps was then diminished as in Table VIII.

Table VIII.

Number of steps by which the force was applied or removed.	Sum of the throws as the force was applied, starting from zero to + or − 0·6 C.G.S. units. U.	Ditto, as the force was removed. D.	U − D, or P.
6	812·5	655·0	157·5
4	825·0	653·0	172·0
3	837·0	654·0	183·0
2	884·5	657·0	227·5
1	908·0	656 0	252·0

The magnitude of the steps has very little influence on the sum of the throws on the *removal* of the force, but a very perceptible one on the sum of the throws on the *application* of the force, so that the dissipation of energy in the complete cycle, as is shown in the fourth‡ column of the table, is considerably *increased* by decreasing the number of steps.

Molecular Accommodation.

When a recently annealed wire is vibrating torsionally the logarithmic decrement of arc is greatest at the very beginning of the vibrations, and gradually diminishes until it becomes sensibly constant. During this period the molecules, to use an expression adopted by Weber, G. Wiedemann and others, are "*accommodating*" themselves to the motion

* This point will be further considered in connection with the relations between the dissipation of energy in torsional and magnetic cycles.

† Six for application and six for removal of both the + and − forces, *i.e.*, twenty-four steps in the complete cycle.

‡ See "Mathematical Formulæ," equation (6), where it is shown that the dissipation of energy is proportional to P.

The author has pointed out that the amount of "accommodation" depends, in the case of *iron*, upon the *length of time* during which the wire vibrates, and not merely upon the *number* of vibrations. Thus a well-annealed iron wire when tested within ten minutes after suspension was found to have a logarithmic decrement of 0·003011 ; after one hour, of 0·001195 ; and after one day, of 0·001078. After the last-mentioned period the logarithmic decrement became sensibly constant ; the wire had apparently "accommodated" itself as far as possible. It was then, however, discovered that on heating the wire to 100° C., oscillating it at this temperature, and then allowing it to cool, next day the logarithmic decrement had considerably diminished. The process of heating and cooling was continued for six days, the wire on each day being maintained at 100° C. for eight hours, and then allowed to cool during the remaining sixteen hours. Ultimately the logarithmic decrement was reduced by this process from 0·001078 to 0·000412. It was by no means necessary to keep the wire oscillating during the whole, or even the greater part of the time, but evidently the reduction was facilitated by oscillations at intervals.

The following experiments were made for the purpose of ascertaining the effect of frequent repetition of a *magnetic cycle* on the dissipation of energy during the cycle, as also the effects of rest, and of the heating and cooling process mentioned above.

Experiment VIII.

A bundle of well-annealed iron wires was, shortly after cooling, put through magnetic cycles of ± 0·3 C.G.S. units of force range. The range of force was purposely taken *slightly* higher than the limit within which A and B have the same values in different cycles (see Table VII.), for in this case the "accommodation" is more striking than with lower ranges.

Table IX.—Molecular Accommodation Produced by Repetition of the Cycle.

Number of cycle.*	Sum of the throws as the force was applied starting from zero to ± 0·3 C.G.S. units. U.	Ditto on the removal of the force. D.	U − D, or P.
1	680·0	583·0	97·0
2	662·5	581·5	81·0
3	657·5	585·5	72·0
4	657·0	585·5	71·5

The fourth column shows that the dissipation of energy rapidly decreased with the first three cycles and then became nearly constant. A rest of eight hours, however, still further reduced the friction.

* In this column *one* preliminary cycle is not included.

TABLE X.—Accommodation of iron aided by rest.

Sum of the throws on the application of the force. U.	Ditto, on the removal of the force. D.	U—D, or P.	Remarks.
657·0	585·5	71·5	Before rest.
654·5	587·0	67·5	After a rest of eight hours.

The next experiment shows that when rest and repetition of the cycle have produced their full effect, the "accommodation" can be still further increased by heating and cooling.

Experiment IX.

A bundle of well-annealed iron was accommodated by rest and by frequent repetition of a cycle whose maximum range of force was $\pm$ 0·3 C.G.S. units. It was then heated to 100° C. for some time, tested at this temperature, suffered to cool for several hours, and then was again tested. The process was repeated for 25 days with a range of $\pm$ 0·3 C.G.S. units of force.

TABLE XI.—Accommodation produced by heating and cooling.

Number of times which the iron had been raised to 100° C.	Sum of the throws on the application of the force. U.	Ditto, on the removal of the force. D.	U—D, or P.
0	485·3	405·0	80·3
1	473·8	401·6	72·2
2	469·2	400·0	69·2
23	398·0	357·0	41·0
24	396·9	356·0	40·9
25	397·0	356·0	41·0

Evidently the effect of repeated heating and cooling on the dissipation of energy in a *magnetic* cycle is quite as striking as that in a *torsional* cycle. There is this difference, however, in the effect of heating and cooling in the two cases. In *torsional* cycles the process has very little effect on the *torsional elasticity*, whereas in *magnetic* cycles the *magnetic susceptibility* is very considerably diminished.

Several experiments in the same direction were now made with different specimens of iron and steel, with the results recorded in Table XII.

TABLE XII.—Accommodating effect of heating and cooling.

Metal.	Remarks.	A in C.G.S. units, before the application of the process.	Ditto, after the process.	B in C.G.S. units, before the process.	Ditto, after the process.
Iron	Ten pieces of wire, each 1 mm. in diameter, annealed by the makers .	126·0	104·8	52·3	26·9
Iron	Ten pieces of the same wire, annealed by passing slowly through a Bunsen's burner	213·0	107·9	76·0	24·9
Piano-steel .	Three pieces, each 1 mm. in diameter, well annealed	135·0	91·6	32·0	16·0
Firth's soft steel	One piece, 1 cm. in diameter, well annealed . .	187·0	181·2	13·0	10·8

There are several points to which it is advisable to call attention in this table. The first is the *very large* reduction of the *constant* permeability, as shown by the values of A (see "Mathematical Formulæ," equation (2)), and the still greater reduction of the dissipation of energy, as shown by the values of B (see "Formulæ," equation (3)), which occur in *bundles of wires* of iron and steel of *small* diameter.

A second point is the comparatively *small* effect of the process on the *rod* of soft steel.*

A third point to be noted is that, though in consequence of difference in the annealing, the values both of A and B for the same specimen may be very different *before* the application of the process, the process tends to bring these values to equality. Heating the wires to a high temperature restored the original values of A and B.

Similar experiments were made with *nickel* and *cobalt*. Both these metals were subjected to the heating and cooling process for several days, but neither of them exhibited any tendency to diminish either in permeability or dissipation of energy.

Nor again did *rest* assist in the "accommodation" which can be produced by repetition of the cycle. Iron, therefore, seems to be unique, both as regards the accommodating effects of rest, and of the heating and cooling process.†

The limit of magnetising force within which the values of A and B are constant (Table VII.) for different ranges of force, can be considerably raised by the heating and cooling process; thus the limit in the case of the iron wires last-mentioned was raised from 0·1 to 0·4 C.G.S. units.

* This did not arise from the fact that the length of the steel rod did not contain the diameter a sufficient number of times. A much longer piece of the same rod gave exactly the same values of A and B.

† This seems to be also true as regards the accommodating effects in *torsional* cycles.

Fatigue of Elasticity.

According to Sir William Thomson* a wire which has been kept vibrating torsionally for several hours or days through a certain range comes to rest much more quickly when left to itself than when set in vibration after it has been at rest for several days, and then immediately left to itself. This increase of dissipation of energy due to repeated vibrations has been shown by the author not to occur provided the vibrations are kept within sufficiently narrow limits;† but, on the contrary, the effect of the vibrations is in this case to *reduce* the dissipation of energy. "Fatigue of elasticity," in fact, *arises from upsetting molecular "accommodation,"* and occurs whenever the molecules are disturbed beyond a very small limit, *whatever be the cause of the disturbance.* Thus, as has been pointed out,‡ the logarithmic decrement of a torsionally oscillating iron wire is subpermanently increased by oscillating it beyond a certain limit, by a very slight mechanical shock, a small rise or fall in temperature, or a slight rotation of the molecules by magnetic stress. The following experiments show that in a *magnetic* cycle we have exactly the same kind of effect produced by the above-mentioned causes.

Experiment X.

A well-annealed iron rod which had previously been accommodated by repeated cycles, followed by rest, was given a slight tap with a hammer. The result is given in Table XIII.

Table XIII.—Upset of Accommodation caused by slight Mechanical Shock.

Sum of the throws obtained by changing the magnetising force from 0 to ± 0·07. U.	Ditto by changing the force from ± 0·07 to 0. D.	U − D = P.	Remarks.
1128·5	1067·5	61·0	Before the tapping
1152·0	1078·0	74·0	Immediately after the tapping and after two preliminary cycles
1145·0	1074·0	71·0	After 2 hours' rest
1138·5	1071·0	67·5	After 7 hours' rest
1124·0	1062·0	62·0	After 48 hours' rest

The upsetting of the "accommodation" by slight mechanical shock, and the restoration of the "accommodation" produced by rest, so well shown in this experiment,

* 'Encycl. Brit.,' 9th edit., Art. "Elasticity," § 30. The phenomenon is designated by Sir William Thomson the "*fatigue of elasticity.*"

† 'Phil. Trans.,' vol. 177 (1886).

‡ *Loc. cit.*

are not confined to *annealed* iron, but take place with unannealed iron and with the hardest steel.

Upset of Accommodation produced by Change of Temperature.

Experiment XI.

A bundle of well-annealed iron wires, which had been thoroughly "accommodated" by the heating and cooling process, was tested with the view of ascertaining whether change of temperature would subpermanently upset the accommodation. This was found to be the case. For whether the iron was raised from the temperature of the room to 100° C. or lowered from 100° C. to the temperature of the room, the dissipation of energy in a cycle was always greater *immediately* after the change than when the temperature was maintained constant for some time. Even a small change of temperature (say two or three degrees) had a sensible effect. This may, in all probability, account for the fact that if the accommodated metal be left for any length of time in the room the "accommodation" is almost always found to have been slightly disturbed, but it may be restored by merely putting the metal through one or two cycles.

Upset of Accommodation produced by Carrying the Magnetising Force beyond Certain Limits.

If a wire has been accommodated by repeated cycles within the limits of force given in Table VII., the accommodation is always disturbed when the force is taken beyond these limits.

Experiment XII.

A well-annealed iron rod was "accommodated" by repeated cycles of $\pm$ 0·074 C.G.S. units range of force. In the hope of increasing the accommodation, the metal was subjected to forces rapidly reversed, beginning with 1·3 C.G.S. units, and slowly diminishing to practically zero, as in the method of demagnetising by reversals. About a thousand reversals took place whilst the force thus varied, but in spite of this the process had just the opposite effect expected of it, as will be seen from the next table.

TABLE XIV.—Upset of Accommodation produced by using the Method of Demagnetising by Reversals.

Sum of the throws obtained by changing the magnetising force from 0 to ± ·074. U.	Ditto by changing the force from ± ·074 to 0. D.	U − D = P.	Remarks.
912·0	880·5	31·5	Before using the method of demagnetising by reversals.
935·0	896·0	39·0	Immediately after using the method and also after two preliminary cycles.
907·5	877·5	30·0	After 48 hours, rest with cycles at intervals.

Several other experiments of the same kind were made with other annealed specimens of iron and steel, and with similar result.

In the last experiment, the upset of "accommodation" produced by using the method of demagnetising by reversals was got rid of with comparative ease, but when the initial force used in the method is at all great, or if the temperature of the metal is at all high, the difficulty of restoring the accommodation may be very much greater.

Experiment XIII.

A bundle of well-annealed iron wires had been partly accommodated by the heating and cooling process. Afterwards, when at a temperature of 100° C., it was subjected to a series of cycles beginning with a maximum force of 1·4 C.G.S. units and ending with zero force as in the method of "demagnetising by reversals." When cool it was again tested with the results given in Table XV.

TABLE XV.—Upset of Accommodation produced by using the Method of Demagnetising by Reversals at the Temperature of 100° C.

Remarks.	Number of days during each of which the heating and cooling process had been applied.	Sum of the throws on changing the force from 0 to ± 0·3. U.	Ditto on removing the force. D.	U − D, or P.
Before demagnetising by reversals	2	469·2	400·0	69·2
After ditto	3	552·5	435·5	117·0
Method of demagnetising by reversals abandoned	4	518·5	420·0	98·5
	8	459·8	391·1	68·7

On consulting the last column in Table XV. we see that the method of demagnetising by reversals is responsible for increasing the dissipation of energy by nearly 70 per cent., and, further, that the upset of the accommodation thus produced requires five repetitions of the heating and cooling process for its removal.*

Providing the forces employed in the method of demagnetising by reversals do not exceed the limits of Table VII., the process of demagnetising by reversals acts beneficially as regards diminution of dissipation of energy in any cycle, as we have already seen, just as preliminary torsional oscillations of small amplitude bring down the logarithmic decrement of arc. Similarly, when the preliminary torsional oscillations exceed a certain limit of amplitude, we have Sir W. THOMSON'S "elastic fatigue," and more or less notable increase of logarithmic decrement. The analogy between "elastic fatigue" in a torsionally oscillating wire and in the same wire when put through magnetic cycles seems to be perfect.

Nickel and cobalt also showed the same disturbance of "accommodation" to be produced by the same causes as in iron.

EFFECT OF ANNEALING ON THE VALUES OF A AND B.

Annealing Iron whilst in the Magnetic Meridian.

It has been already mentioned that the metals annealed by the author were, during the process, lying in a direction at right angles to the magnetic meridian, whilst those annealed by Mr. ANDREWS (the long rods) had, by chance, been placed with their axes making an angle of about 45° with the meridian. Before, therefore, attempting any comparison of the values of A and B for different specimens of iron and steel, it seemed as well to ascertain how far a specimen of iron annealed in the magnetic meridian would differ from the same specimen annealed at right angles to the meridian. It is true that the horizontal component of the earth's magnetic force is not great (about 0·18 C.G.S. units), but it must be remembered that we are in many cases dealing with forces which are less; and, further, that at high temperatures the magnetic permeability of iron for feeble forces may be very much greater than its permeability at the temperature of the room. Very careful investigation, however, showed only a slight difference between the values of A and B for the same specimen when annealed in the two different positions.

The next table exhibits the effects of annealing at various temperatures.

* *Five whole days*, on each of which the metal was kept at 100° C. for eight hours, with cycles at intervals, followed by sixteen hours of cooling, with cycles at intervals, little more than sufficed for the purpose.

Table XVI.—Annealing at different temperatures.

Metal.	A in C.G.S. units.	B in C.G.S. units.	Remarks.
Soft iron wire . . .	129·5	42·2	Annealed by the makers (Johnson and Nephew)
,, ,, . . .	213·0	76·0	Annealed by passing several times slowly through the flame of a Bunsen's burner
,, ,, . . .	240·0	86·5	Annealed in Fletcher's furnace at 1000° C.
,, ,, . . .	77·7	1·8	Hardened by torsion
,, ,, . . .	185·4	18·61	Re-annealed by passing several times slowly through the flame of a Bunsen's burner
,, ,, . . .	96·1	8	Hardened by hammering
,, ,, . . .	241·7	64·4	Again annealed in Fletcher's furnace at 1000° C.
Firth's soft steel . .	170·9	8·1	Annealed by Mr. Andrews at 566° C.
,, ,, . .	187·0	13·0	Annealed in Fletcher's furnace at 1000° C.
Nickel wire . . .	6·24	0·04	Hard drawn
,, ,, . . .	63·3	4·21	Annealed by Bunsen's burner
,, ,, . . .	63·3	4·21	Annealed by Mr. Andrews at 580° C.
,, ,, . . .	111·95	6·90	Annealed in Fletcher's furnace at 1000° C.

The above table shows the importance when annealing of raising the temperature sufficiently high, and maintaining the high temperature for a sufficient time, also of cooling slowly.* That a high temperature is required is well exhibited by the behaviour of Johnson and Nephew's iron wire. This had been raised by the makers to a temperature above dull red, and the wire was rendered by their annealing exceedingly soft and, moreover, for high, or rather comparatively high, magnetising forces, of great magnetic permeability; but annealing in the Fletcher's furnace at the temperature of 1000° C. nearly doubled the permeability for very minute magnetising forces, and more than doubled the internal friction. The annealing, by passing the wire slowly several times through the flame of a Bunsen's burner, so as to heat all parts equally to a bright red, does not seem satisfactory, partly, it may be, because a sufficiently high temperature is not reached, and partly, doubtless, because the cooling is too rapid; in every instance, annealing by Fletcher's furnace at the temperature of 1000° C., produced higher values of both A and B than annealing by a Bunsen's burner. This last fact is especially noticeable in the nickel wire, where the annealing with the furnace nearly doubled the value of A in the wire which had previously been annealed with the burner, as well as by Mr. Andrews at the temperature of 580° C. Professor Ewing has examined the magnetic permeability of nickel wire with very low magnetising forces, and he finds the value of A for *annealed* nickel to be 22·4.† This value is very much less than that obtained by the author for the thoroughly annealed

* The slow cooling is, at any rate, necessary for iron; for nickel and cobalt it may not be necessary.

† 'Phil. Trans.,' vol. 179 (1888), A, p. 332. Professor Ewing finds the *susceptibility* to be 1·7; this would make the *permeability* equal to $1 + 4\pi \times 1{\cdot}7$ or 22·4.

nickel wire, and may, perhaps, be due to imperfect annealing* of Professor EWING's nickel; the difference cannot certainly be ascribed to the presence of iron in the author's specimen.† The influence of permanent strain on the magnetic properties of nickel is evidently very great, the value of A for the drawn metal being only $\frac{1}{18}$th of the value for the thoroughly annealed metal, whilst the internal friction in the former is only about $\frac{1}{200}$th of the internal friction in the latter.

It would seem that the different specimens of wrought iron in ordinary use will, if *properly annealed*, be found to have not very different permeabilities for very low magnetising forces, as may be seen from the following table. The specimens mentioned in this table were all annealed by the author in the manner already described, whilst lying in a direction at right angles to the magnetic meridian, the temperature of annealing being about 1000° C.‡

TABLE XVII.

Substance.	A in C.G.S. units.	B in C.G.S. units.	Mean value of A.	Mean value of B.
Wrought iron rod, history unknown . . .	240·0	64·7		
Soft iron rod furnished by Mr. ANDREWS .	244·5	135·0		
JOHNSON and NEPHEW's wrought iron wire .	240·0	86·5	241·5	87·6
Soft steel rod furnished by Mr. ANDREWS .	187·0	13·0		
Soft piano-steel wire	135·0	32·0	146·0	22·5
Nickel wire	111·95	6·90	111·95	6·90
Cobalt strip	65·14	1·26	65·14	1·26

The numbers given in the above table must be understood to be obtained 24 hours after annealing and after a large number of preliminary cycles. The magnetising forces used were all well within the limit up to which the values of A and B are the same in cycles of different ranges of magnetising force.

The values of B for the different specimens of wrought iron are not nearly so much alike as those of A; but it is not improbable that if the different specimens were subjected to the heating and cooling process already mentioned they would be found to become much more nearly alike.

It is evident that both for permeability for very minute magnetising forces and for dissipation of energy the descending order for well annealed metals would stand thus:—

* Professor EWING seems to have annealed his wire by passing it slowly through a Bunsen's burner.

† See the analysis of the nickel previously given.

‡ This temperature seems to be sufficiently high for iron, nickel, and cobalt; some experiments made at about 1200° C. did not sensibly increase the permeability.

Iron.
Steel.
Nickel.
Cobalt.

The Temporary Effect of Change of Temperature on the Values of A and B.

The permanent diminution of the values of A and B, which can be produced by repeated heating to 100° C. and then cooling, have been already alluded to in Experiment IX. The temporary effects were studied at the same time as the permanent ones, with the results given in Table XVIII.

It will be noticed that the effect on the permeability is greatest in annealed iron wire which has not been "accommodated" by the heating and cooling process, and least in the same wire which has been thus accommodated.

In all cases, except for freshly annealed iron, the increase of the value of B produced by rise of temperature can be more than accounted for by increase of permeability, so that, if we subjected the metals to forces which produced the same magnetic *induction* at the two temperatures, the dissipation of energy in the cycle would be *less* for the hot than for the cold metals.

For magnetising forces below the limits given in Table VII. the temporary changes of both A and B are, in all cases, quite *independent of the value of the force.*

Table XVIII.

Substance.	A_t and A_0 represent the values of A at $t°$ C. and 0° C. respectively.	B_t and B_0 represent the values of B at $t°$ C. and 0° C. respectively.	Increase of A per cent. produced by rise of temperature from 0° C. to 100° C.	Ditto of B ditto.	A_0 in C.G.S. units.	B_0 ditto.	Remarks.
Johnson and Nephew's wrought iron wire	$A_t = A_0(1 + \cdot002890t)$	$B_t = B_0(1 + \cdot00739t)$	28·90	73·96	232	80	Freshly annealed
Ditto	$A_t = A_0(1 + \cdot000717t)$	$B_t = B_0(1 + \cdot001005t)$	7·17	10·05	107	22·5	After repeated heating and cooling
Steel rod of Mr. Andrew's . . .	$A_t = A_0(1 + \cdot001244t)$	$B_t = B_0(1 + \cdot002624t)$	12·44	26·24	185	12·7	Ditto
Piano-steel wire . .	$A_t = A_0(1 + \cdot001232t)$	$B_t = B_0(1 + \cdot000593t)$	12·32	5·93	89	15·9	Ditto
Nickel wire	$A_t = A_0(1 + \cdot002789t)$	$B_t = B_0(1 + \cdot00535t)$	27·89	53·47	109	6·6	Ditto
Cobalt strip . . .	$A_t = A_0(1 + \cdot001450t)$	$B_t = B_0(1 + \cdot002979t)$	14·50	29·79	64	1·22	Ditto

The Effects of Sudden Cooling on the Values of A and B.

The effects of sudden cooling are exhibited in Table XIX. There are three points to be specially noted. The first is that the permeability of steel for *very minute* forces, though considerably diminished, is not nearly so much affected by sudden cooling as is the permeability for *moderate* or *great* forces. The second is that the diminution of A does not seem to depend so much on the *amount* of hardening* as does the permeability for greater forces or the other physical properties of steel. And the third is the large effect of the sudden cooling in the case of soft *iron*. The value of B, and consequently the dissipation of energy, for steel is enormously diminished by the sudden cooling, whilst that of soft iron is also considerably lessened.

Table XIX.

Substance.	A in C.G.S. units.	B in C.G.S. units.	Remarks.
†Soft steel rod .	187·0	13·0	Before chilling
,, ,, .	73·1	0·18	After sudden chilling from a temperature of 780° C.
,, ,, .	72·2	0·32	After sudden chilling from nearly a white heat
†Soft iron rod .	244·5	135·0	Before chilling
,, ,, .	159·6	43·1	After sudden chilling from nearly a white heat

The Effects of Permanent Extension, Hammering and Torsion on the Values of A and B.

The next three tables show the effects of permanent extension, hammering transversely, and torsion on well annealed iron wire of ·1 cm. diameter.

Table XX.—Permanent Extension.

Percentage of extension.	A.	B.
0	240·9	68·8
1	118·7	25·6
5	88·4	16·7
10‡	77·7	1·8

* The effect of cooling from a white heat (about 1200° C.) is sensibly the same as that in cooling from 780° C.

† These rods had been previously annealed by the author, at a temperature of 100° C.

‡ On attempting to stretch further the wire broke.

TABLE XXI.—Transverse Hammering.

Percentage increase of length produced by the hammering.	A.	B.
0	240·9	68 8
5	96·1	8·7

TABLE XXII.—Permanent Torsion.

Number of turns of permanent torsion in a length of 160 cm.	A.	B.
0	240·9	68·8
10	125·5	36·4
100	98·0	14·7
300	90·2	9·8
350*	95·9	6·9
400†	97·0	16·3

The values of A and B given in these tables were all obtained when the metals had been allowed to rest for several hours after the strain had been imparted. When the metals were tested *immediately* after straining the permeability and the internal friction were always very sensibly greater.

Evidently permanent strain imparted by any of the processes mentioned above diminishes considerably the permeability for very feeble forces, and very much more considerably the dissipation of energy.

In all cases annealing in Fletcher's furnace at a temperature of 1000° C., restored the original values of A and B.

THE TEMPORARY EFFECT OF LOADING ON THE VALUES OF A AND B.

Experiment XIV.

A well annealed iron 1 mm. in diameter was loaded by small additions at a time, and with long rests between, with 20 kilos., a permanent elongation of 2 per cent. being thereby produced.‡ After the greatest load had been applied and removed a great many times until the values of A and B had become constant at the different

* The last 50 turns were applied in the opposite direction to the previous ones.

† The last 50 turns were applied, 10 at a time, alternately in opposite directions.

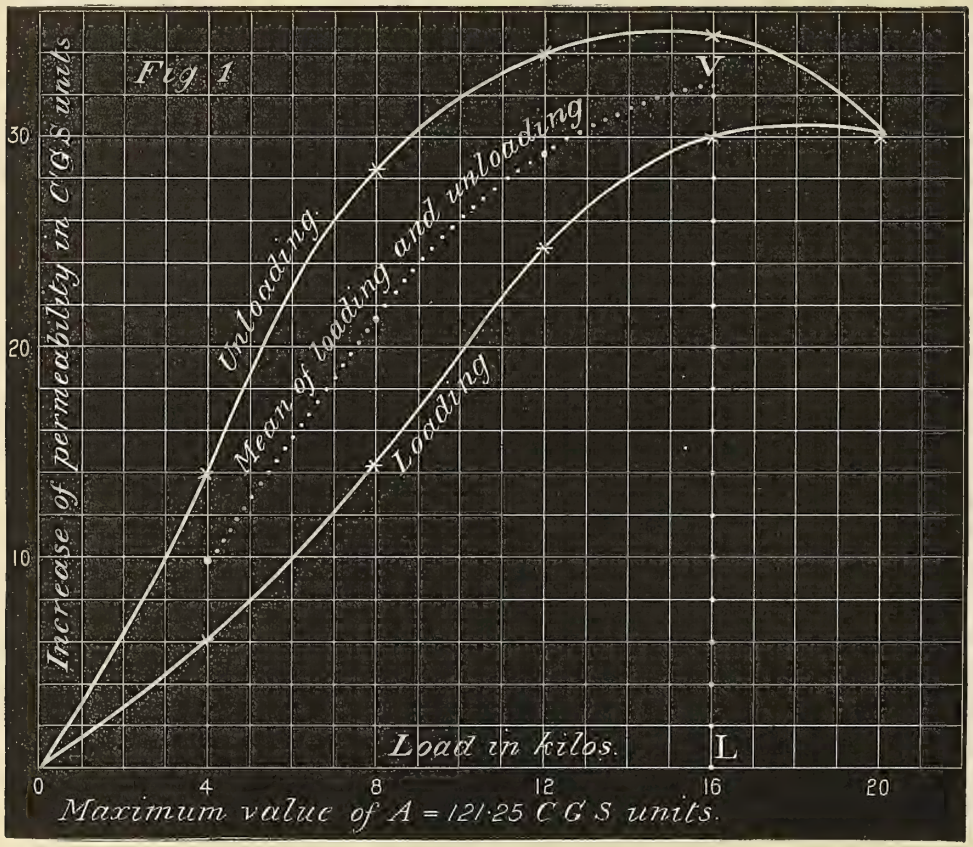
‡ A much greater permanent elongation would have been produced by the load had this been applied in the ordinary manner, *i e.*, without long intervals of rest between each addition

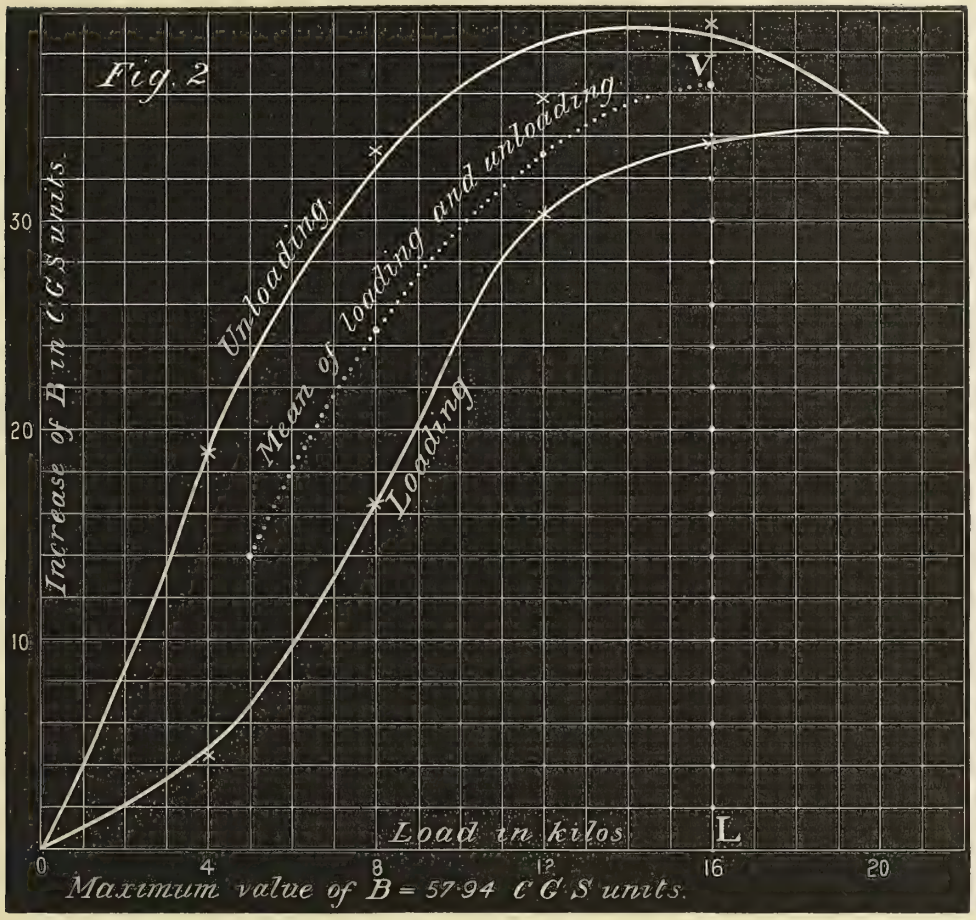
stages of loading and unloading the observations recorded in the next table were made. The loading and unloading were performed as carefully as possible so as to avoid jarring the wire, which was all the time lying in a direction at right angles to the magnetic meridian.

TABLE XXIII.

Load in kilos.	A in C.G.S. units. Loading.	Ditto. Unloading.	Mean value of A observed.	Ditto. A calculated.	B in C.G.S. units. Loading.	Ditto. Unloading.	Mean value of B observed.	Ditto. B calculated.	$\frac{B}{A^2}$
0	89·17	89·17	89·17	..	22·36	22·36	22·36	..	·0028
4	95·22	103·00	99·11	98·93	26·85	41·30	34·08	34·13	·0035
8	104·62	117·78	111·20	111·33	38·91	55·67	47·29	47·35	·0038
12	114·14	122·95	118·55	118·77	52·76	58·15	55·46	55·39	·0039
16	118·96	123·88	121·42	121·25	55·95	59·85	57·90	57·94	·0039
20	118·96	118·96	118·96	..	56·65	56·65	56·65	..	·0041

The results given in the table represent the means of several sets of observations made at each of the various stages of loading and unloading. There is an evident lagging of strain behind stress which is much more conspicuous in the values of B than in those of A. The means of the two values obtained at the same stages of





loading and unloading at each stage probably represent what would be the values of A and B if there were no such lagging. In the figs. 1 and 2, which supplement Table XXIII., the *dotted* curves show the relations which exist between the changes of these *mean* values and change of load. The values of both A and B increase with the load until the latter reaches 16 kilos.; further increase of load then causes decrease of A and B.*

The dotted curves of figs. 1 and 2 seem to be strictly parabolic, the vertex of the parabola being at V, and the axis a vertical drawn through V. Of course, the values of A and B corresponding to the loads 0 and 20 kilos. will not represent what would be the values of A and B if lag were absent, and they are therefore omitted in the comparison between observation and calculation given in columns 4 and 5, and 8 and 9 of the table. The calculated values of A and B are deduced respectively from the two formulæ

* A large number of independent observations were made which need not be recorded here, and in which the loads were altered by smaller amounts at a time; these observations seemed to fix 16 kilos. as *almost exactly* the load for which A and B were greatest.

$$A = 121{\cdot}25 - 0{\cdot}1550n^2$$
$$B = 57{\cdot}94 - 0{\cdot}1654n^2$$

where n is the number of kilos. in the actual load above or below 16 kilos.

The temporary increase of the value of B produced by the loading is greater than can be accounted for by mere increase of permeability, for if the increase of the value of B could be accounted for by increase of permeability, the ratio $B : A^2$ should be constant, whereas we see from the last column of Table XXIII. that this ratio increases with the load.

The above experiment serves but little more than to confirm the observations of Villari, Sir William Thomson, Ewing, and others, that for small magnetising forces the magnetic permeability of iron is temporarily increased by moderate amounts of longitudinal traction. It does, however, go a little further, for it proves not only that the magnetic permeability of *iron* for *infinitesimal* magnetising forces is increased, but also that the *dissipation of energy* is increased. Now, Sir William Thomson has shown that the permeability of *nickel* is *diminished*, for moderate magnetising forces, by longitudinal traction ; and it seems very probable, therefore, that the value of A for nickel will be diminished. But will the molecular friction—will the value of B—be also diminished? The author hopes to be able shortly to answer this question.

It had been rather expected that the "accommodation" by preliminary cycles at the various stages of loading and unloading might almost entirely do away with the lagging of strain behind stress. But though this lagging was considerably diminished, as subsequent experiments showed, by the "accommodating" process, yet, as we have seen, it was very sensible. In order to still further do away with the lag, the wire was now not only "accommodated" by preliminary cycles, but also tapped, or rather shaken, at each stage of the loading and unloading. The tapping still further diminished the lag, but by no means abolished it. Moreover, the dotted curves shown in figs. 1 and 2 remained sensibly unaltered, *i.e.*, the values which A and B would have if lag were absent remained the same.

Relations between the Dissipations of Energy in Torsional and Magnetic Cycles.

The following table shows in what respects the dissipations of energy in torsional cycles agree with those in magnetic cycles, and in what respects they differ :—

TABLE XXIV.

Torsional cycles.	Magnetic cycles.
1. The logarithmic decrement of arc is independent of the amplitude, and therefore the dissipation of energy in the cycle is proportional to the *square* of the maximum torsional force in the cycle.	The dissipation of energy is proportional to the *cube* of the maximum magnetising force in the cycle.
2. The logarithmic decrement of arc is independent of the vibration-period.	The dissipation of energy in the cycle does not depend upon the number of steps by which the maximum magnetising force in the cycle is reached.
3. The logarithmic decrement of arc is independent of the length and diameter of the wire.	The dissipation of energy per cubic unit of volume does not depend upon the dimensions.*
4. The logarithmic decrement of arc can be diminished by repeated oscillation within sufficiently small limits, by rest (in the case of iron), and (in the case of iron-wire) by repeated heating to 100° C. and then cooling.	The dissipation of energy can be diminished by repeatedly putting the metal through cycles of sufficiently small range of magnetising force, by rest (in the case of iron), and (in the case of iron-wire) by repeated heating to 100° C. and then cooling.
5. The logarithmic decrement of a very small arc is increased by recent oscillations, through arcs exceeding certain limits; it is also increased, either permanently or subpermanently, by mechanical shocks, by sudden changes of temperature, or by molecular disturbances due to magnetisation.	The dissipation of energy in a cycle of very small range of magnetising force is increased by previously putting the metal through cycles whose ranges of force exceed certain limits; it is also increased, either permanently or subpermanently, by mechanical shocks, by sudden changes of temperature, or by molecular disturbances due to torsional oscillation.†
6. The logarithmic decrement of arc is permanently *increased*, and the torsional elasticity is permanently *decreased* by excessive permanent extension and by permanent torsion.	The dissipation of energy is permanently *decreased*, and the magnetic elasticity is permanently *increased* by permanent extension, torsion, and hammering.
7. The logarithmic decrement of iron and nickel is *much less* at 100° C. than at 0° C.; the torsional elasticity is also *slightly less* at 100° C. than at 0° C.	The dissipation of energy of iron, nickel, and cobalt is *greater* at 100° C. than at 0° C.; the magnetic elasticity is *considerably less* at 100° C. than at 0° C.
8. The logarithmic decrement and the torsional elasticity of iron are *both independent* of the load.	The dissipation of energy of iron is *increased*, and the magnetic elasticity *decreased* by loading.

It will be seen from 1 that the dissipations of energy in the two kinds of cycles differ in the important particular that in torsional cycles the dissipation varies as the *square* of the maximum torsional force, whilst in magnetic cycles the dissipation varies as the *cube* of the maximum magnetising force. In both kinds of cycles the dissipation varies as PX, where P is the molecular set first on this side and then on that, as the wire vibrates torsionally in the one case, or as it is magnetised first in one direction and then in the opposite in the other case, and X is the maximum force, in the one case torsional and in the other magnetic; but for torsional cycles P varies as X

* Provided, of course, the length contains the diameter a sufficient number of times to enable us to neglect the effect of the ends.

† This last, though not mentioned in the present memoir, has nevertheless been determined by the author to be the case.

and in magnetic cycles as X^2. In attempting to account for this difference we might, perhaps, be inclined to ask ourselves whether, in spite of their minuteness, the magnetising forces employed have been *sufficiently* minute—whether, if we operated with forces which were, say, one-hundredth or one-thousandth of the least of those used in the present inquiry, we should not find P varying as X instead of as X^2. On the other hand, it is easy to show that the dissipations of energy in the torsional cycles employed by the author are *comparable in amount* with those encountered in true magnetising cycles; and if we are to reduce the range of the magnetising force to the extent mentioned above, the dissipations of energy in the two sets of cycles would quite cease to be comparable.

As regards 2 and 3, the dissipations in the two sets of cycles are analogous. We may notice, also (4), that in both torsional and magnetic cycles what have been called the "accommodating" influences are the same; and they are not only alike in kind, but comparable in amount. Further (5), the various causes which upset the "accommodation" are the same in the two kinds of cycles, and produce comparable effects.

When, however, we turn to both the permanent and temporary effects of mechanical stress (6 and 8), and to the temporary effect of change of temperature (7), we find *very essential* differences in the two kinds of cycles. How far these differences may be due to differences in the effects on the magnetic and torsional elasticities remains to be seen.

Summary.

1. When a rod of iron, nickel, or cobalt is made to pass through a complete magnetic cycle of sufficiently minute range of magnetising force, the instantaneous changes of induction can be expressed by an equation of the form

$$y = Ax + Bx^2,$$

where x represents the total *change* of magnetising force as the force passes from its greatest value to zero, if the force be always applied in the same direction, or from its greatest value in one direction to its greatest value in the opposite, if the same force be alternately employed in opposite directions; y represents the corresponding *change* of induction; and A and B are constants throughout the cycle.

2. The values of A and B are the same for different ranges of magnetising force.

3. The constant A is a measure of the magnetic permeability for infinitesimal magnetising force; the constant B is a measure of the dissipation of energy in the cycle.

4. The dissipation of energy in any cycle may be decreased by repetition of the cycles; the molecules may be said to be "accommodated" by this process.

5. The molecular "accommodation" of freshly-annealed iron can be increased by repeatedly raising the metal to 100° C., and then allowing it to cool.

6. The "accommodation" of iron, nickel, and cobalt is upset by slight mechanical shock, by small changes of temperature, or by magnetisation beyond certain limits.

7. The values of A and B for iron, nickel, and cobalt may be very largely decreased by permanent mechanical strain. In the case of iron and steel they may also be largely decreased by sudden cooling.

8. The values of A and B for iron are decreased by temporary loading.

9. The values of A and B for iron, nickel, and cobalt are increased when the temperature is raised from 0° C. to 100° C.

10. In both torsional and magnetic cycles of sufficiently minute range the dissipation of energy is independent, apart from the effects of self-induction and time-lag, of the rate at which the cycle is performed.

11. The dissipation of energy in a magnetic cycle is proportional to the *cube* of the maximum magnetising force in the cycle, but in a torsional cycle the dissipation is proportional to the *square* of the maximum torsional force.

12. As regards "accommodating" influences, and also as regards those influences which upset "accommodation," the dissipations of energy in torsional and magnetic cycles are strictly analogous.

13. The temporary and permanent effects of mechanical stress on the dissipation of energy and the effects of change of temperature are very different in torsional and magnetic cycles.

X. *On the Reflection and Refraction of Light at the Surface of a Magnetized Medium.*

By A. B. Basset, *M.A., F.R.S.*

Received December 17, 1890,—Read January 8, 1891.

1. The object of this investigation is to endeavour to ascertain how far the electromagnetic theory of light, as at present developed, is capable of giving a theoretical explanation of Dr. Kerr's experiments on the effect of magnetism on light.

In the first series of experiments* polarized light was reflected from the polished pole of an electromagnet, and it was found that when the circuit was closed, so that the reflecting surface became magnetized perpendicularly to itself, the reflected light exhibited certain peculiarities, which disappeared when the circuit was broken.

In the second series of experiments† the reflector was a polished plate of soft iron laid upon the poles of a horse-shoe electromagnet, so that the direction of magnetization was parallel, or approximately so, to the reflecting surface; and it was found that the effect of the current was analogous to, though by no means identical with, the effect produced in the first series of experiments. It was also found that when the incidence was normal, or when the plane of incidence was perpendicular to the direction of magnetization, no effect was produced.

In both series of experiments it was found that the effects produced by magnetization materially varied with the angle of incidence. It was also found that these effects were in most cases reversed, when the direction of the magnetizing current was reversed; that is to say, if the intensity of the reflected light was strengthened by a right-handed current, it was weakened by a left-handed one.

Since the effects produced by the current were in most cases reversed when the direction of the current was reversed, it follows that the *first power* of the magnetic force must enter into the expression for the intensity of the reflected light.

It will be noticed that in all these experiments a metallic reflector was employed, and consequently the results were complicated by the influence of metallic reflection. It therefore seems hopeless to attempt to construct a theory which shall furnish a complete explanation of these phenomena, until a perfectly satisfactory electromagnetic theory of metallic reflection has been obtained, and, so far as I am aware,

* 'Phil. Mag.,' May, 1877.

† *Ibid.*, March, 1878.

no such theory has been discovered. Lord RAYLEIGH* has shown that CAUCHY'S formulæ may be derived from GREEN'S theory of elastic media, by assuming that the effect of metallic reflection is represented by a term proportional to the velocity; and in a subsequent paper† he has shown that the same formulæ may be obtained from the electromagnetic theory by taking into account the conductivity of the metallic reflector. The results obtained by either of these theories are open to various objections, which Lord RAYLEIGH has discussed in the papers referred to; and he considers "that much remains to be done before the electrical theory of metallic reflection can be accepted as complete."

There are, however, several non-metallic reflecting media (such as strong solutions of certain chemical compounds of iron), which are capable, when magnetized, of producing an effect upon light; and the theoretical explanation of the magnetic action of such media is accordingly free from the difficulties surrounding metallic reflection. It would be exceedingly desirable that experiments upon magnetic solutions should be made; and in view of the possibility of such experiments I have thought it worth while to develop a theory applicable to them. Whether the results of the present paper will stand the test of experiment cannot be finally decided until the experiments alluded to have been made;‡ but it must not excite surprise if the results of the theory do not agree very well with KERR'S experiments, since the conditions of the problem are materially different.

2. I shall now describe four of KERR'S experiments, which it will be necessary to consider more fully when discussing the results obtained from the mathematical theory; and I shall then proceed to give a brief abstract of the results to which this theory leads, stating the points of agreement and disagreement.

In Experiments I. and II. the magnetization is perpendicular to the reflector.

Experiment I.—Light polarized in or perpendicularly to the plane of incidence is allowed to fall on the polished pole of an electromagnet, and the analyser is placed in the position of extinction. When the circuit is closed, so that the reflector becomes magnetized, the light immediately reappears; when the circuit is broken the light disappears, and again reappears when the current is reversed.

The light reflected whilst the circuit is closed is not plane polarized, since it cannot be extinguished by rotating the analyser; also the analyser's position of extinction before closing is the position of minimum intensity after closing.

Experiment II.—The arrangements are the same as in the last experiment, and the analyser is turned from the position of extinction through a small angle towards the right-hand of the observer,§ giving a faint restoration of light. When the circuit is

* Hon. J. W. STRUTT, 'Phil. Mag.,' May, 1872.

† 'Phil. Mag.,' August, 1881.

‡ See, however, § 16, July, 1891.

§ The terms right-handed and left-handed refer to an observer who is looking at the point of incidence through the analyser.

closed, so that the reflector becomes a negative pole, the intensity is increased; but if the current is reversed, so that the reflector becomes a positive pole, the intensity is diminished. The weakening effect of the second operation is always less than the strengthening effect of the first, and its effect diminishes as the angle through which the analyser is turned is diminished. In this experiment the angle of incidence lay between 60° and 80°. Similar results were also obtained when the incidence was normal.

In Experiments III. and IV. the direction of magnetization is parallel to the reflector, and the plane of incidence is parallel to the lines of magnetic force.

Experiment III.—The incident light is polarized *in* the plane of incidence, and the analyser is placed in the position of extinction, and is then turned through a small angle. The circuit is now closed, and it is found that the light restored from extinction by a small right-handed rotation of the analyser is always strengthened by a right-handed current, and weakened by a left-handed one.

The intensity of these optical effects of magnetization varies with the angle of incidence. The effects increase from grazing incidence up to about 65° to 60°, and then diminish as the angle of incidence decreases.

Experiment IV.—The incident light is polarized *perpendicularly* to the plane of incidence, and the arrangements are the same as in the last experiment. At an incidence of 85° the light restored by a right-handed rotation of the analyser is strengthened by a right-handed current and weakened by a left-handed one, and the effects are undistinguishable from those of the last experiment. As the angle of incidence decreases the effects diminish, and finally disappear at an angle of 75°. As the angle of incidence still further decreases the effects reappear, but are reversed; for the light restored by a right-handed rotation is now weakened by a right-handed current and strengthened by a left-handed one. The effects increase from 75° to about 60°, and then gradually diminish.

I shall now describe the results of the present paper, in which the reflecting surface is supposed to be a magnetized transparent medium.

As regards the points of agreement I find (i.) that the reflected light is elliptically polarized; (ii.) that when the magnetization is parallel to the reflecting surface no effect is produced when the incidence is normal, or when the plane of incidence is perpendicular to the lines of magnetic force; (iii.) when the plane of incidence is parallel to the lines of magnetic force, and the light is polarized *in* the plane of incidence, the magnetic term increases from grazing incidence to a maximum value, and then diminishes to normal incidence.

The points of disagreement are that in all cases the intensity is unchanged when the magnetizing current is reversed; and also that no results analogous to those of the fourth experiment are obtained.

It is not improbable that this disagreement is due to the disturbing effect of metallic reflection, inasmuch as it is easy to see, from Lord RAYLEIGH'S proof of

Cauchy's formulæ for metallic reflection, how a term in the intensity, which is proportional to the first power of the magnetic force, and which would therefore be the most important term, might come in, when the reflector is a metal; but the question is one which can only be decided by experiment, and it is therefore very desirable that experiments on magnetic solutions should be made.

Theory of Magnetic Action on Light.

3. We must now consider the theories which have been proposed to explain the action of magnetism on light.

One of the difficulties of Maxwell's theory* of magnetic action is, that it is inconsistent with his previous theory of isotropic and doubly refracting media. In the latter theory, he has assumed that the disturbance which produces the sensation of light is represented by the *electric displacement*, whereas, in the theory of magnetic action, he has supposed that light is produced by the vibrations of an elastic medium or ether, and that when this medium is acted upon by magnetic force, a state of vortex motion is set up, which modifies the motion which produces light.

4. Professor Fitzgerald† has modified Maxwell's theory, by supposing that the effect of magnetic force is to introduce into the kinetic energy an additional term

$$4\pi C \iiint \left(\frac{d\xi}{d\omega} \dot{f} + \frac{d\eta}{d\omega} \dot{g} + \frac{d\zeta}{d\omega} \dot{h} \right) dx\, dy\, dz \quad \ldots\ldots \quad (1),$$

where

$$\frac{d}{d\omega} = \alpha \frac{d}{dx} + \beta \frac{d}{dy} + \gamma \frac{d}{dz} \quad \ldots\ldots\ldots \quad (2),$$

and f, g, h are the components of electric displacement, whilst α, β, γ are the components of the external magnetic force. The quantities ξ, η, ζ are the time integrals of the components of the periodic portions of the magnetic force, and, accordingly, f, g, h are proportional to the curl of ξ, η, ζ, so that

$$4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz} \text{ \&c.} \quad \ldots\ldots\ldots \quad (3).$$

If, therefore, we substitute the values of f, g, h from (3) in (1), and integrate by parts, we shall obtain

* 'Electricity and Magnetism,' vol. 2, ch. xxi.

† 'Phil. Trans.,' 1880, p. 691.

$$C\iint\left\{(m\dot{\zeta}-n\dot{\eta})\frac{d\xi}{d\omega}+(n\dot{\xi}-l\dot{\zeta})\frac{d\eta}{d\omega}+(l\dot{\eta}-m\dot{\xi})\frac{d\zeta}{d\omega}\right\}dS$$

$$+C\iiint\left\{\dot{\xi}\frac{d}{d\omega}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\dot{\eta}\frac{d}{d\omega}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\dot{\zeta}\frac{d}{d\omega}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}dx\,dy\,dz \quad . \quad (4).$$

The volume integral part of this expression is of the same form as MAXWELL'S expression, but the similarity between the two theories ends here; for in MAXWELL'S theory ξ, η, ζ are the displacements of an elastic medium, whereas, in FITZGERALD'S theory, they are the time integrals of magnetic force. It also appears that the additional terms in the latter theory cannot be accounted for on MAXWELL'S hypothesis of molecular vortices, inasmuch as this would require us to assume that the components α', β', γ', of the total magnetic force, satisfy the equations

$$\alpha'=\alpha+d\xi/d\omega, \qquad \beta'=\beta+d\eta/d\omega, \qquad \gamma'=\gamma+d\zeta/d\omega,$$

where ξ, η, ζ are the time integrals of the periodic parts of the magnetic force;—a relation which cannot apparently be justified on any physical grounds whatever.

The equations which are satisfied at the surface of two different media are also not deduced in a very satisfactory manner, nor is their physical interpretation fully explained. Professor FITZGERALD has also failed to recognise, that when light is incident upon a magnetic medium, there are always (except in special cases), two refracted waves which are circularly polarized in opposite directions, and whose *directions*, as well as their velocities of propagation, are different. It is quite true, that the angle between the directions of the two waves is very small, and would be difficult of observation; but since it depends upon the *first power* of the external magnetic force, we are not at liberty to neglect it.

5. The theory which I propose to develop in the present paper, depends upon the following considerations:—

It was proved experimentally by HALL,* that when a current passes through a conductor which is placed in a magnetic field, an electromotive force is produced, whose intensity is proportional to the product of the current and the magnetic force, and whose direction is at right angles to the plane containing the current and the force. Now, Professor ROWLAND† has assumed that this result holds good in a dielectric which is under the action of a strong magnetic force; if, therefore, we adopt this hypothesis, we must substitute the time variations of the electric displacement for the current, and the equations of electromotive force become

* 'Phil. Mag.,' March, 1880.

† *Ibid.*, April, 1881, p. 254.

$$\left.\begin{aligned} \mathrm{P} &= -\frac{d\mathrm{F}}{dt} - \mathrm{C}(\gamma\dot{g} - \beta\dot{h}) - \frac{d\psi}{dx} \\ \mathrm{Q} &= -\frac{d\mathrm{G}}{dt} - \mathrm{C}(\alpha\dot{h} - \gamma\dot{f}) - \frac{d\psi}{dy} \\ \mathrm{R} &= -\frac{d\mathrm{H}}{dt} - \mathrm{C}(\beta\dot{f} - \alpha\dot{g}) - \frac{d\psi}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where α, β, γ are components of the total magnetic force.

When the magnetic field is disturbed by the passage of a wave of light, α, β, γ may be supposed to have the same values as before disturbance, since their variations when multiplied by $\dot{f}$, $\dot{g}$, $\dot{h}$ are terms of the second order, which may be neglected. Since we shall confine our attention to the propagation of light in a uniform magnetic field, α, β, γ may be regarded as constant quantities.

I shall, therefore, assume that when light is transmitted through a medium, which, when under the action of a strong magnetic force, is capable of magnetically affecting light, the equations of electromotive force are represented by (5), where C is called Hall's constant, and is a quantity which depends on the nature of the medium, and possibly, also, on the period of vibration. Since we shall require to use the letters α, β, γ to denote that portion of the magnetic force which is due to optical causes, we shall write these equations in the form

$$\left.\begin{aligned} \mathrm{P} &= -\frac{d\mathrm{F}}{dt} - p_3\dot{g} + p_2\dot{h} - \frac{d\psi}{dx} \\ \mathrm{Q} &= -\frac{d\mathrm{G}}{dt} - p_1\dot{h} + p_3\dot{f} - \frac{d\psi}{dy} \\ \mathrm{R} &= -\frac{d\mathrm{H}}{dt} - p_2\dot{f} + p_1\dot{g} - \frac{d\psi}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $p_1 = \mathrm{C}\alpha$, &c. All the other equations of the field are the same as Maxwell's, with the exception that we do not suppose that $d\mathrm{F}/dx + d\mathrm{G}/dy + d\mathrm{H}/dz$ is zero.

6. In order to obtain the equations of electric displacement, let us consider a medium which is magnetically isotropic but electrostatically anisotropic. Let k be the magnetic permeability; K_1, K_2, K_3 the three principal specific inductive capacities; also let

$$k\mathrm{K}_1 = \mathrm{A}^{-2}, \quad k\mathrm{K}_2 = \mathrm{B}^{-2}, \quad k\mathrm{K}_3 = \mathrm{C}^{-2}.$$

$$\Omega = \mathrm{A}^2\frac{df}{dx} + \mathrm{B}^2\frac{dg}{dy} + \mathrm{C}^2\frac{dh}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\frac{d}{d\omega} = p_1\frac{d}{dx} + p_2\frac{d}{dy} + p_3\frac{d}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

From the last two of (6) we obtain

$$\frac{da}{dt} + p_1 \frac{df}{dy} + p_3 \frac{df}{dz} - p_1 \frac{dg}{dy} - p_1 \frac{dh}{dz} = \frac{dQ}{dz} - \frac{dR}{dy}.$$

Substituting the values of P, Q, R from the equations $P = 4\pi f/K_1$, &c., and recollecting that

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

we obtain

$$\frac{da}{dt} = 4\pi k \left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dy} \right) - \frac{df}{d\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

with two similar equations.

Now,

$$4\pi k \ddot{f} = 4\pi k \dot{u} = \frac{d}{dt} \left(\frac{dc}{dy} - \frac{db}{dz} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

substituting the values of $\dot{a}$, $\dot{b}$, $\dot{c}$ from (10) we obtain

$$\left.\begin{aligned} \frac{d^2 f}{dt^2} &= A^2 \nabla^2 f - \frac{d\Omega}{dx} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{g}}{dz} - \frac{d\dot{h}}{dy} \right) \\ \frac{d^2 g}{dt^2} &= B^2 \nabla^2 g - \frac{d\Omega}{dy} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{h}}{dx} - \frac{d\dot{f}}{dz} \right) \\ \frac{d^2 h}{dt^2} &= C^2 \nabla^2 h - \frac{d\Omega}{dz} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx} \right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

These are the equations satisfied by the components of electric displacement.

Professor FITZGERALD has not obtained the equations of electric displacement, but he has obtained the equations satisfied by the components of magnetic force (see p. 702); and by forming these equations by means of (10) and (11) it will be found that the results agree. It is not, however, easy to see what is the physical interpretation of those boundary conditions which involve the differential coefficients of the time integrals of the components of magnetic force; and it will be seen later on that they disagree with those given in the present paper, which are based upon physical considerations. A term in the kinetic energy which involves a volume integral, can usually be transformed in a variety of ways by integration by parts, into another volume integral and a surface integral; and although it is immaterial, so far as equations of motion are concerned, which volume integral we use, the boundary conditions will depend upon the surface integral. Accordingly, if we omitted a surface integral term from the expression for the kinetic energy and were to apply the principle of least action, it is evident that we should obtain the correct equations of motion, but the boundary conditions would be erroneous.

The boundary conditions $\xi_1 = \xi$, $\eta_1 = \eta$ (p. 706) appear to be perfectly satisfactory, since they imply that the component of magnetic force parallel to the reflecting surface is continuous; and the rejection of the condition $\zeta_1 = \zeta$ can be justified on the ground that it implies that the component of magnetic *force* perpendicular to the reflecting surface is continuous, whereas it is the corresponding component of magnetic *induction* that is continuous.

7. We shall now confine our attention to isotropic media. In this case $A = B = C = U$, where $U^{-2} = kK$; hence (10) becomes

$$\frac{da}{dt} = 4\pi k U^2 \left(\frac{dg}{dz} - \frac{dh}{dy}\right) - \frac{df}{d\omega}.$$

Let

$$f = S\lambda, \quad g = S\mu, \quad h = S\nu, \quad S = \epsilon^{2\iota\pi/V\tau.(lx + my + nz - Vt)} \quad . \quad . \quad . \quad (13),$$

then

$$\frac{da}{dt} = \frac{8\iota\pi^2 U^2 k}{V\tau}(n\mu - m\nu)\,S - \frac{2\iota\pi}{V\tau}(lp_1 + mp_2 + np_3)\,\dot{S}\lambda,$$

whence

$$\alpha = -\frac{4\pi U^2}{V}(n\mu - m\nu)\,S - \frac{2\iota\pi}{Vk\tau}(lp_1 + mp_2 + np_3)\,S\lambda.$$

Accordingly if $\mathfrak{H}$ denote the component of the external magnetic force perpendicular to the wave-front, the equations of magnetic force become

$$\left.\begin{aligned} \alpha &= -\frac{4\pi U^2}{V}(ng - mh) - \frac{2\iota\pi}{Vk\tau} C\mathfrak{H} f \\ \beta &= -\frac{4\pi U^2}{V}(lh - nf) - \frac{2\iota\pi}{Vk\tau} C\mathfrak{H} g \\ \gamma &= -\frac{4\pi U^2}{V}(mf - lg) - \frac{2\iota\pi}{Vk\tau} C\mathfrak{H} h \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where C is HALL'S constant.

Propagation of Light.

8. We are now prepared to consider the propagation of light in a magnetized medium.

Let us suppose that plane waves of light are incident upon the surface of separation of air and a magnetized medium. Let the axis of x be the normal, and be drawn into the first medium, and let the axis of z be perpendicular to the plane of incidence; also let the direction of magnetization be parallel to the axis of x. Then $p_2 = p_3 = 0$, and none of the quantities are functions of z; whence the equations of motion become

$$\left.\begin{aligned}\frac{d^2f}{dt^2} &= \mathrm{U}^2\nabla^2 f - \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx\,dy}\\ \frac{d^2g}{dt^2} &= \mathrm{U}^2\nabla^2 g + \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx^2}\\ \frac{d^2h}{dt^2} &= \mathrm{U}^2\nabla^2 h + \frac{p}{4\pi k}\frac{d}{dx}\left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15),$$

where p is written for p_1.

Let

$$f = \mathrm{A}'\mathrm{S}, \quad g = \mathrm{A}''\mathrm{S}, \quad h = \mathrm{A}\mathrm{S}, \quad \mathrm{S} = \epsilon^{2\iota\pi/\mathrm{V}\tau.(lx + my - \mathrm{V}t)}.$$

Substituting in (15) we obtain

$$\begin{aligned}(\mathrm{U}^2 - \mathrm{V}^2)\,\mathrm{A}' &= \frac{\iota p}{2k\tau}\,lm\mathrm{A},\\ (\mathrm{U}^2 - \mathrm{V}^2)\,\mathrm{A}'' &= \frac{\iota p}{2k\tau}\,l^2\mathrm{A},\\ (\mathrm{U}^2 - \mathrm{V}^2)\,\mathrm{A} &= \frac{\iota pl}{2k\tau}(\mathrm{A}'m - \mathrm{A}''l).\end{aligned}$$

From these equations we deduce

$$\mathrm{V}^2 = \mathrm{U}^2 \pm \frac{pl}{2k\tau} \quad \ldots\ldots\ldots \quad (16),$$

whence

$$\mathrm{A}' = \pm\,\iota m\mathrm{A}, \qquad \mathrm{A}'' = \mp\,\iota l\mathrm{A}.$$

Hence, if V_1, V_2 denote the two values of V corresponding to the upper and lower signs, we see that two waves are propagated with velocities V_1, V_2.

It must also be borne in mind that the directions of the two refracted waves corresponding to an incident wave are, in general, different. To see this, let the suffixes 1 and 2 refer to the two refracted waves, and let the incident wave be

$$h = \epsilon^{2\iota\pi/\mathrm{V}\tau.(lx + my - \mathrm{V}t)},$$

then the displacements in one of the refracted waves will be

$$f_1 = \iota m_1\mathrm{A}_1\mathrm{S}_1, \quad g_1 = -\,\iota l_1\mathrm{A}_1\mathrm{S}_1, \quad h_1 = \mathrm{A}_1\mathrm{S}_1,$$

where

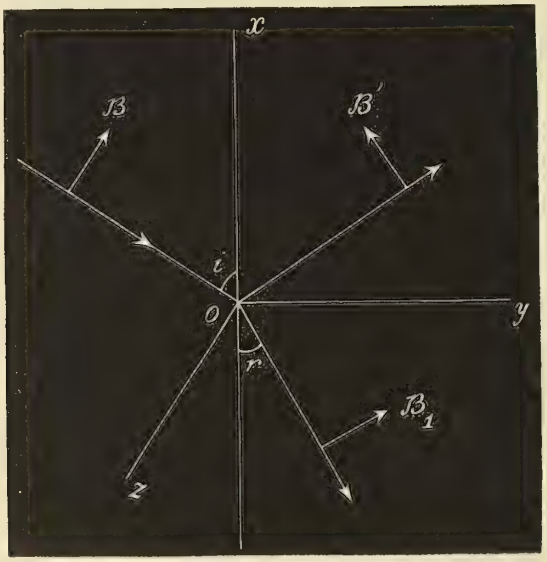
$$\mathrm{S}_1 = \epsilon^{2\iota\pi/\mathrm{V}_1\tau.(l_1x + m_1y - \mathrm{V}_1t)},$$

and the displacements in the other wave will be obtained by changing the suffix from 1 to 2, and changing the signs of f_1, g_1. Now, if r_1, r_2 be the angles of refraction,

$m_1 = \sin r_1$, $m_2 = \sin r_2$; and, since the coefficient of y must be the same in all three waves, we must have

$$\frac{V}{\sin i} = \frac{V_1}{\sin r_1} = \frac{V_2}{\sin r_2},$$

which shows that r_1 is different from r_2.



Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the component displacements in the plane $z = 0$, then since $l_1 = -\cos r_1$, it follows that

$$\mathfrak{B}_1 = f_1 \sin r_1 + g_1 \cos r_1 = \iota A_1 S_1.$$

Similarly

$$\mathfrak{B}_2 = -\iota A_2 S_2.$$

The component displacements perpendicular to the wave-fronts are evidently zero; whence, in real quantities, the displacements in the two waves are

$$h_1 = A_1 \cos \frac{2\pi}{V_1 \tau}(l_1 x + m_1 y - V_1 t),$$

$$\mathfrak{B}_1 = -A_1 \sin \frac{2\pi}{V_1 \tau}(l_1 x + m_1 y - V_1 t),$$

and

$$h_2 = A_2 \cos \frac{2\pi}{V_2 \tau}(l_2 x + m_2 y - V_2 t),$$

$$\mathfrak{B}_2 = A_2 \sin \frac{2\pi}{V_2 \tau}(l_2 x + m_2 y - V_2 t),$$

and, consequently, the two waves are circularly polarized in opposite directions.

The Boundary Conditions.

9. When light is reflected and refracted at the surface of separation of two isotropic or crystalline media, the boundary conditions are (i) that the components of the electromotive and magnetic forces *parallel* to the surface of separation must be continuous; (ii) that the components of electric displacement and magnetic induction *perpendicular* to the surface of separation must likewise be continuous. We have, therefore, *six* equations to determine *four* unknown quantities; but inasmuch as two pairs of these equations are identical, the total number reduces to four, which is just sufficient to determine the four unknown quantities. If, however, we were to assume these six conditions in the case of a magnetized medium, we should find that we should be led to inconsistent results, and we shall, therefore, proceed to prove the boundary conditions.

Since the electric displacement and the magnetic induction both satisfy the equation

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 ,$$

which is an equation of the same form as the equation of continuity of an incompressible fluid in hydrodynamics, it follows that the components of the electric displacement and magnetic induction perpendicular to the surface of separation must be continuous.

To obtain the other conditions, let us suppose as before, that the plane $x = 0$ is the surface of separation, and that the plane $z = 0$ contains the direction of propagation. Then, since the coefficients of y and t in the exponential factor must be the same in all four waves, d/dy and d/dt of any continuous function will also be continuous, and conversely. Since none of the quantities are functions of z,

$$4\pi\dot{f} = \frac{d\gamma}{dy},$$

which shows that γ is continuous.

Since the continuity of γ follows from that of f, the conditions of continuity of both these quantities will be expressed by the same equation.

Since

$$\alpha = \frac{d\mathrm{H}}{dy},$$

it follows that H is continuous, whence if the accents refer to the second medium, we obtain from (6)

$$\mathrm{R}' + p_2\dot{f}' - p_1\dot{g}' = \mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

This equation shows that the electromotive force parallel to z is discontinuous. This circumstance may, at first sight, appear somewhat strange, and may perhaps be

regarded as an objection to the theory; but since the p's are exceedingly small quantities, the discontinuity is also very small. We have, moreover, assumed that the transition from one medium to the other is *abrupt*, whereas, if we were better acquainted with the conditions at the confines of two different media, we should probably find that this was not the case; and under these circumstances there would be a rapid but continuous change in the component of the electromotive force parallel to the boundary, in passing from one medium to the other.

We have, therefore, as yet, only obtained two independent boundary equations. Now, we shall presently see that when plane polarized light is reflected and refracted at the surface of a magnetized medium, the reflected light is elliptically polarized; whilst, as we have already shown, the two refracted waves are circularly polarized in opposite directions. We have, therefore, four unknown quantities to determine, viz., the amplitudes of the two components of the reflected vibration, and the amplitudes of the two refracted waves. We, therefore, require two more equations. To find a third equation, we may assume, as has been done by previous writers, that the component of magnetic force parallel to the axis of y is continuous. A fourth equation will be obtained from the condition of continuity of energy; for since there is no conversion of energy into heat, or any form of energy other than the electrical kind, it follows that the rate of increase of the electrostatic and electrokinetic energies within any closed surface must be equal to the rate at which energy flows in across the boundary.

[We must now obtain an expression for the energy.*

It is a general principle of dynamics, that if equations are given which are sufficient to completely determine the motion of a system, the principle of energy can be deduced from these equations. The proper form of the principle of energy in the case of a dielectric medium is this:—*Describe any closed surface in the medium, then the rate at which energy increases within the surface, is equal to the rate at which energy flows in across the boundary.* If E be the electric energy per unit of volume, the rate at which energy increases within the surface is $\iiint \dot{\mathrm{E}}\, dx\, dy\, dz$, and, consequently, this quantity must be capable of being expressed as a surface integral taken over the boundary; and any form of $\dot{\mathrm{E}}$ which is not capable of being so expressed must certainly be wrong. If the medium were a conductor, in which there is a conversion of energy into heat, $\iiint \dot{\mathrm{E}}\, dx\, dy\, dz$ would not be expressible in the form of a surface integral,† but this case need not be considered, since we are dealing with a transparent dielectric.

Since $\mathrm{P} = 4\pi f/\mathrm{K}_1 = 4\pi k\mathrm{A}^2 f$, equation (10) may be written in the form

$$\frac{da}{dt} = \frac{d\mathrm{Q}}{dz} - \frac{d\mathrm{R}}{dy} - \frac{\dot{df}}{d\omega}.$$

* Rewritten, July 1, 1891.

† *See* POYNTING, 'Phil. Trans.,' 1884, p. 343.

Multiply this equation and the two corresponding ones by α, β, γ; then add and integrate throughout any closed surface, and we shall obtain

$$\tfrac{1}{2}k\frac{d}{dt}\iiint(\alpha^2+\beta^2+\gamma^2)\,dx\,dy\,dz$$
$$=\iiint\left\{\alpha\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+\beta\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+\gamma\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)\right\}dx\,dy\,dz$$
$$-\iiint\left(\alpha\frac{df}{d\omega}+\beta\frac{dg}{d\omega}+\gamma\frac{dh}{d\omega}\right)dx\,dy\,dz \quad . \quad (17\text{A}).$$

Let

$$W=2\pi k\iiint(A^2f^2+B^2g^2+C^2h^2)\,dx\,dy\,dz \quad . \quad . \quad . \quad . \quad (17\text{B}),$$

then

$$\frac{dW}{dt}=4\pi k\iiint(A^2f\dot{f}+B^2g\dot{g}+C^2h\dot{h})\,dx\,dy\,dz$$
$$=\iiint(P\dot{f}+Q\dot{g}+R\dot{h})\,dx\,dy\,dz.$$

Substituting the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ in terms of α, β, γ, and integrating by parts, we obtain

$$\frac{dW}{dt}=\frac{1}{4\pi}\iint\{l(R\beta-Q\gamma)+m(P\gamma-R\alpha)+n(Q\alpha-P\beta)\}\,dS$$
$$-\frac{1}{4\pi}\iiint\left\{\alpha\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+\beta\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+\gamma\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)\right\}dx\,dy\,dz \quad . \quad (17\text{C}).$$

If in the identity

$$\dot{f}(p_3\dot{g}-p_2\dot{h})+\dot{g}(p_1\dot{h}-p_3\dot{f})+\dot{h}(p_2\dot{f}-p_1\dot{g})=0,$$

we substitute the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ from the equations of $4\pi\dot{f}=d\gamma/dy-d\beta/dz$, &c., in the coefficients of the terms in brackets, and integrate by parts, we shall find that the last volume integral in (17A) is equal to

$$-\iint[l\{(p_2\dot{f}-p_1\dot{g})\beta-(p_1\dot{h}-p_3\dot{f})\gamma\}$$
$$+m\{(p_3\dot{g}-p_2\dot{h})\gamma-(p_2\dot{f}-p_1\dot{g})\alpha\}$$
$$+n\{(p_1\dot{h}-p_3\dot{f})\alpha-(p_3\dot{g}-p_2\dot{h})\beta\}]\,dS \quad . \quad . \quad . \quad . \quad (17\text{D}).$$

Accordingly (17A) becomes on substitution from (17B), (17C) and (17D)

$$\frac{d}{dt}\iiint\left\{\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k(\mathrm{A}^2f^2+\mathrm{B}^2g^2+\mathrm{C}^2h^2)\right\}dx\,dy\,dz$$
$$=\frac{1}{4\pi}\iint[l\{(\mathrm{R}+p_2\dot{f}-p_1\dot{g})\beta-(\mathrm{Q}+p_1\dot{h}-p_3\dot{f})\gamma\}$$
$$+m\{(\mathrm{P}+p_3\dot{g}-p_2\dot{h})\gamma-(\mathrm{R}+p_2\dot{f}-p_1\dot{g})\alpha\}$$
$$+n\{(\mathrm{Q}+p_1\dot{h}-p_3\dot{f})\alpha-(\mathrm{P}+p_3\dot{g}-p_2\dot{h})\beta\}]dS \quad . \quad . \quad (18).$$

The physical interpretation of this equation is, that the rate at which something increases within the closed surface, must be equal to the rate at which something flows into the surface. This cannot be anything else but energy; we are therefore led to identify the expression

$$\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k(\mathrm{A}^2f^2+\mathrm{B}^2g^2+\mathrm{C}^2h^2)$$

as representing the energy of the electric field per unit of volume. The first term represents the electrokinetic energy, and the second term the electrostatic energy.

The above expressions are the same as those obtained by MAXWELL by a different method, and it thus appears that the expressions for each species of energy are not altered by the additional terms which have been introduced into the general equations of electromotive force.

The right hand side of (18) represents the rate at which work is done by the electric and magnetic forces which act upon the surface of S.]

In the optical problem which we are considering, the bounding surface is the plane $x=0$, whence if the quantities in the magnetized medium be denoted by accented letters, the condition of continuity of energy becomes

$$\mathrm{R}\beta-\mathrm{Q}\gamma=(\mathrm{R}'+p_2\dot{f}'-p_1\dot{g}')\beta'-(\mathrm{Q}'+p_1\dot{h}'-p_3\dot{f}')\gamma'.$$

Since $\beta=\beta'$ and $\gamma=\gamma'$, it follows from (17) that this equation reduces to

$$\mathrm{Q}=\mathrm{Q}'+p_1\dot{h}'-p_3\dot{f}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

which shows that the components of the electromotive force *in* the plane of incidence are also discontinuous.

The boundary conditions are therefore the following; (i.) continuity of electric displacement perpendicular to the reflecting surface, which is equivalent to continuity of magnetic force parallel to z; (ii.) continuity of magnetic induction perpendicular to the reflecting surface, which is equivalent to equation (17); (iii.) continuity of magnetic force parallel to y; (iv.) equation (19), which follows partly from (i.), (ii.), and (iii.), and partly from the condition that the flow of energy must be continuous.

We have therefore four equations and no more, to determine the four unknown quantities.

Reflection and Refraction when the Reflecting Surface is Magnetized Normally.

10. We are now prepared to calculate the amplitudes of the reflected and refracted waves.

Let us first suppose that the reflecting surface is magnetized normally; then $p_2 = p_3 = 0$. Also let A, B be the amplitudes of the two components of the incident vibrations perpendicular to, and in the plane of incidence; then the displacements in the four waves may be written

$$\begin{aligned} h &= \mathrm{AS}, & \mathfrak{B} &= \mathrm{BS}, & &\text{incident wave;}\\ h' &= \mathrm{A'S'}, & \mathfrak{B}' &= \mathrm{B'S'}, & &\text{reflected wave;}\\ h_1 &= \mathrm{A_1S_1}, & \mathfrak{B}_1 &= \iota \mathrm{A_1S_1}, & &\text{1st refracted wave;}\\ h_2 &= \mathrm{A_2S_2}, & \mathfrak{B}_2 &= -\iota \mathrm{A_2S_2}, & &\text{2nd refracted wave.} \end{aligned}$$

Also

$$\begin{aligned} l &= -\cos i, & l' &= \cos i, & l_1 &= -\cos r_1, & l_2 &= -\cos r_2;\\ m &= \sin i, & m' &= \sin i, & m_1 &= \sin r_1, & m_2 &= \sin r_2. \end{aligned}$$

The boundary conditions (i.), (ii.), (iii.), (iv.), at the end of § 9, furnish the following equations:—

$$(\mathrm{B} + \mathrm{B'})\,\mathrm{V} = \iota\,(\mathrm{A_1V_1} - \mathrm{A_2V_2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$(\mathrm{A} + \mathrm{A'})\,\mathrm{V}^2 = \mathrm{U}^2 k\,(\mathrm{A_1} + \mathrm{A_2}) - \frac{p}{2\tau}\,(\mathrm{A_1}\cos r_1 - \mathrm{A_2}\cos r_2) \quad . \quad . \quad (21),$$

$$(\mathrm{A} - \mathrm{A'})\,\mathrm{V}\cos i = \mathrm{U}^2\left(\frac{\mathrm{A_1}}{\mathrm{V_1}}\cos r_1 + \frac{\mathrm{A_2}}{\mathrm{V_2}}\cos r_2\right) - \frac{p}{2k\tau}\left(\frac{\mathrm{A_1}}{\mathrm{V_1}}\cos^2 r_1 - \frac{\mathrm{A_2}}{\mathrm{V_2}}\cos^2 r_2\right). \quad (22),$$

$$(\mathrm{B} - \mathrm{B'})\,\mathrm{V}^2\cos i = \iota \mathrm{U}^2 k\,(\mathrm{A_1}\cos r_1 - \mathrm{A_2}\cos r_2) - \frac{\iota p}{2\tau}\,(\mathrm{A_1} + \mathrm{A_2}) \quad . \quad . \quad (23),$$

where p is written for p_1.

We shall now simplify these equations by introducing an auxiliary angle R, such that

$$\frac{\mathrm{V}}{\sin i} = \frac{\mathrm{V_1}}{\sin r_1} = \frac{\mathrm{V_2}}{\sin r_2} = \frac{\mathrm{U}}{\sin \mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Hence, R is the angle of refraction when the second medium is unmagnetized; and accordingly r_1 and r_2 will differ from R by a small quantity which depends upon p.

Since the magnetic effects are small, we shall neglect squares and higher powers of p, and we may, therefore, in the terms multiplied by p, put $r_1 = r_2 = \mathrm{R}$.

Let $q = p/4k\tau$, then from (16)

$$\mathrm{V}_1 = \mathrm{U} - \frac{q \cos \mathrm{R}}{\mathrm{U}}, \quad \mathrm{V}_2 = \mathrm{U} + \frac{q \cos \mathrm{R}}{\mathrm{U}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25);$$

also from (24) and (25) we obtain

$$\cos r_1 = \cos \mathrm{R} + \frac{q \sin^2 \mathrm{R}}{\mathrm{U}^2}, \quad \cos r_2 = \cos \mathrm{R} - \frac{q \sin^2 \mathrm{R}}{\mathrm{U}^2} \quad . \quad . \quad . \quad . \quad (26).$$

Substituting these values in equations (20) to (23) and reducing, they finally become

$$\left.\begin{aligned} (\mathrm{B} - \mathrm{B}')\, \mathrm{V}^2 \cos i &= \iota \mathrm{U}^2 k\, (\mathrm{A}_1 - \mathrm{A}_2) \cos \mathrm{R} + \iota q k\, (\sin^2 \mathrm{R} - 2)\, (\mathrm{A}_1 + \mathrm{A}_2) \\ (\mathrm{B} + \mathrm{B}')\, \mathrm{V} &= \iota \mathrm{U}\, (\mathrm{A}_1 - \mathrm{A}_2) - \frac{\iota q \cos \mathrm{R}}{\mathrm{U}}\, (\mathrm{A}_1 + \mathrm{A}_2) \\ (\mathrm{A} - \mathrm{A}')\, \mathrm{V} \cos i &= \mathrm{U}\, (\mathrm{A}_1 + \mathrm{A}_2) \cos \mathrm{R} - \frac{q \cos 2\mathrm{R}}{\mathrm{U}} (\mathrm{A}_1 - \mathrm{A}_2) \\ (\mathrm{A} + \mathrm{A}')\, \mathrm{V}^2 &= \mathrm{U}^2 k\, (\mathrm{A}_1 + \mathrm{A}_2) - 2 q k\, (\mathrm{A}_1 - \mathrm{A}_2) \cos \mathrm{R} \end{aligned}\right\} \quad (27).$$

These equations determine the amplitudes of the reflected and refracted waves, when the magnetization is perpendicular to the reflecting surface.

From the first two of (27) we get

$$\iota q k\, (\mathrm{A}_1 + \mathrm{A}_2) = \mathrm{BV}\, (\mathrm{U}k \cos \mathrm{R} - \mathrm{V} \cos i) + \mathrm{B}'\mathrm{V}\, (\mathrm{U}k \cos \mathrm{R} + \mathrm{V} \cos i),$$

$$\iota \mathrm{U}^2 k\, (\mathrm{A}_1 - \mathrm{A}_2) = \mathrm{BV}\, \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) - \mathrm{V} \cos i \cos \mathrm{R}\} + \mathrm{B}'\mathrm{V}\, \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) + \mathrm{V} \cos i \cos \mathrm{R}\}.$$

Substituting in the last two, we obtain

$$\begin{aligned} &(\mathrm{A} - \mathrm{A}') \cos i \\ &= -\iota \mathrm{B} \left[\frac{\mathrm{U} \cos \mathrm{R}}{qk} (\mathrm{U}k \cos \mathrm{R} - \mathrm{V} \cos i) - \frac{q \cos 2\mathrm{R}}{\mathrm{U}^3 k} \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) - \mathrm{V} \cos i \cos \mathrm{R}\} \right] \\ &\quad - \iota \mathrm{B}' \left[\frac{\mathrm{U} \cos \mathrm{R}}{qk} (\mathrm{U}k \cos \mathrm{R} + \mathrm{V} \cos i) - \frac{q \cos 2\mathrm{R}}{\mathrm{U}^3 k} \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) + \mathrm{V} \cos i \cos \mathrm{R}\} \right] \quad (28) \end{aligned}$$

and

$$\begin{aligned} &(\mathrm{A} + \mathrm{A}')\, \mathrm{V} \\ &= -\iota \mathrm{B} \left[\frac{\mathrm{U}^2}{q} (\mathrm{U}k \cos \mathrm{R} - \mathrm{V} \cos i) - \frac{2q \cos \mathrm{R}}{\mathrm{U}^2} \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) - \mathrm{V} \cos i \cos \mathrm{R}\} \right] \\ &\quad - \iota \mathrm{B}' \left[\frac{\mathrm{U}^2}{q} (\mathrm{U}k \cos \mathrm{R} + \mathrm{V} \cos i) - \frac{2q \cos \mathrm{R}}{\mathrm{U}^2} \{\mathrm{U}k\, (2 - \sin^2 \mathrm{R}) + \mathrm{V} \cos i \cos \mathrm{R}\} \right] \quad (29). \end{aligned}$$

Solving these equations, we obtain

$$A' = \frac{A(Uk\cos i - V\cos R)}{Uk\cos i + V\cos R} + \frac{2\iota qkBV\cos i}{U(Uk\cos R + V\cos i)(Uk\cos i + V\cos R)} \quad (30),$$

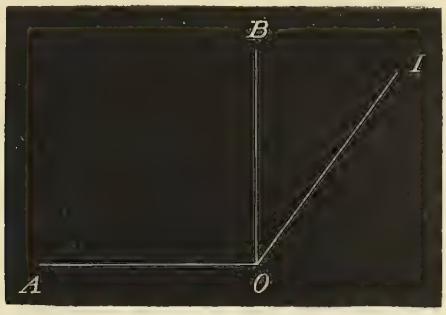
$$B' = -\frac{B(Uk\cos R - V\cos i)}{Uk\cos R + V\cos i} + \frac{2\iota qkAV\cos i}{U(Uk\cos R + V\cos i)(Uk\cos i + V\cos R)} \quad (31).$$

11. We shall now discuss these results.

Equations (30) and (31) give the amplitudes of the two components of the reflected light, and we see that the magnetic terms vanish at grazing incidence, but do not vanish for any other incidence.

The equations may be written in the form.

$$\left.\begin{aligned} A' &= A\alpha + \iota qB\beta \\ B' &= B\gamma + \iota qA\beta \end{aligned}\right\} \quad (32).$$



In the figure let I be the point of incidence, IO the normal to the reflected wave, and let O be the observer; also let OA, OB be drawn at right angles to OI, perpendicular to and in the plane of incidence respectively. Let ξ, η be the displacements along OA, OB; also let $\phi = (2\pi/\lambda)(x\cos i + y\sin i - Vt)$. Then by (32)

$$\left.\begin{aligned} \xi &= A\alpha\cos\phi - qB\beta\sin\phi \\ \eta &= B\gamma\cos\phi - qA\beta\sin\phi \end{aligned}\right\} \quad (33),$$

which shows that the reflected light is elliptically polarized.

Let us now suppose that the incident light is polarized *in* the plane of incidence, so that $B = 0$, and let the principal section of the analyser coincide with OB. Then the intensity of the reflected light after it has passed through the analyser is proportional to $A^2\beta^2q^2$, and is therefore independent of the direction of the magnetizing current, and vanishes when the current is cut off.

These results are in accordance with the first experiment.

Let the analyser be now turned through a small angle ϵ towards the right hand of the observer. From (33) we see that the intensity of the reflected light after emerging from the analyser is proportional to

$$A^2 (\alpha^2\epsilon^2 + \beta^2 q^2),$$

from which it appears that the effect of the current is always to increase the intensity, and that the intensity is independent of the direction of the current.

This result is not in accordance with the second experiment, since in that experiment it was found that if a current in one direction strengthened the reflected light, a current in the opposite direction weakened it. In order that theory and experiment should agree, it would be necessary that the amplitudes of the two reflected vibrations should satisfy equations of the form

$$A' = A (\alpha + q\alpha') + \iota q B\beta, \qquad B' = B (\gamma + q\gamma') + \iota q A\delta \ . \ . \ . \quad (34),$$

where α', γ' are quantities which may be either real or complex, but not purely imaginary. Under these circumstances the principal part of the magnetic effect would be represented by a term proportional to q, which would change sign when the current is reversed.

I do not, however, think that this failure of the theory to agree with KERR's second experiment is a conclusive objection to the theory itself, inasmuch as he used a metallic reflector, and his results were accordingly complicated by the influence of metallic reflection; and it is by no means improbable that the terms α', γ' in equation (34), which are necessary to explain the phenomena observed, are the result of metallic reflection, and disappear when the reflector is a non-metallic substance.

These considerations show the importance of endeavouring to carry out a series of experiments upon the reflection of light from non-metallic media which are capable of magnetically affecting light. If it should be found that the intensity of the reflected light is independent of the direction of the magnetizing current, when such a medium is substituted for the metallic pole, the evidence in favour of the theory will be considerably strengthened; but if it should be found that the effect depends upon the direction of the current, the present theory must be abandoned.

Reflection and Refraction when the Reflecting Surface is Magnetized parallel to itself.

12. We must now consider the case in which the magnetic force is parallel to the reflecting surface.

Let the axes of y coincide with the direction of the lines of magnetic force, then $p_1 = p_3 = 0$; and let us first suppose that the plane of incidence is the plane of xz. Since none of the quantities in this case are functions of y, it follows that the last

terms in equations (12) are zero, and therefore magnetization produces no optical effect.

In the next place suppose that the incidence is normal; then all the quantities are functions of x and t only; whence the last terms in equations (12) vanish in this case also, so that magnetization produces no optical effect.

These results are in accordance with KERR's experiments.

13. We shall now consider the case in which the plane of incidence is parallel to the direction of magnetization.

From (12) it follows that the equations of motion in this case are

$$\left.\begin{aligned} \frac{d^2f}{dt^2} &= U^2\nabla^2 f - \frac{p}{4\pi k}\frac{d^2\dot{h}}{dy^2} \\ \frac{d^2g}{dt^2} &= U^2\nabla^2 g + \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx\,dy} \\ \frac{d^2h}{dt^2} &= U^2\nabla^2 h + \frac{p}{4\pi k}\frac{d}{dy}\left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots (35),$$

where p is written for p_2. Solving these equations in the same way as (15) we obtain

$$V^2 = U^2 \pm \frac{pm}{2k\tau} \quad \ldots\ldots\ldots (36),$$

$$A' = \pm \iota m A, \qquad A'' = \mp \iota l A.$$

Accordingly, the component displacements in the two refracted waves are

$$\begin{aligned} h_1 &= A_1 S_1, & \mathfrak{B}_1 &= \iota A_1 S_1; \\ h_2 &= A_2 S_1, & \mathfrak{B}_2 &= -\iota A_2 S_2. \end{aligned}$$

The boundary conditions (i.), (ii.), (iii.) and (iv.) of § 9 give rise to the following equations, viz.,

$$(B + B')V = \iota(A_1 V_1 - A_2 V_2) \quad \ldots\ldots\ldots (37),$$

$$(A + A')V^2 = U^2 k(A_1 + A_2) + \frac{p}{2\tau}(A_1 \sin r_1 - A_2 \sin r_2) \quad \ldots\ldots (38),$$

$$(A - A')V\cos i = U^2\left(\frac{A_1\cos r_1}{V_1} + \frac{A_2\cos r_2}{V_2}\right) + \frac{p\sin R}{2Uk\tau}(A_1\cos r_1 - A_2\cos r_2) \quad (39),$$

$$(B - B')V^2\cos i = \iota U^2 k(A_1\cos r_1 - A_2\cos r_2) \quad \ldots\ldots\ldots (40).$$

From (36) and (24) we obtain

$$V_1 = U + \frac{q\sin R}{U}, \qquad V_2 = U - \frac{q\sin R}{U} \quad \ldots\ldots (41),$$

$$\cos r_1 = \cos R - \frac{q\sin^3 R}{U^2\cos R}, \qquad \cos r_2 = \cos R + \frac{q\sin^3 R}{U^2\cos R} \quad \ldots (42).$$

Substituting in equations (37) to (40), and rewriting, we obtain

$$\left.\begin{aligned}
(\mathrm{B}-\mathrm{B}')\,\mathrm{V}^2\cos i &= \iota \mathrm{U}^2 k\cos \mathrm{R}\,(\mathrm{A}_1-\mathrm{A}_2) - \frac{\iota q k \sin^3 \mathrm{R}}{\cos \mathrm{R}}(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{B}+\mathrm{B}')\,\mathrm{V} &= \iota \mathrm{U}\,(\mathrm{A}_1-\mathrm{A}_2) + \frac{\iota q \sin \mathrm{R}}{\mathrm{U}}(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{A}-\mathrm{A}')\,\mathrm{V}\cos i &= \mathrm{U}\cos \mathrm{R}\,(\mathrm{A}_1+\mathrm{A}_2) + \frac{q\cos 2\mathrm{R}\tan \mathrm{R}}{\mathrm{U}}(\mathrm{A}_1-\mathrm{A}_2)\\
(\mathrm{A}+\mathrm{A}')\,\mathrm{V}^2 &= \mathrm{U}^2 k\,(\mathrm{A}_1+\mathrm{A}_2) + 2qk\sin \mathrm{R}\,(\mathrm{A}_1-\mathrm{A}_2)
\end{aligned}\right\}\quad (43\text{–}46).$$

These are the equations which determine the amplitudes of the reflected and refracted light, when the lines of magnetic force are parallel to the reflecting surface, and the plane of incidence is parallel to them.

From the first two of (43–46) we obtain

$$\iota qk\tan \mathrm{R}\,(\mathrm{A}_1+\mathrm{A}_2) = \mathrm{BV}\,(\mathrm{U}k\cos \mathrm{R} - \mathrm{V}\cos i) + \mathrm{B}'\mathrm{V}\,(\mathrm{U}k\cos \mathrm{R} + \mathrm{V}\cos i),$$

$$\iota \mathrm{U}^2 k\,(\mathrm{A}_1-\mathrm{A}_2) = \mathrm{BV}\,(\mathrm{U}k\sin^2 \mathrm{R} + \mathrm{V}\cos i\cos \mathrm{R}) + \mathrm{B}'\mathrm{V}\,(\mathrm{U}k\sin^2 \mathrm{R} - \mathrm{V}\cos i\cos \mathrm{R}).$$

Substituting in the last two of (43–46) we obtain

$$\begin{aligned}
(\mathrm{A}-\mathrm{A}')\cos i = &-\iota \mathrm{B}\left\{\frac{\mathrm{U}\cos^2 \mathrm{R}}{qk\sin \mathrm{R}}(\mathrm{U}k\cos \mathrm{R} - \mathrm{V}\cos i) + \frac{q\cos 2\mathrm{R}\tan \mathrm{R}}{\mathrm{U}^3 k}(\mathrm{U}k\sin^2 \mathrm{R} + \mathrm{V}\cos i\cos \mathrm{R})\right\}\\
&-\iota \mathrm{B}'\left\{\frac{\mathrm{U}\cos^2 \mathrm{R}}{qk\sin \mathrm{R}}(\mathrm{U}k\cos \mathrm{R} + \mathrm{V}\cos i) + \frac{q\cos 2\mathrm{R}\tan \mathrm{R}}{\mathrm{U}^3 k}(\mathrm{U}k\sin^2 \mathrm{R} - \mathrm{V}\cos i\cos \mathrm{R})\right\}
\end{aligned}$$

$$\begin{aligned}
(\mathrm{A}+\mathrm{A}')\,\mathrm{V} = &-\iota \mathrm{B}\left\{\frac{\mathrm{U}^2\cot \mathrm{R}}{q}(\mathrm{U}k\cos \mathrm{R} - \mathrm{V}\cos i) + \frac{2q\sin \mathrm{R}}{\mathrm{U}^2}(\mathrm{U}k\sin^2 \mathrm{R} + \mathrm{V}\cos i\cos \mathrm{R})\right\}\\
&-\iota \mathrm{B}'\left\{\frac{\mathrm{U}^2\cot \mathrm{R}}{q}(\mathrm{U}k\cos \mathrm{R} + \mathrm{V}\cos i) + \frac{2q\sin \mathrm{R}}{\mathrm{U}^2}(\mathrm{U}k\sin^2 \mathrm{R} - \mathrm{V}\cos i\cos \mathrm{R})\right\},
\end{aligned}$$

whence

$$A' = \frac{A\,(Uk\cos i - V\cos R)}{Uk\cos i + V\cos R} - \frac{2\iota qkBV\tan R\cos i}{U\,(Uk\cos i + V\cos R)\,(Uk\cos R + V\cos i)} \quad (47),$$

$$B' = -\frac{B\,(Uk\cos R - V\cos i)}{Uk\cos R + V\cos i} + \frac{2\iota qkAV\tan R\cos i}{U\,(Uk\cos i + V\cos R)\,(Uk\cos R + V\cos i)} \quad (48).$$

14. In the third experiment, the incident light is polarized *in* the plane of incidence, so that $B = 0$. We therefore see that the component perpendicular to the plane of incidence is unaffected by magnetization, but that the effect of the latter is to introduce a component in the plane of incidence, which is proportional to the magnetic force, and which is given by (48). Now B′ may be put into the form

$$\frac{2\iota qkA\mu\sin 2i\sin^2 i}{V^2(\mu^2 - \sin^2 i)^{\frac{1}{2}}(\sin i\cos R + k\sin R\cos i)(\sin 2i + k\sin 2R)},$$

where μ is the index of refraction; and this vanishes when $i = 0$ or $i = \frac{1}{2}\pi$. Accordingly, the magnetic term increases from grazing incidence to a maximum value and then decreases to normal incidence.

This result is in partial agreement with the third experiment, since it was found that the magnetic effect attained a maximum value when i was equal to about 60°; but it disagrees, owing to the fact that the intensity depends on the square of the magnetic force, and is, therefore, unaffected when the current is reversed.

In the fourth experiment, the incident light was polarized perpendicularly to the plane of incidence, so that $A = 0$; and we see from (47) and (48) that the effects are of the same kind as when the incident light is polarized in the plane of incidence. This result altogether disagrees with the fourth experiment.

The theory which I have attempted to develop, cannot, I think, be considered altogether unsatisfactory, since it certainly explains some of KERR's experimental results; at the same time the theory must be regarded as a somewhat tentative one, until experiments upon magnetic solutions have been made.

[15.* In a recent paper,† I have attempted to show that the rotatory properties of quartz may be accounted for by assuming that the electromotive force and the electric displacement are connected together by equations of the form

$$\left.\begin{aligned} P &= 4\pi f/K_1 + p_3\dot{g} - p_2\dot{h} \\ Q &= 4\pi g/K_2 + p_1\dot{h} - p_3\dot{f} \\ R &= 4\pi h/K_3 + p_2\dot{f} - p_1\dot{g}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (49),$$

* Partly rewritten, July 1, 1891.

† 'Phil. Mag.,' August, 1890, p. 152.

where p_1, p_2, p_3 are absolute constants which depend upon the rotatory properties of the medium.

This assumption leads to equations (12) of the present paper; and, consequently, if we introduce the additional terms into (6), the equations of motion given in the paper in the 'Philosophical Magazine'* will be of the same form as those of the present paper.

I am disposed on reconsideration to modify the former theory in the above manner; in which case the properties of quartz would be accounted for by the hypothesis that the molecular structure of quartz produces an effect upon light, which may be explained by supposing that the general equations of electromotive force may be represented by (6). There is, however, an important distinction between the two classes of phenomena, for, in the case of a magnetized medium, p_1, p_2, p_3 are proportional to the components of magnetic force, and consequently change sign when the magnetic force is reversed; but, in the case of quartz, p_1, p_2, p_3 are absolute constants. Moreover, if the axis of z be the axis of the crystal, p_1 and p_2 must each be zero, because a wave of light, which is transmitted perpendicularly to the axis, experiences no rotation. In the case of syrup, turpentine, and other isotropic substances which produce rotatory polarization, p_1, p_2, p_3 must be equal to one another.

Under these circumstances we should anticipate that when light is reflected from a plate of quartz, whose faces are parallel or perpendicular to its axis, the reflected light would exhibit certain peculiarities, which are analogous to those produced by a magnetized transparent medium; but in working out the problem we must recollect that in quartz there are two principal wave velocities, and this circumstance would lead to certain modifications in the final results of equations (30), (31), (47), and (48).

The effect produced by a single reflection would probably be too small to be detected; but possibly some peculiarities might be observed, if (as Dr. Kerr has suggested to me) the method of multiple reflections were employed. In any case the subject is one which opens up a wide field for experimentalists.]

[†16. When glass is placed in a strong field of magnetic force, and a ray of plane polarized light is transmitted in the direction of the magnetic force, it is known that the plane of polarization is rotated in the *positive direction;* that is to say, the rotation takes place in the same direction as the amperean currents which would produce the force. It therefore follows that Hall's constant must be negative for glass. The experiments of Kundt‡ on the transmission of light through thin metallic films, show that the direction of rotation in iron, cobalt, and nickel is the same as in

* Equations (10) of that paper, in which the factor $(4\pi\mu)^{-1}$, or $(4\pi k)^{-1}$ in the notation of the present paper, was inadvertently omitted from the last term.

† Added July 1, 1891.

‡ 'Berlin Sitzungsberichte,' July 10, 1884; translated, 'Phil. Mag.,' series 5, vol. 18, p. 308.

glass; if, therefore, magnetic rotation by metals could be explained by HALL'S effect, it ought to follow that HALL'S constant is negative for all three metals. When HALL'S *constant* is negative, the additional electromotive force, or HALL'S *effect*, is positive; for if under these circumstances the conductor is a plate which coincides with the plane of (xy), and the magnetic force and the primary current are respectively parallel to the axes of z and y, the additional electromotive force will be parallel to the positive direction of the axis of x. It appears from experiment that HALL'S effect is positive for iron and cobalt, and negative for nickel.* Sir W. THOMSON† has, however, shown that, under certain conditions, the behaviour of nickel with respect to magnetic force is opposite to that of iron, and it is therefore not surprising that a similar peculiarity should exist with regard to the connection between HALL'S constant and the rotation of the plane of polarization.

At the time that this paper was written I was unacquainted with the experiments of KUNDT referred to above. Most of them relate to metals, and cannot, therefore, be compared with the theoretical results of this paper. KUNDT has, however, made one series of experiments which is capable of comparison, in which light was incident upon a plate of glass whose sides were not quite accurately parallel, so that the rays which had undergone two refractions at the anterior surface, and one reflection at the posterior surface, were well separated from those which were reflected at the anterior surface. When the plate was magnetized certain effects were observed, which are described by KUNDT, and which are analogous to those obtained by KERR in the case of metals; and in particular,‡ when the lines of magnetic force are perpendicular to the faces of the plate, and the incident light is polarized in the plane of incidence, the plane of polarization is always rotated in the positive direction (*i.e.*, in the direction of the amperean currents); but when the light is polarized perpendicularly to the plane of incidence, the rotation is positive from 0° to the polarizing angle, and negative up to 90°.

To give a complete mathematical investigation of these experimental results would lead to some rather complicated expressions; I shall therefore confine myself to the case in which the lines of magnetic force are perpendicular to the faces of the plate, and the incidence is sensibly normal, and shall calculate the intensity of the light which has undergone two refractions at the anterior surface, and one reflection at the posterior.

Let the incident vibration be $f = 0$, $g = 0$, $h = 2\mathrm{A}\epsilon^{-2\iota\pi t/\tau}$; we shall find it convenient to resolve this into the two circularly polarized waves,

* 'HALL, 'Phil. Mag.,' series 5, vol. 12, p. 157; VON ETTINGHAUSEN and NERNST, 'Amer. Journ. Science,' series 3, vol. 34, p. 151.

† 'Phil. Trans.,' 1879, p. 55.

‡ 'Phil. Mag.,' series 5, vol. 18, p. 326.

$$h = \mathrm{AS}, \qquad g = \iota \mathrm{AS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$h = \mathrm{AS}, \qquad g = -\iota \mathrm{AS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51),$$

where $\mathrm{S} = \epsilon^{-2\iota\pi t/\tau}$.

Since we are dealing with glass, we may put $k = 1$; consequently for normal incidence we obtain from (30), (31), and (50)

$$\mathrm{A}' = \mathrm{A}(\alpha - q\beta), \qquad \mathrm{B}' = -\iota \mathrm{A}',$$

where

$$\alpha = \frac{\mathrm{U} - \mathrm{V}}{\mathrm{U} + \mathrm{V}}, \qquad \beta = \frac{2\mathrm{V}}{\mathrm{U}(\mathrm{U} + \mathrm{V})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

If, however, the incident wave is polarized in the opposite direction, and is therefore represented by (51), we shall obtain

$$\mathrm{A}' = \mathrm{A}(\alpha + q\beta), \qquad \mathrm{B}' = \iota \mathrm{A}',$$

in which the sign of q is reversed.

In order to calculate the intensity of the refracted light let us first confine our attention to the incident wave given by (50). Then if we put $i = \mathrm{R} = 0$ in (27), we shall obtain $\mathrm{A}_2 = 0$,

$$\mathrm{A}_1 = \frac{2\mathrm{AV}^2}{\mathrm{U}} \left\{ \frac{1}{\mathrm{U} + \mathrm{V}} + \frac{q(2\mathrm{U} + \mathrm{V})}{\mathrm{U}^2(\mathrm{U} + \mathrm{V})^2} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

If, however, we considered the other wave (51), we should obtain $\mathrm{A}_1 = 0$,

$$\mathrm{A}_2 = \frac{2\mathrm{AV}^2}{\mathrm{U}} \left\{ \frac{1}{\mathrm{U} + \mathrm{V}} - \frac{q(2\mathrm{U} + \mathrm{V})}{\mathrm{U}^2(\mathrm{U} + \mathrm{V})^2} \right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

in which the sign of q is again reversed.

In order to calculate the intensities, when light propagated in glass is reflected at the surface in contact with air, we may use Stokes's principle of reversion,* and apply it separately to each of the two circularly polarized waves, into which the incident wave may be conceived to be resolved. Let Ab, Ac be the amplitudes of the reflected and refracted waves, when the wave (50) passes from air into glass; and let Ae, Af be the amplitudes when the wave passes from glass into air, then

$$b + e = 0, \quad b^2 + cf = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

Also if we denote by accented letters the corresponding quantities for the other

* 'Math. and Phys. Papers,' vol. 2, p. 91.

wave (51), the values of b', c', e', f' will be obtained from those of b, c, e, f by writing $-q$ for q.

By (52) and (53), the values of b, c, e, f are

$$\left.\begin{aligned} b &= \frac{U-V}{U+V} - \frac{2qV}{U(U+V)^2} = -e \\ c &= \frac{2V^2}{U}\left\{\frac{1}{U+V} + \frac{q(2U+V)}{U^2(U+V)^2}\right\} \\ f &= \frac{2U^2}{V}\left\{\frac{1}{U+V} - \frac{q(2V+U)}{U^2(U+V)^2}\right\} \end{aligned}\right\} \quad \ldots\ldots (56),$$

which can also be verified by independent calculation.

If r be the thickness of the plate, the emergent light is

$$h = A\left(cef\epsilon^{4\iota\pi r/V_1} + c'e'f'\epsilon^{4\iota\pi r/V_2}\right)\epsilon^{-2\iota\pi t/\tau},$$

$$g = \iota A\left(cef\epsilon^{4\iota\pi r/V_1} - c'e'f'\epsilon^{4\iota\pi r/V_2}\right)\epsilon^{-2\iota\pi t/\tau},$$

whence, if

$$\lambda = \alpha(1-\alpha^2), \quad \mu = \beta(3\alpha^2 - 1),$$

the real parts become

$$-h = A(\lambda + q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right),$$

$$-g = -A(\lambda + q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right).$$

Let

$$\phi = \frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right), \quad \eta = \frac{2\pi r}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) \quad \ldots\ldots (57);$$

then

$$-h = 2A\lambda\cos(\phi+\eta)\cos\eta - 2Aq\mu\sin(\phi+\eta)\sin\eta,$$

$$-g = -2A\lambda\cos(\phi+\eta)\sin\eta - 2Aq\mu\sin(\phi+\eta)\cos\eta.$$

Let the analyser be placed in the position of extinction, and be then turned through a small angle ϵ, which will be considered positive when the analyser is turned towards the *right hand* of an observer who is looking through it; then the vibration on emerging from the analyser is

$$2A\lambda\cos(\phi+\eta)\sin(\eta-\epsilon) + 2Aq\mu\sin(\phi+\eta)\cos(\eta-\epsilon)$$

Since q^2 is to be neglected, the intensity of the emergent light is

$$4A^2\lambda^2(\sin\eta - \epsilon\cos\eta)^2 \quad \ldots\ldots\ldots (58),$$

and will therefore be approximately zero when

$$\epsilon = \tan \eta.$$

Now from (16),

$$V_1^2 = U^2 - 2q$$
$$V_2^2 = U^2 + 2q$$
$$q = C\alpha/4\tau,$$

where C is HALL'S constant, and α is the external magnetic force; whence the last of (57) becomes

$$\eta = \frac{\pi r C \alpha}{U}.$$

Since C is negative for glass, it follows that η, and therefore ϵ, is negative; whence the plane of polarization is rotated in the *same* direction as the amperean currents, which agrees with KUNDT'S experimental result.

It also follows from (58) that the effect depends upon the direction of the current; for if the current is flowing in the same direction as that in which the analyser is rotated, its effect is to weaken the light, but if it flows in the opposite direction its effect is to strengthen the light.

Hence the effect produced by the glass plate is the *opposite* of that which is produced by a metallic magnetic pole.]

XI. *On the Causes which Produce the Phenomena of New Stars.*

By J. Norman Lockyer, *F.R.S.*

Received November 28, 1890,—Read April 16, 1891.

Contents.

	PAGE.
I. Introduction	398
II. Early views of the origin of Novæ	399
The Nova of 1572	399
The Nova of 1604	400
Tycho Brahe's hypothesis	401
Support of the hypothesis by Kepler	401
Newton's view	402
III. Modern views	402
Zöllner, 1865	402
Huggins and Miller, 1866	403
Johnstone-Stoney, 1868	403
Vogel, 1877	404
Lohse, 1877	404
Lockyer, 1877	405
Monck, 1885	405
IV. Meteoritic theory of the origin of Novæ	405
General statement	405
Origin of the mixed spectra of Novæ	407
Relation between Novæ and variables of Group II.	407
V. Discussion of the spectra of Novæ	408
Nova Coronæ, 1866	408
Nova Cygni, 1876-77	414
Nova Andromedæ, 1885	424
VI. Sequence of phenomena in the spectra of Novæ	434
Sequence of spectra	434
Variations in magnitude	437
VII. Colour phenomena	440
Changes of colour in condensing swarms	440
Mixed colours in Novæ	440
The Nova of 173 A.D.	441
Nova Cassiopeiæ, 1572	441
Nova Coronæ	442
Nova Cygni	443
Nova Andromedæ	443
General conclusions with regard to colour	444

10.12.91.

PAGE.
VIII. Carbon radiation in Novæ 445
IX. General conclusions 447

List of Diagrams.

Fig. 1. Spectra of Nova Coronæ 410
Fig. 2. Integration of the spectra of Coggia's comet and planetary nebula, compared with spectrum of Nova Coronæ 413
Fig. 3. Spectra of Nova Cygni 416
Fig. 4. Vogel's observations of Nova Cygni 418
Fig. 5. Integration of the spectra of comets and nebula, compared with that of Nova Cygni 423
Fig. 6. Integration of the spectra of nebula 4373 and γ Argûs, compared with that of Nova Cygni 423
Fig. 7. Integration of the spectra of a planetary nebula and Lalande 13412, compared with that of Nova Cygni 423
Fig. 8. Integration of the spectra of the nebula in Andromeda and of hydrogen, compared with that of Nova Andromedæ 433
Fig. 9. Sequence of spectra in Novæ 435
Fig. 10. Light curves of Novæ 439

I. Introduction.

In communications to the Society during the last three years I have produced evidence to show that many so-called stars, instead of being bodies like the Sun, are composed of swarms of meteorites. I have also discussed the spectroscopic evidence which suggests that such stars are increasing their temperature, and, taking a normal case of an undisturbed swarm, I have shown, by means of a temperature curve, the spectra produced by the same mass of meteorites in its evolution from a nebula to a condensed and nearly cold body. In considering this question the appearance of so-called "New Stars" was referred to, and it was suggested that such appearances might be due to the collision of meteor swarms or streams in space, an idea which I first put forward in 1877 with regard to Nova Cygni.

It became obvious then that a complete discussion of these phenomena would afford a valuable test of the general hypothesis, for the reason that such bodies, instead of going forward along the temperature curve, should go back as they cooled and became invisible. All the observations, therefore, have been brought together, and have been discussed from this point of view. These discussions form the subject matter of the present communication. The investigation has particular reference to the sequence of the spectra of Novæ, from their appearance to their diminution to invisibility. In the absence of spectroscopic observations the changes of colour that have been recorded afford a means of arriving at some idea of the physical constitution of the body observed.

For the purposes of this paper I define a Nova as a body which suddenly appears then diminishes its brightness and finally disappears *as a star*.

II. Early Views of the Origin of Novæ.

The first published views relating to the possible causes of the phenomena presented by stars suddenly coming into visibility we owe to Tycho Brahe and Kepler.

These were promulgated with reference to, and in explanation of, the new stars which appeared in 1572 and 1604.

It will be convenient to begin our consideration of the views in question by giving a condensed account of the observations which had suggested them.

The Nova of 1572 *observed by* Tycho Brahe.

This, the first Nova of which anything like a complete record exists, appeared in Cassiopeia, in 1572. The accompanying phenomena were minutely described by Tycho Brahe,* who first saw it on November 11, 1572. The Nova appeared destitute of nebulous surroundings, and only differed from other stars in the vivacity of its scintillations. When it was first observed, it appeared more brilliant than Sirius, α Lyræ, or Jupiter, and even rivalled the splendour of Venus at greatest brillancy, being, like Venus, visible in the daytime. At the beginning of December a diminution in brightness was noticed. This regularly continued until, in March, 1574, the Nova had disappeared.

The gradual diminution of the star's luminosity may be seen from the following photometric observations made by Tycho Brahe :—

November, 1572. Brighter than Sirius, α Lyræ, or Jupiter, and only comparable to Venus when most brilliant.
December, 1572. As bright as Jupiter.
January, 1873. Less bright than Jupiter.
February and March, 1573. Equal to a first magnitude star.
April and May, 1573. Equal to a second magnitude star.
July and August, 1573. Equal to a third magnitude star.
October and November, 1573. Equal to a fourth magnitude star
End of November, 1573. Equal to a fifth magnitude star.
Between December, 1573, and February, 1574. Equal to a sixth magnitude star.
March, 1574. Disappeared.

Curves have been plotted to show the decrease in magnitude of this and three other Novæ. These will be found on p. 439 of this paper.

Changes of colour accompanied the changes of brightness. When the star first became conspicuously visible it was white, like Venus and Jupiter. It then acquired a yellow colour which merged into red. In the first months of 1573, Tycho Brahe compared it to Mars and α Orionis, and considered it to be much like Aldebaran.

* Tycho Brahe, 'Progymnasmata,' lib. 2, p. 300.

Later on in the same year, and especially towards May, a leaden hue was observed, like that exhibited by the planet Saturn. This continued until January, 1574, when the colour appeared to become less clear and of a less pure whiteness as the star slowly disappeared.

The Nova of 1604 *observed by* Kepler.

The famous Nova which appeared in 1604 is associated with the name of Kepler, as that of 1572 is with Tycho Brahe. It was first observed on October 10, by Bronowski, a pupil of Kepler's. The latter astronomer saw the Nova on October 17, and wrote a circumstantial account of his subsequent observations.

The following is a tabulated account of the estimations of its magnitude during the fifteen months that the star was visible:—

1604, October. Brighter than first magnitude stars, and also Saturn, Mars, and Jupiter.
1604, November 6. Could be seen at twilight, although Jupiter was not visible.
1604–5, December 24 to January 3. Greater that Antares, but less than Arcturus.
1605, March 20. Less than Saturn, but greater than third magnitude stars.
1605, April 21. Equal to a third magnitude star.
1605, August 12 and 14. Equal to a fourth magnitude star.
1605, September 13. Less than a fourth magnitude star.
1606, March. Disappeared.

Kepler does not record a sequence of changes in colour like that undergone by the Nova of 1572, although he alludes to the yellow, saffron, purple, and red tints of the star when it was near the horizon. It will be seen from the following extract* that he refers to the scintillations as being like the colours reflected by the faces of a diamond, but this was also the case with Tycho Brahe's Nova, and is distinct from the gradual colour changes of the latter star.

"Mini, qui potissimum respiciebam ad colorum Iridis varietatem, quos pro scintillationis alio atque alio habitu evomuit, placuit uti exemplo Adamantis multanguli, qui Solis radios inter convertendum, ad spectantium oculos, variabili fulgore revibraret: simul ad causam scintillationis, in Opticis meis allatam, alludere volui.

"De coloribus Mæstlinus planè mecum, eos ad momenta singula variari, ex flava mox croceam, è vestigio purpuream et rufam, *ut plurimum candidam videri, ubi ex vaporibus paulo altius elevetur.*"

Although many other Novæ have been observed, none have matched the splendour of those of 1572 and 1604, and of none have such circumstantial accounts been written. I now proceed to a statement of the explanation of the phenomena put forward by the respective observers.

* 'De Stella Nova in Pede Serpentarii,' Prague, 1606, p. 5.

Tycho Brahe's *Hypothesis.*

Tycho Brahe considered that new stars were formed from the cosmical vapour, which was supposed to have reached a certain degree of condensation in the Milky Way. In the words of this astronomer :—*

"Cœli materia ut subtillissima nostroque visui et planetarum circuitibus pervia, in unum tamen globum condensata compactaque, et lumine, si non proprio saltem solari, illustrata, hanc stellam effingere potuit. Quæ quoniam citra communem naturæ ordinem quasi monstrosa extitit, parem cum cæteris perseverantiam obtinere nequibat; veluti neque novæ ex elementis constantes generationes et monstra diu durant.

"Et quamvis in tota cælestis mundi vastitate materia pro conformatione alicuius stellæ ascititiæ, meo judicio abundè suppetat; tamen nusquam copiosius et pleniùs, quam juxta Viam Lacteam, quam substantiam quandam cælestem à materia reliquarum stellarum non discrepantem, sed diffusam certisque locis expansam, non in unum corpus discretim, prout in stellis fit, conglobatam esse statuo : hincque factum judico quod nova hæc in ipso Galaxiæ margine constiterit."

It will be seen from the above extract from Tycho Brahe's work that the fact that the Nova appeared on the edge of the galaxy was used to give weight to the hypothesis of stellar formation advanced by him. Indeed, some observers imagined they could see the *hiatus* or opening out of which the Nova came. The disappearance of the star was supposed to be due either to some action in itself or to its dissipation by the light of the Sun and stars.

When Tycho Brahe advanced the above theory, the tails of comets were looked upon as similar in constitution to the Milky Way.

Support of the Hypothesis by Kepler.

Kepler agreed with Tycho in considering that new stars were created from the ethereal substance of which the Milky Way was composed. The circumstance that Mira Ceti, which was looked upon as a Nova, appeared in a part of the heavens distant from the Milky Way, was explained by saying that the nebulous material was not exclusively confined to the galaxy, as supposed by Tycho Brahe, but pervaded all space.†

A fact deemed of considerable importance was that both Tycho Brahe's and Kepler's Novæ became suddenly and strikingly visible, and did not appear gradually to increase in brightness. Indeed, it was thought that all new stars must exhibit the maximum of brilliancy at their first appearance, and Kepler went so far as to use the statement made by Antonius Laurentinus Politianus, that he had seen the Nova

* 'Progymnasmata,' p. 794.

† 'De Stella Nova in Pede Serpentarii,' Kepler, pp. 110–112.

of 1604 increase in brightness, as an argument against his having seen the star at all. The following are the words used by LAURENTINUS :—

"Apparuit parva, et postea de die in diem crescendo apparuit magnitudine, et lumine non multò inferior Venere, superior Jove."*

NEWTON'S *View*.

The first Nova that attained any brilliancy, after that of 1604, appeared near β Cygni in June 1669, and was observed by ANTHELM.†

This Nova fluctuated in brightness between the 3rd and 5th magnitudes, and finally disappeared altogether. It is most probable that observations of this star drew NEWTON'S attention to the subject, and led him to the idea that "Novæ" were produced by the appulse of comets, thus propounded in 1686 in the 'Principia.'‡

"Sic etiam stellæ fixæ, quæ paulatim expirant in lucem et vapores, cometis in ipsas incidentibus refici possunt, et novo alimento accensæ pro stellis novis haberi. Hujus generis sunt stellæ fixæ quæ subito apparent, et sub initio quam maxime splendent, et subinde paulatim evanescunt. Talis fuit stella in cathedra Cassiopeiæ quam Cornelius Gemma octavo Novembris, 1572, lustrando illam cœli partem nocte serena minime vidit; at nocte proxima (Novem. 9) vidit fixis omnibus splendidiorem et luce sua vix cedentem Veneri. Hanc TYCHO BRAHÆUS vidit undecimo ejusdem mensis ubi maximè splenduit; et ex eo tempore paulatim decrescentem et spatio mensium sexdecim evanescentem observavit."

III. MODERN VIEWS.

In dealing with the period between NEWTON'S time and the present, I propose to give, as shortly as possible, some of the most important views expressed, during the last quarter of a century, by observers of the phenomena we are now considering.

ZÖLLNER, 1865.

According to the hypothesis advanced by ZÖLLNER,§ all stars, at a certain period of their formation, become covered with a cold, non-luminous crust. If the glowing mass bursts forth, the chemical combinations which have formed on the surface, under the influence of a low temperature, are again decomposed with a resulting development of considerable heat and light. Hence the great brilliancy of a new star must not be ascribed merely to the bursting forth of a glowing mass, but also to the combustion of the substances which form the shell.

* 'De Stella Nova in Pede Serpentarii,' KEPLER, p. 3.

† 'Phil. Trans.,' vol. 5, p. 2028.

‡ P. 525, Glasgow edition, 1871.

§ 'Photometrische Untersuchungen,' 1865, p. 251.

Huggins *and* Miller, 1866.

Drs. Huggins and Miller's observations of the Nova that appeared in Corona Borealis in 1866 led them to the following general conclusions :*

"It is difficult to imagine the present physical condition of this remarkable object. There must be a photosphere of matter in the solid or liquid state emitting light of all refrangibilities. Surrounding this must exist also an atmosphere of cooler vapours, which give rise, by absorption, to the groups of dark lines.

"Besides this constitution, which it possesses in common with the sun and the stars, there must exist the source of the gaseous spectrum. That this is not produced by the faint nebulosity seen about the star is evident by the brightness of the lines, and the circumstance that they do not extend in the instrument beyond the boundaries of the continuous spectrum. The gaseous mass from which this light emanates must be at a much higher temperature than the photosphere of the star, otherwise it would appear impossible to explain the great brilliancy of the lines compared with the corresponding parts of the continuous spectrum of the photosphere. The positions of two of the bright lines suggest that this gas may consist chiefly of hydrogen. . . . . The character of the spectrum of this star, taken together with its sudden outburst in brilliancy and its rapid decline in brightness, suggests to us the rather bold speculation that, in consequence of some vast convulsion taking place in this object, large quantities of gas have been evolved from it, that the hydrogen present is burning by combination with some other element and furnishes the light represented by the bright lines, also that the flaming gas has heated to vivid incandescence the solid matter of the photosphere. As the hydrogen becomes exhausted all the phenomena diminish in intensity and the star rapidly wanes."

Johnstone-Stoney, 1868.

In a paper on the physical constitution of the sun and stars, Mr. Johnstone-Stoney, in 1868, discussed the origin of double stars. He then remarked† that the astonishing phenomena witnessed in T Coronæ were precisely what should arise towards the end of a process he describes. This process may be understood from the following quotation :—

"The stars having been intensely heated by previous perihelion passages, and having begun to shrink, would, at ordinary times, present a spectrum subdued by an abundance of very dark lines; but immediately after one of the last occasions upon which their atmospheres brush against one another, the outer constituent of their atmospheres [hydrogen] and the outer constituent alone, would be raised by the friction to brilliant incandescence, which would reveal itself by the temporary sub-

* 'Roy. Soc. Proc.,' vol. 15, p. 148.

† 'Roy. Soc. Proc.,' vol. 17, p. 53.

stitution of four intensely bright for four dark dydrogen lines in a spectrum which everywhere else continues to be filled with dark lines. And, moreover, these dark lines would for a while be rendered faint by the fierce heat radiated upon the outer parts of the atmosphere of each star by its companion."

Vogel, 1877.

Observations of the new star in Cygnus (1876–77) led Professor Vogel to make the following remarks :—*

"It is generally supposed that the bright lines in some star spectra are due to gases which burst out from the interior of the luminous body at a temperature higher than the surface, as similar lines are occasionally seen in the spectra of sun-spots, and are caused by the eruption of incandescent hydrogen from the hot interior over the cooler surface of the spot. But this is not the only explanation. It may be assumed that the outer shell of a star, consisting of glowing gases, as in the case of our sun, has a lower temperature than the nucleus, although it is at a relatively high temperature.

"On the first supposition, it is impossible for the appearance to last for any length of time. The gas bursting forth from the heated interior would communicate part of its heat to the cooler surface, thus raising the temperature of the latter until the difference of temperature between the incandescent gas and the surface is insufficient to produce the bright lines, which consequently disappear.

"This theory satisfactorily explains the apparition of so-called *new* stars, having bright lines in their spectra, and which suddenly appear and speedily disappear altogether or lose most of their brilliancy, if Zöllner's hypothesis be accepted as a further explanation."

It will be seen that Vogel advanced no new hypothesis, but supported that of Zöllner by spectroscopic observations.

Lohse, 1877.

Dr. Lohse advanced a theory to account for "new stars," founded upon his observations of Nova Cygni.† His suggestions are summed up as follows :—

(1.) "The lighting up of new stars may probably be looked upon as the result of the innate affinity of chemical matter.

(2.) "The conditions and manner of illumination may be regarded as follows :—By the progressive cooling of the mass of a luminous body (fixed star) which consists of heated vapours and gases, an atmospheric envelope is erected which absorbs the light so much, that the star cannot be seen at all, or only very faintly, from the earth. As this body continues to give out heat, at length the degree of coolness is reached which

* 'Berlin Akad. Monatsberichte,' 1877, p. 256.

† 'Berlin Akad. Monatsberichte,' 1877, p. 826.

is necessary for the formation of chemical combinations. The greater portion of the body is composed of elements which then combine, producing, by their combination, heat and light, and thus making the star visible to a great distance, and for a long or short space of time."

Lockyer, 1877.

In this year, discussing the phenomena of Nova Cygni, I advanced the view that meteoritic collisions were, in all probability, the cause of them. This will be discussed in a subsequent part of this communication.

Monck, 1885.

Mr. W. H. S. Monck suggested,* in relation to the new star in Andromeda, that, "as shooting-stars are known to be dark bodies rendered luminous for a short time by rushing through our atmosphere, new stars are dark (or faintly luminous) bodies which acquire a short-lived brilliancy by rushing through some of the gaseous masses which exist in space."

IV. The Meteoritic Theory of the Origin of Novæ.

General Statement.

In a former communication to the Royal Society† I stated the conclusions I had arrived at as a result of spectroscopic work as follows: "New stars, whether seen in connection with nebulæ or not, are produced by the clash of meteor swarms, the bright lines seen being low temperature lines of elements, the spectra of which are most brilliant at a low stage of heat." This general theory, however, does not exclude other causes, such as the sudden illumination produced by collisions of two dark bodies, but I pointed out that no case of this kind has happened within human ken. This hypothesis was first, I think, advanced by myself in 1877, when I wrote as follows in connection with Nova Cygni ‡ "Of all the lines chronicled by Cornu and Vogel, only one remains, that, namely, which the latter observer showed to be constantly increasing in brightness, while all the rest were waning, and which, moreover, as Vogel also distinctly showed, is coincident in position in the spectrum with that observed in the majority of the nebulæ.

"It should have been perfectly clear to those who thought about such matters, that the word 'star' in such a case is a misnomer, from a scientific point of view, although no word would be better to describe it in its popular aspect. The word is a misnomer,

* 'Observatory,' October, 1885.

† 'Roy. Soc. Proc.,' vol. 43, p. 154.

‡ 'Nature,' vol. 16, p. 413.

for this reason. If any star, properly so called, were to become a 'world on fire,' were to 'burst into flames,' or, in less poetical language, were to be driven either into a condition of incandescence absolutely, or to have its incandescence increased, there can be little doubt that thousands or millions of years would be necessary for the reduction of its light to the original intensity.

"Mr. Croll has recently shown that if the incandescence observed came for instance from the collision of two stars, each of them half the mass of the sun, moving directly towards each other with a velocity of 476 miles per second, light and heat would be produced which would cover the present rate of the sun's radiation for a period of 50,000,000 years.

"A very different state of affairs this from that which must have taken place in any of the Novæ from the time of Tycho to our own, and the more extreme the difference, the less can we be having to deal with anything like a star properly so called.

"The very rapid reduction of light in the case of the new star in Cygnus was so striking that I at once wrote to Mr. Hind to ask if any change of place was observable, because it seemed obvious that if the body which thus put on so suddenly the chromospheric spectrum were single, *it might only weigh a few tons or even hundredweights*, and being so small might be very near us. We are driven, then, from the idea that these phenomena are produced by the incandescence of large masses of matter because, if they were so produced, the running down of brilliancy would be exceedingly slow.

"Let us consider the case, then, on the supposition of small masses of matter. Where are we to find them? The answer is easy: in those small meteoric masses which an ever-increasing mass of evidence tends to show, occupy all the realms of space.

"The Nova now exists as a nebula, so far as its spectrum goes, and the fact not only goes far to support the view I have suggested as against that of Zöllner, but it affords collateral evidence of the truth of Thomson and Tait's hypothesis of the true nature of nebulæ.

"The nebular hypothesis in its grandeur and simplicity remains untouched by these observations, the facts, so far from being in direct opposition to it, help us, I think, all the better to know exactly what a nebula is.

"There is another point of extreme interest to the spectroscopist, if we accept the bright line observed in the star by Dr. Copeland and others to be veritably the chief nebula line.

"It is clear from Dr. Vogel's diagram, that this line brightened relatively with each decrease in the brilliancy of the hydrogen lines. On December 8th, 1876, it was much fainter than F, while by March 2nd, 1877, F was a mere ghost by the side of it. On any probable supposition the temperature must have been higher at the former date."

The Origin of the Mixed Spectra of Novæ.

The discussion of the observations which have been made of the changes that take place in the spectra of new stars, has already shown that the sequence of phenomena is strikingly similar to that which occurs in cometary spectra after perihelion passage. In general, however, there will be a difference : namely, that in comets there is usually only one swarm to be considered, whereas in new stars, there are two, which may or may not be equally dense. In new stars, we have accordingly the integration of two spectra, and the spectrum we see will depend upon the densities and relative velocities of the two swarms. At one part of the mixed swarm the temperature must generally be considerably higher than at another, in consequence of the greater number of collisions occurring locally, and the temperature would be lowest where the outliers were engaged.

In new stars, therefore, it is possible for us to have the radiation spectra of vapours corresponding to sparse swarms of meteorites (nebulæ, bright-line "stars," and comets away from perihelion), and the mixed radiation of carbon, with the fluting absorption of metallic vapours, corresponding to bodies of Group II., and to comets nearer perihelion. That is to say, we may have the radiation lines of hydrogen and fluting of magnesium at 500, *plus* the radiation of carbon and fluting or line absorption of manganese, lead, iron, &c., a condition which cannot occur when only a single swarm is in question.

The mixed spectroscopic phenomena, which should be seen on the collision of two swarms, were noted in my paper of November, 1887,* as follows :—"We shall, in fact, have in one part the conditions represented in Class III*a*., and in the other, such a condition as we get in γ Cassiopeiæ."

In the present paper I give detailed discussions of all the Novæ which have been spectroscopically examined, and the results fully bear out these views.

The Relation between Novæ and Variables of Group II.

If the two swarms which produce a new star by collision are such that the mean distance between the meteorites in the resulting "mixed swarm" is about equal to that between the meteorites in a body of Group II., say α Orionis, mixed radiation of carbon and metallic fluting absorption will preponderate in the spectrum observed. At the same time, the sparser portions of the swarms give us hydrogen radiation. This was the state of things in Nova Coronæ, a detailed discussion of which is given later on. The general spectrum observed was one similar to that of α Orionis, but in addition, the presence of bright hydrogen lines was noted. This is a condition which cannot occur at any stage in the condensation of a single swarm, because a swarm dense enough to give the α Orionis type of spectrum would be too dense to give hydrogen radiation. In a dense swarm absorption preponderates, whilst in sparse

* 'Roy. Soc. Proc.,' vol. 43, p. 147.

swarms radiation preponderates, the interspaces being flooded usually with incandescent vapours of magnesium and hydrogen.

In variables of the Mira type, we have almost a reproduction of the conditions of Nova Coronæ. As I suggested in the Bakerian Lecture, the variability in this class of stars is in all probability due to a swarm of meteorites revolving in a more or less elliptical orbit around a central swarm, the maximum luminosity occurring at periastron passage. At maximum, therefore, in such a variable, the luminosity proceeds from a mixed swarm, exactly in the same way as in a new star. At the maxima of Mira and other long-period variables, bright hydrogen lines have been observed, although the spectra are of the Group II. type.

In the detailed discussions which follow, Nova Orionis has been omitted, as it has turned out to be nothing more than a long-period variable of Group II.

V. Discussion of the Spectra of Novæ.

Nova Coronæ, 1866.

The spectrum of this star was observed on several dates by Messrs. Huggins and Miller, Wolf and Rayet, and Stone and Carpenter. The observations of Huggins and Miller were made on May 16, 17, 18, 19, 21, and 23; those of Wolf and Rayet on May 20; and those of Stone and Carpenter on May 19, 20, 23, 24, 28, and June 7. There is a general agreement as to the positions of the lines determined by the different observers, and the principal difference observed as the star faded was simply a diminution of the number of lines.

In their first observation on May 16, Messrs. Huggins and Miller* noted four bright lines and suspected a fifth. Two of these were found by direct comparison to be the C and F lines of hydrogen, but the positions of the other two were only roughly plotted on a diagram. There are not sufficient data for accurate reduction of the wave-lengths of these lines, but by a curve they have been found to be at approximately 468 and 473. Of the more refrangible of the two it is remarked: "The appearance of this line suggested that it was either double or undefined at the edges." The fifth line suspected was probably not far from G of the solar spectrum, and the presence of C and F make it highly probable that this was the hydrogen line at G.

In addition to these bright lines, many absorption lines and bands were observed. These absorptions were evidently very similar to those seen in stars like α Orionis. This was noted by Messrs. Huggins and Miller, and I have since compared those in the blue end of the spectrum with the lines photographed in α Orionis by Professor Pickering, who kindly furnished me with the data necessary for the reduction to wave-lengths. This shows a very close relation between the two spectra. The

* 'Roy. Soc. Proc.,' vol. 15, p. 147.

spectrum was again observed by Messrs. Huggins and Miller on May 17 and 19, but no remarkable difference was noted.

On May 19, the spectrum was also observed by Messrs. Stone and Carpenter.* Four bright lines were seen, but only three of them were fixed by measurements. The position of the fourth was approximately determined by estimation. One of the lines was undoubtedly F, and the wave-lengths of the others, as deduced from a curve, I find to be about—

501	Position of line estimated.
471	,, ,,
467	,, ,,

The data for reduction, however, are quite insufficient to give anything like accurate results. The C line was not recorded.

In the account of his observations of Nova Cygni, Vogel refers to the observations† of Stone and Carpenter; he reduced the positions of the lines to wave-lengths with the following results:—

502
467
463

He points out, however, that on account of insufficient data, these cannot be regarded as accurate.

The line 500 was seen only by Messrs. Stone and Carpenter. The other two lines are, in all probability, identical with those observed by Messrs. Huggins and Miller, for which I have determined the wave-lengths 468 and 473. Traces of absorption lines were also observed, thus confirming the observation of Messrs. Huggins and Miller. Another observation was made by Messrs. Stone and Carpenter on May 20, but to them the spectrum did not differ from that of the previous day.

MM. Wolf and Rayet,‡ however, made an observation of the spectrum of this star on the same day. They record: "The light of the new star gives a complete, very pale spectrum, on which are seen a certain number of brilliant bands. The brightest and largest of these bands appeared in a continued manner almost at the edge of the yellow and the green. It is preceded on the yellow side by a rather dark space, then by a bright, but weak line. A third line, which seems to correspond to D, is seen in the yellow, pretty bright, and towards the orange. Then, after the brightest line towards the violet end, the green is seen well marked, then a darker space a little darker than we spoke of before, and another line as bright as the principal band.

* 'R. Astron. Soc. Monthly Notices,' vol. 26, p. 295.

† 'Berlin Akad. Monatsber.,' 1877, p. 243.

‡ 'Comptes Rendus,' vol. 62, p. 1108.

The rest of the spectrum is pale, with ill-defined edges, and with nothing marked about it that we could distinguish."

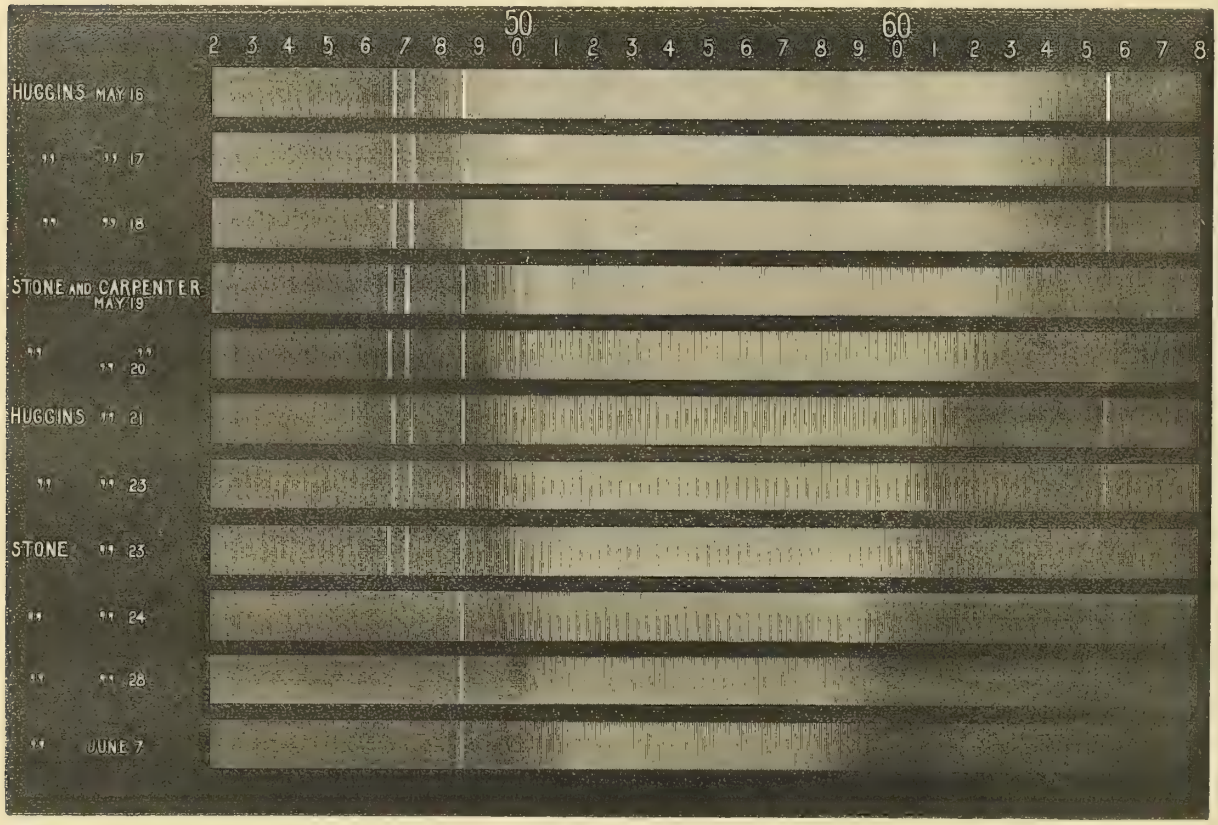
It is greatly to be regretted that the positions of the lines were not measured. Without these it is quite impossible to connect them with the other observations. It is remarkable that the line in the yellow, which may have been D or D_3, was not recorded by any of the other observers.

Messrs. Huggins and Miller again observed the spectrum on May 21 and 23. On these dates the bright lines appeared more brilliant than on former occasions, owing to the dimming of the continuous spectrum.

On May 23 Mr. Stone again saw the four lines which he had observed previously, but on May 24 and 28 and June 7, he was only able to see the F line. It was the opinion of both Mr. Stone and Mr. Carpenter that the bright lines and the continuous spectrum faded at the same rate.

All these observations, with the exception of the one by Wolf and Rayet, are shown on the accompanying map.

Fig. 1.



Spectra of Nova Coronæ.

The observations are also shown in another way in the following table :—

Date.	Observer.						Remarks.
May 16, 17, and 19	Huggins and Miller	Hγ suspected	468, 473	486	..	Hα (656)	468, 473 deduced from the curve, the former being "either double or undefined at the edges." Many absorption lines seen
May 19 and 20 . .	Stone and Carpenter	..	467, 471	486	501	..	467, 471, 501, determined by curve. Absorption lines also seen
May 20	Wolf and Rayet . .	..	..	..	..	..	Bright lines seen but not measured
May 21 and 23 . .	Huggins and Miller	Hγ suspected	468, 473	486	..	Hα (656)	The lines appeared more brilliant than before
May 23	Stone	..	467, 471	486	..	..	
May 24, 28, and June 7	Stone	..	..	486	..	..	The F line was "desperately faint"
Suggested origins		H (434)	Carbon (468–474)	H (486)	Mg (500)	H (656)	

It will be seen that the F line was constantly recorded, whilst C was only observed by Messrs. Huggins and Miller. The two lines in the blue, near 468 and 473, were constantly seen up to May 23, when the star was of magnitude 7·2. The line near 501 was only seen by Messrs. Stone and Carpenter; it was not recorded after May 23. The *dark* lines are left out in the map.

There can be no question as to the origin of two of the lines—that in the red and that in the blue-green. They were undoubtedly due to hydrogen. The line near 501 was, in all probability, the same as that which was seen to brighten in Nova Cygni as the star faded. This was also suggested to Vogel in referring to his observations of Nova Cygni. There is other evidence to show that this was identical with the chief line in the spectrum of the nebulæ.

The two lines in the blue present a little more difficulty. The more refrangible one near 468 may have been due to the carbon fluting, which, under certain conditions, has its maximum intensity about 468,* especially as Dr. Huggins stated that its "appearance suggested that it was either double or undefined." The one near 473 was probably the less refrangible edge of the same compound fluting of carbon.

In some of the observations of comets (*e.g.*, Comet III., 1881, on June 28) the blue band has been recorded with two maxima, one at 468 and the other at 474. This strengthens the view that the two lines seen in Nova Coronæ may have had the same origin. In comets, the more refrangible one is the least defined, and this was also the case in the Nova. The two maxima were also seen in Nova Cygni (see p. 415), but in that case the less refrangible one was only visible for a short time. I have very little doubt that both in comets and Novæ the two maxima were due to carbon. In the laboratory experiments in connection with this point photographs have been obtained, which show the compound group with a bright maximum of intensity about 468, at the same time leaving the least refrangible maximum (474) fairly bright, and the two intermediate ones much weaker. It is not difficult to understand that under difficult conditions of observation such a group should be mapped as two lines, one corresponding to the maximum at 468 and the other to that at 474.

Whatever the origin of the two lines, the fact of their being common to comets and Novæ is very significant.

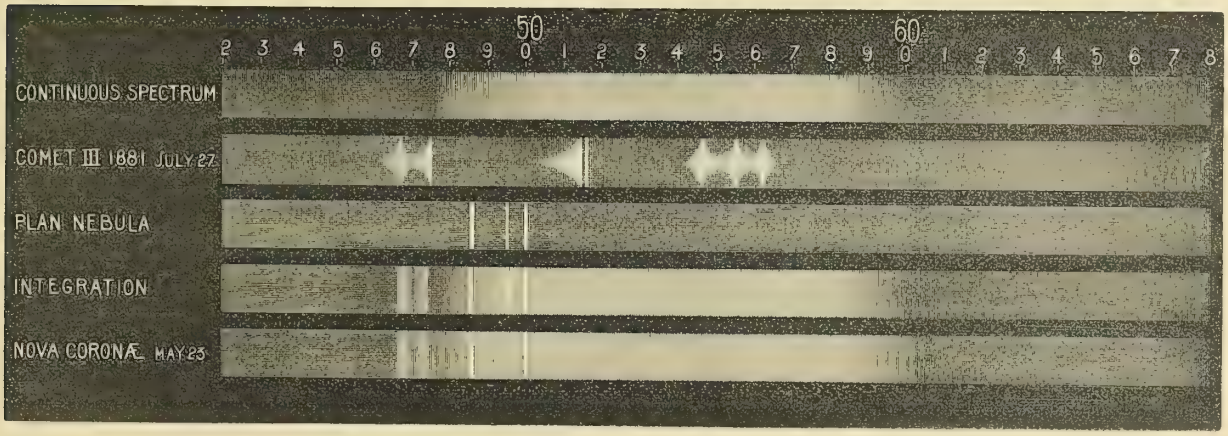
It is rather remarkable that in Nova Coronæ the F line should remain visible for a longer time than the line near 500, because in Nova Cygni it died away first, leaving 500 very bright to the end. It must be remembered, however, that the later observations by Mr. Stone were made under very difficult conditions, the F line itself being "desperately faint," so that the 500 line may very well have been overlooked. If the observations could have been continued longer, it is not impossible that the line 500 might have been seen after F. The 500 line would also be more liable to be masked by continuous spectrum than the F line, so that the presence of a brighter continuous

* 'Roy. Soc. Proc.,' vol. 45, p. 167.

spectrum than in Nova Cygni may explain why F should be seen last in one case and 500 in the other.

I have already pointed out that in Novæ we have not simply to deal with one state of condensation of a swarm. At least two swarms are concerned, and these are not necessarily of the same density. It therefore becomes interesting to attempt to reproduce the spectrum of a Nova by integrating the spectra of two or more swarms of meteorites, each of which is at a stage of condensation different from the others. Such an attempt is shown in fig. 2, where the spectra of Coggia's comet and the

Fig. 2.



Integration of the spectra of Coggia's comet and planetary nebula compared with spectrum of Nova Coronæ.

planetary nebula G.C. 4373 are integrated. The comet at this stage showed the carbon spectrum, the blue band having two maxima. In the resulting integration the green cometary bands are almost masked by continuous spectrum, whilst the blue one remains visible, exactly as it does in the bright-line stars.* The nebula line 495 was probably too faint to be detected. The line 500 was also faint, and this and 495 were probably partly masked by the continuous spectrum, which would be slightly brighter in that part of the spectrum than at F. The continuous spectrum would be brightened by the collision of two swarms, and hence it is necessary to add continuous spectrum in the integration.

The observed changes of magnitude of this Nova during its period of visibility were as follows :—

May	12	. . .	2
,,	13	. . .	2·3
,,	14	. . .	3
,,	15	. . .	3·4
,,	16	. . .	4·2

* 'Roy. Soc. Proc.,' vol. 44, p. 33.

May	17	4·7
,,	18	5
,,	19	5·3
,,	20	6·2
,,	21	6·3
,,	22	7·3
,,	23	7·5
..	24	8·4
..	27	8·5
,,	28	8·3
,,	29	8·5
June	7	8·9

Nova Cygni, 1876–77.

This star was discovered by Schmidt at Athens on November 24, 1876. Its magnitude was then 3·0, and its colour a reddish-yellow. On December 5 the magnitude was 5·9, and it steadily diminished in brightness, until on March 10 its magnitude was 8·3. In 1882 Dr. Copeland found that it had decreased to the 14th magnitude.

The spectrum of the star was observed on several occasions by Vogel, Lohse, Copeland, and Cornu. The earliest spectroscopic observations appear to have been made by M. Cornu, who recognised bright lines in the spectrum on December 2, but was unable to make any measurements. Two days after, a more complete examination of the spectrum was made, and on this occasion the positions of the lines were measured. Eight lines were bright enough to be recorded, but no dark lines were seen. He says,* "The dark lines, if they exist, must be very fine, and must have escaped me, on account of the very feeble light of the star." Vogel suggests that Cornu did not see the dark lines because he employed too much dispersion. The following are the wave-lengths of the bright lines which he observed, the Greek letters indicating the relative intensities, α being the brightest :—

α	δ	γ	β	ζ	η	θ	ϵ
661	588	531	517	500	483	451	435.

The lines α, δ, η, and ϵ were doubtless C, D, F, and G respectively, and M. Cornu points out that the remaining lines, with the exception of the one near λ 500, are nearly coincident with the lines which occur most frequently in the spectrum of the solar chromosphere. The line γ is regarded by M. Cornu as possibly coincident with the coronal line 5315·9. β he believed to correspond with the b lines of mag-

* 'Comptes Rendus,' vol. 83, p. 172.

nesium. He further states that "the feeble band θ corresponds also to a band, $\lambda = 447$, of the chromosphere; one is thus led to think that the line δ corresponds rather to the bright line of the chromosphere, $\lambda = 587$ (helium), than to that of sodium, 589. If this interpretation be accurate, the bright lines of the spectrum of the star comprehend inclusively the brightest and most frequent lines of the chromosphere."

In the diagram of the spectrum which accompanies M. CORNU's paper, there are shown three maxima of brightness which are not referred to in his description. These have been reduced by a curve, and their wave-lengths found to be approximately 546, 563, and 635. These agree very well with lines observed by VOGEL.

The observations of VOGEL and LOHSE were made independently with the Berlin refractor. LOHSE's chief object was "to study the total character of the star's spectrum and to note the changes undergone by it in the course of its appearance." He made no measurements, but simply gave diagrams, showing approximately the relative positions and appearance of the various lines and bands. VOGEL, on the other hand, gave particular attention to the determination of wave-lengths, but he also made drawings indicating the general appearance of the spectra on different dates.

LOHSE's observations commenced on December 5, and, referring to this observation, he says:— *

"Particularly striking was a broad bright line in the red, bordered towards the yellow by a broad dark band, which ran towards the yellow and broke off into lines showing the red ground between them and giving the band a brownish appearance. I saw, besides, several bright spots in the yellow, green, and blue, but, on account of their width, I regarded them, not as bright lines, but as spots free from lines in the continuous spectrum. A bright spot in the blue-green looked particularly striking, on account of its being bordered by a broad very dark band running towards the blue.

"The spectrum was traversed by numerous bands.

"The positions of the lines were based not on measurement but on estimation."

VOGEL's observation of the star on this date, December 5, is not mapped with the others given by him, but there appears to be little difference between his observation and that of Dr. LOHSE; he says:—†

"The spectrum was very bright, with numerous dark bands, the most striking of which were a very dark one in the green and a very broad one in the blue. . . .

"Besides the dark lines and bands several bright lines were seen in the spectrum, one of which, in the red, was specially striking on account of its brilliancy; another very bright line was on the border of the green and blue, and two were in the blue. In the yellow and green there were some bright lines or bands, but I could not ascertain whether they were really bright lines or only places in the spectrum which appeared bright by contrast with the dark absorption bands. . . .

"For the two parts in the blue the wave-lengths 474 and 470 were estimated, the

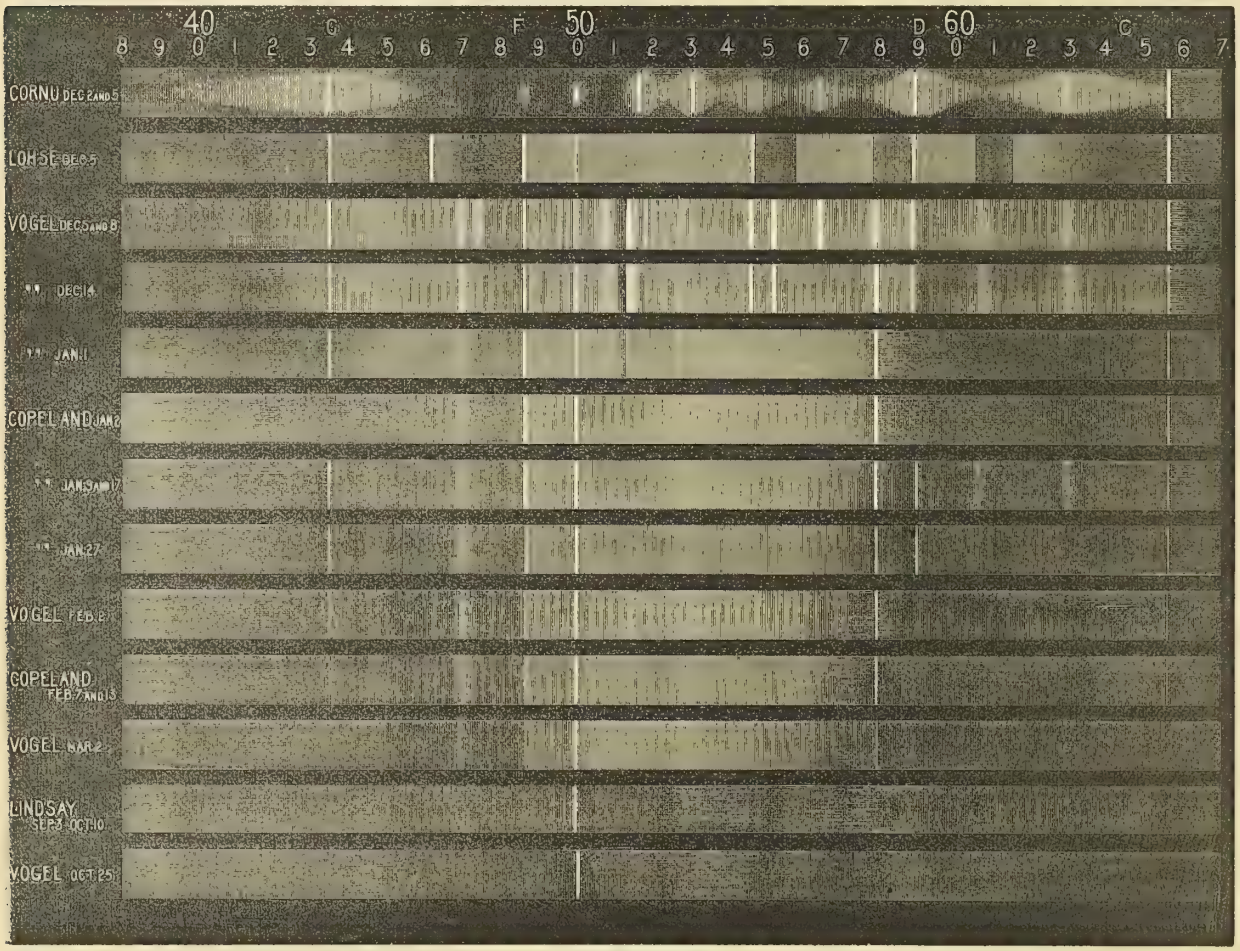
* 'Berlin Akad. Monatsber.,' 1877, p. 826.

† 'Berlin Akad. Monatsber.,' 1877, p. 243.

bright lines or bands found in the green were estimated at the wave-lengths 512 and 498."

Vogel's observation of December 5 appears to differ mainly from Lohse's in the blue end, Lohse only recording one line in place of the two observed by Vogel near λ 474 and 470. The subsequent confirmations of these lines by Vogel, however, can leave very little doubt as to their presence. Vogel's observation of this date was not quite completed on account of clouds.

Fig. 3.



Spectra of Nova Cygni.

Lohse's results are only represented by rough diagrams, and it is not possible to accurately reduce to wave-lengths the positions of the lines which he observed. By comparison with Vogel's observations, however, it is not difficult to determine the lines which he observed. There were certainly C, D, F, and G. The line in the green which brightened as the star faded was undoubtedly the line near λ 500, which is seen in the spectra of nebulæ. Another bright line between 500 and D is found on reduction by a curve to be near λ 546, and this agrees with one of the lines observed by Vogel.

Instead of the two lines 470 and 474, observed by Vogel, Lohse only records one,

and this is found to be near λ 462. In his later observations VOGEL only recorded one line in place of the two previously observed, at a mean wave-length of 467.

These observations are shown on the map, fig. 3, and it will be seen that there is really no great divergence. All three observers agree as to the chief lines.

The next observation was made by VOGEL on December 8. The hydrogen lines C, F, and G were again seen, and, in addition, other lines were measured. The bright line less refrangible than F was estimated to be near λ 498, and three other lines in the green near 527, 514, and 508.

Two lines in the yellow had their wave-lengths estimated at 588 and 570,* but these depended on one measurement only, and their positions are, therefore, somewhat uncertain. With regard to the line near 588, VOGEL says :—

"The first of these lines agrees very closely with D_3 (λ 587·5), and it would not be surprising if this line appeared bright in the spectrum with the hydrogen lines." The two lines in the blue, previously estimated at 470 and 474, were repeatedly seen, but not measured. A broad faint line was seen in the violet, and VOGEL suggests that this was the hydrogen line at G (λ 434). He also points out that although the line 514 agrees most nearly with the magnesium line *b* (wave-length for the middle, 517·5), the difference between the lines is such that the probability of the magnesium lines appearing bright in the star's spectrum is very small.

Further observations were made by VOGEL on December 14, 22, 26, 27; and January 1, 2, 6, 16, 17, and 18; February 2, 17; March 2 and 10. All these observations for which the description is complete, are shown in fig. 4, and it will be seen that the principal changes are :—(1) a general diminution in the number and brightness of the lines; and (2) the brightening of the line in the green near λ 500 as the other lines disappeared.

VOGEL thus summarises the lines which he observed, pointing out that the difficulty of the observations and the unfavourable weather prevented very exact measurements.

(1) The hydrogen lines, $H\alpha$, $H\beta$ } certainly.
$H\gamma$, most probably.

(2) A line of the wave-length 499 ± 1 max. This line comes into pretty close agreement with the brightest line of the nitrogen spectrum under ordinary pressure; it is the same line which is brightest in the spectrum of the nebulæ.

(3) A faint line at λ 580.

(4) A similar one at 467. (This agrees pretty well with a group of lines close together in the spectrum of air.)

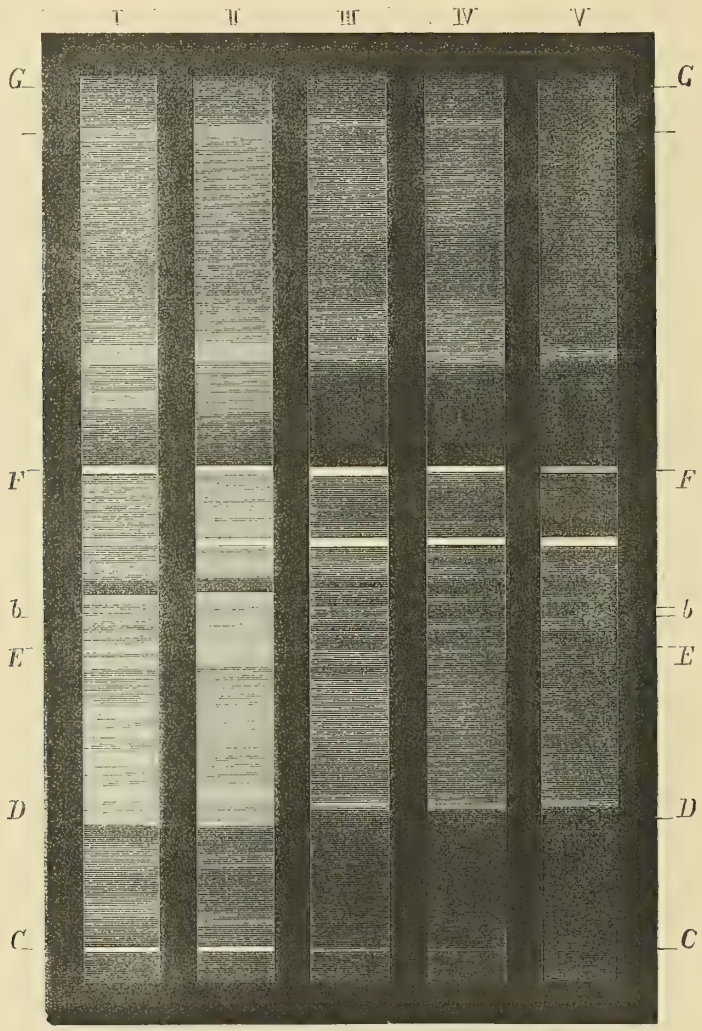
(5) There were also bright lines seen in the region of *b* and E, but nothing certain could be ascertained about their position. At the first observation on December 5, two lines were measured in the blue (474 and 470); these were again seen on

* This is evidently a misprint for 580. No line is shown at 570 in the map, but one at 580 is afterwards referred to.

December 8, but in later observations only the second was perceived as a faint band (λ 467).

For further details, the drawings must be referred to, for, as VOGEL states, they show several details which can hardly be put into words.

Fig. 4.



VOGEL'S observations of Nova Cygni.

VOGEL refers to CORNU'S observation, and states that he cannot agree that "the atmosphere of the star possesses exactly the same composition as that of the chromosphere of the sun." His chief objection to this view is that the line near 500, which is not in the chromosphere, was distinctly seen along with other bright lines in the star's spectrum, and eventually became the strongest line.

Dr. COPELAND* also made a large series of observations, but he was not able to commence before January 2, 1877. He continued his observations to February 16, and recommenced on September 2.

On January 2, five bright lines were recorded. The measurements show that two

* 'COPERNICUS,' vol. 2, p. 102.

of these lines were the C and F lines of hydrogen. Two series of measurements were made, and for the other three lines the following wave-lengths were determined :—

1st series.	2nd series.	
577·9	579·0	Bright band fading rapidly on both sides.
502·4	505·1	Bright well-defined lines.
463·4	466·8	Faint band.

There can be little doubt that these correspond to the lines measured by Vogel at 580, 499, and 470 (some measures 467).

On January 9, seven lines were measured, and it was further stated that "the space between the red line (C) and the next one (D) was certainly columnar, and probably contained two maxima." These were probably identical with those seen by Vogel, and they have therefore been inserted in the map (fig. 3).

The following summary of results is given by Dr. Copeland :—*

Number of line.	1	1*a*	2	3	4	5	7
Date.							
1877, Jan. 2	657·8	..	579·9	502·4	486·1	463·4	
„ „ 2	654·7	..	579·0	505·1	488·4	466·8	
„ „ 8	656·2	..	..	502·9	486·1	464·8	
„ „ 9	656·2	589·5	575·9	500·7	486·1	461·6	437·6
„ „ 27	..	..	..	500·8	486·1	461·5	
„ Feb. 13	681·5	..	574·9	502·3			
„ „ 13	670·8	..	574·9				
„ „ 13	..	..	574·0				
Adopted means	656·2	589·5	577·5	502·3	486·1	463·5	437·6
Mean error	..	..	± 1·1	± 0·8	..	± 1·0	

The star was not again observed until September 2, 1877 (the estimated magnitude being 10·5), when the startling fact was at once apparent that the spectrum was restricted practically to a single line. The measurements show that this was the line near 500. On September 6, it was observed that in addition to the chief line there were traces of one or two lines on the violet side of the chief line, but very close to it. On this date it was noted that the star did not give a sharp image, but that the "extreme diameter could not be above 2 seconds or 1½ seconds." On October 1 the star was examined with a view to detecting any very faint continuous spectrum, but nothing of the kind was visible. On October 10, Lord Lindsay noted that "since the last measures a decided change is to be seen, as the light is more spread out. The mean brightness is still as before, but is divisible into two lines very close together with a dark gap, and then another very faint line." Measurements of the three lines were made with the following results :—

* 'Copernicus,' vol. 2, p. 111.

499·5 ± 1·1
492·2 ± 1·3
491·8 ± 1·0

The latter two " were too feeble and too near together to be effectively measured separately." But regarding this, COPELAND remarks that " the faint extension of the spectrum of this star towards the violet measured on October 10, 1877, was seen with so much difficulty that it would be unwise to regard it as in any certain degree indicating the presence of other nebular lines."

On October 25, 1877, VOGEL succeeded in making another observation of the star's spectrum.* He says :—" The spectrum was almost monochromatic; it consisted of one bright line, on both sides of which a very faint continuous spectrum could be seen." This was measured at wave-length 499. This observation was again confirmed on February 18, 1878.

COPELAND remarked,† " Bearing in mind the history of this star from the time of its discovery by SCHMIDT, it would seem certain that we have an instance before us in which a star has changed into a planetary nebula of small angular diameter," and again,‡ " It seems, therefore, not unreasonable to assume that in the case of a planetary nebula, the whole light of which did not equal the light of a 10th magnitude star, the spectrum might be reduced to a single line, and that this line would then be the one of which the wave-length may be taken at 500·4."

Hence we see that on and after September 3, 1877, the only line seen with certainty was the line near 500, which had gradually increased in brightness since the first observation made on December 5, 1876. All the important spectroscopic observations of Nova Cygni are shown in fig. 3. It is evident that the hydrogen lines were seen by all the observers, and notwithstanding the slightly divergent measures, they have all been inserted in their proper places. This also applies to the line near 500. The yellow line has also been taken to be D. There is a little more difficulty with the lines in the blue, as CORNU and LOHSE differ from VOGEL and COPELAND. CORNU measured a blue line as 451; LOHSE's is approximately estimated at 462, while VOGEL measured a line at the mean place 467. COPELAND measured the line as 463·5, but he is evidently of opinion that VOGEL's wave-length is the more correct, on the ground that it agrees with one of the bright bands in the spectra of the bright-line stars in Cygnus. No accurate measures were made of the groups of lines in the green indicated in VOGEL's map, and the positions of these have been estimated by reference to those stated to be near 514, 527, and 508.

SECCHI also made some observations of the spectrum,§ and he was of opinion that

* 'Berlin Akad. Monatsber.,' 1878, p. 302.
† 'Astr. Nachr.,' 2158.
‡ 'COPERNICUS,' vol. 2, p. 113.
§ 'Astr. Nachr.,' 2116.

one of the bright lines coincided with hydrogen, one with magnesium (*b*), and a third with sodium (D). Referring to this, VOGEL says: "In this he is certainly mistaken, for on January 8 the lines in the neighbourhood of the magnesium group were quite faint, and at D there was not a single bright line to be seen. The bright lines which he noticed had the wave-lengths 500 and 580 mm., and were pretty far distant from the sodium and magnesium lines." It will be noticed that SECCHI'S observation agrees to a certain extent with that of CORNU.

Other observations were made by Mr. BACKHOUSE,* but these simply confirm the presence of bright lines.

With regard to the origins of the lines, we have first unquestionably the bright lines of hydrogen. These gradually dimmed as the star diminished in brightness. Again, there was a line near λ 500, apparently coincident with the chief line in the spectra of the nebulæ, which brightened as the other lines faded. I have given evidence in other papers† to show that this probably owes its origin to the magnesium fluting at the same wave-length, which is seen so very brightly when magnesium is burned in air.

The next in order of importance is, perhaps, the line or band in the blue between F and G, which VOGEL at first saw double (470 and 474), and afterwards single, with a mean wave-length of 467. COPELAND described it as a "faint band," and gave its mean position as 463·5. It is more than likely that this was the compound fluting of carbon in the blue, seen at first with two maxima, as in Nova Coronæ and Comet III., 1881, &c., and afterwards with a single maximum at 468. This band agrees in position with the bright band seen in the spectra of bright-line stars (*e.g.*, WOLF and RAYET'S stars in Cygnus, &c.).

VOGEL also refers to this relation. The two maxima first seen, however, were not recognised by VOGEL as being the same as those observed in Nova Coronæ, although he pointed out that the green line 500 was common to the two as well as the hydrogen lines. In Nova Coronæ there were simply the brightest lines which were afterwards seen in Nova Cygni.

Another important line is the one at 579, about which there is no disagreement between COPELAND and VOGEL. This is undoubtedly the same line which is seen in the bright line stars in Cygnus, as pointed out by COPELAND, and I have previously ascribed it to an iron line which is seen brightest in the flame spectrum.‡

One of the lines in the green is probably also an iron line, 527, the next in intensity to the line at 579. This died out in the Nova before the line 579, and this is what should happen, for in the laboratory 579 is seen without 527 in the coolest part of the flame. The three lines or bands near 564, 552, and 546, are quite

* 'Nature,' vol. 15, p. 295.

† 'Roy. Soc. Proc.,' vol. 43, p. 113, and vol. 48, p. 197.

‡ 'Roy. Soc. Proc.,' vol. 44, p. 33.

familiar in cometary spectra. The first two of them are probably the 1st and 3rd members of the citron carbon fluting, the intermediate one being masked, as in Coggia's comet, on June 13, 1874.* The one at 546 is the fluting due to lead, which has frequently been recorded in cometary spectra.

The two bands shown in Vogel's map, at wave-lengths 513 and 517, are probably also carbon bands seen in comets. The less refrangible one is the brighter, exactly as it was in Coggia's comet, on the occasion just referred to.

On December 5, 8, and 14, Vogel recorded four faint lines between F and the line 500. One of these is doubtless the nebula line 495; the two between this and F are probably identical with two shown in my photograph of the spectrum of the nebula in Orion.† The other line has not been identified.

Another line of considerable interest is that observed by Vogel and Cornu, near λ 531, *i.e.*, near the corona line. In Vogel's observations, the line is always associated with the iron lines E and 519, and as great accuracy is not claimed for the measures, it is quite possible that it is simply the third brightest flame line of iron at 5327. In the map it is represented as being fainter than 527, and this is quite consistent with this origin. Cornu represents the line without E or 579, and suggests that it is identical with the corona line.

The line near 635, recorded by three observers, is no doubt the same as that seen in the spectra of the bright line stars of Cygnus. No terrestrial equivalent has yet been found, but a line in this position is seen in the spectrum of the Limerick meteorite.

It is thus seen that practically all the lines and bands in Nova Cygni can be explained by reference to laboratory work.

It is extremely interesting to attempt to build up the spectrum on any given date by integrating the spectra of comets and nebulæ, or nebulæ and bright-line stars, which we have reason to believe to be swarms of meteorites. Some of these integrations are shown in the diagram (figs. 5, 6, and 7). As in Novæ we have to deal with collisions of swarms of meteorites of different densities, in such integrations it is necessary to take swarms of different degrees of condensation.

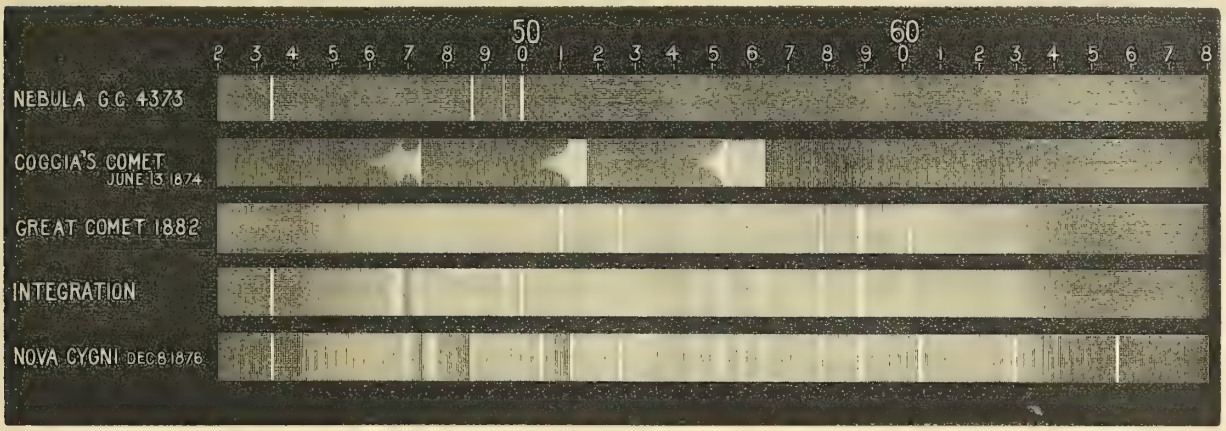
The first integration shown (fig. 5) is for the spectrum of December 8, as observed by Vogel. In this case the spectrum is reproduced almost line for line by adding together the spectra of a nebula, Coggia's Comet (June 13, 1874), and Great Comet of 1882 at perihelion (brightest lines only). The only lines of importance which are in the Nova and not in the integration are C and a line near 635, and we know that these are associated with the spectra of meteor-swarms; C occurs in γ Cassiopeiæ and β Lyræ, and the line at 635 occurs in the bright-line stars in Cygnus. This integration strengthens the view that Nova Cygni was produced by the collision of swarms of different densities. In such a case there would first be the collisions

* 'Roy. Soc. Proc.,' vol. 45, p. 175.

† 'Roy Soc. Proc.,' vol. 48, p. 200.

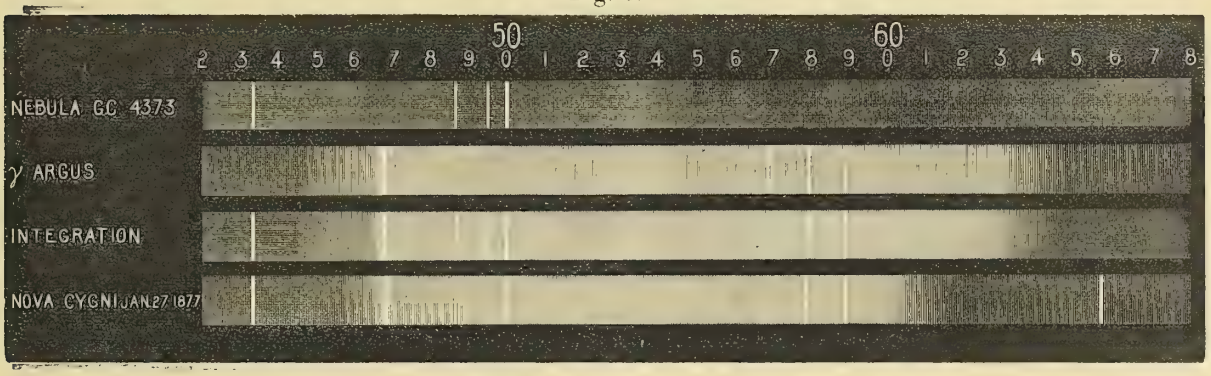
between the two sets of outliers, then the denser part of the smaller swarm would enter the outliers of the larger, and finally the densest parts of both swarms would

Fig. 5.



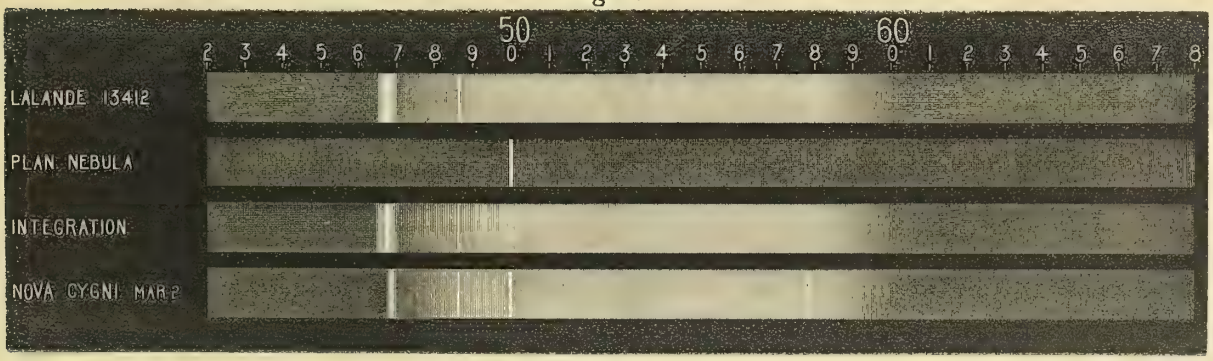
Integration of the spectra of comets and nebula, compared with that of Nova Cygni.

Fig. 6.



Integration of the spectra of nebula 4373 and γ Argûs compared with that of Nova Cygni.

Fig. 7.



Integration of the spectra of a planetary nebula and LALANDE 13,412, compared with that of Nova Cygni.

come together; so that, in this way, the resulting temperature effects may be very complicated.

The spectrum of the Nova on a later date, January 27, 1877, is considerably simpler than that of December 8, and may be almost perfectly reproduced by integrating the spectrum of a planetary nebula and that of γ Argûs (fig. 6). All the lines in the Nova on this date are shown in the integration, with the exception of the C line. The integration, however, in this case shows the lines at 495 and 568, which were not recorded in the Nova; but their absence was probably only due to difficulties of observation. The line at 495 is excessively faint in some nebulæ.

The spectrum of the Nova on March 2 was still simpler than on former days, and can be fairly reproduced by adding together the spectrum of the bright-line star, Lalande 13,412, and that of a planetary nebula which shows only the chief line of the nebulæ. This is shown in fig. 7. The temperature in this case is obviously very much lower. We get only the lowest temperature line of iron at 579, of magnesium 500, the F line of hydrogen, and a slight indication of carbon. The absence from the Nova of the line near 540 is probably also due to difficulties of observation.

It will thus be seen that the complete discussion of Nova Cygni generally bears out the statement I ventured to make after the preliminary inquiry in November, 1887, viz.: "This star passed through all the changes of temperature represented by stars with bright lines, comets, and nebulæ."

The complete series of observations of the magnitude of this star was as follows:—*

Date.	Magnitude.	Date.	Magnitude.
1876.		1876.	
November 24	3	December 12	6·7
,, 25	3·1	,, 13	6·8
,, 26	3·1	,, 14	6·9
,, 27	3·2	,, 15	7·0
,, 28	3·8	1877.	
,, 29	4·7	January 1	7·2
,, 30	5	,, 2	7·4
December 1	5·2	,, 16	>7·6
,, 2	5·4	,, 18	>7·6
,, 3	5·6	February 2	7·6
,, 4	5·8	March 2	8·5
,, 5	5·9	,, 10	8·3
,, 7	6·3	October 25	<10
,, 8	6·5	1878.	
,, 9	6·6	February 13	<11
,, 11	6·7	,, 18	11·5

Nova Andromedæ, 1885.

The spectrum of Nova Andromedæ was first examined by Dr. Copeland on September 1, 1885, when it was noted† "the spectrum was continuous from end to end, and only on close examination could slight condensations, indicative of bright lines,

* 'Astr. Nachr.,' vol. 89, 1877, p. 42. 'Berlin Akad. Monatsber.,' 1877.

† 'Monthly Notices, R. A. S.,' vol. 47, p. 50.

be detected. The spectrum was not considered to differ strikingly from that of the nebula." At this time the star was "yellowish" in colour, on September 3 it was "full yellow." A close spectroscopic examination, however, was not possible until September 10, when Dr. COPELAND wrote:—"With the unmagnified dispersion of a direct-vision Vogel spectroscope the spectrum extended from W.L. 670 mmm. to 453 mmm., or, from between B and C to half-way between F and G. When the spectrum was sufficiently narrow all the colours were visible, with a suspicion of brighter points in the line. An attempt was made to measure these with the Grubb spectroscope and a flint prism of only 40° refracting angle. The instrumental change cut down the spectrum to the limits of 600 mmm. and 456 mmm., with a maximum at 544·41 mmm., and a suspicion of a bright line, but hardly more at 482·2 mmm. With the same apparatus the spectrum appeared *quite continuous* on September 11, but again showed traces of bands on the 13th, and was slightly banded on the 15th. Traces of a condensation of light were seen at W.L. 471·6 mmm. on September 20." Using a special acute prism of only 15° angle, it was noted: "Only traces of two brighter points towards the yellow end of the spectrum could be made out on September 30, the rest of the spectrum appearing absolutely continuous." On October 1, these two lines and another fainter one were measured, the following being the wave-lengths found, and their mean values:—

Band 1.	Band 2.	Band 3.
557·1	510·1	493·1
541·8	519·5	489·9
541·5	512·4	484·5
Mean wave-length 546·8	514·0	489·2
Red end of spectrum, 575·8. Violet end of spectrum, 467·6.		

Dr. COPELAND thought that some of the discordances of the above measures were due to the indeterminateness of the objects measured, and noted that "making due allowance for this uncertainty, it seems probable that the three 'bright' bands of wave-lengths 546·8, 514·0, and 489·2, are identical with the three brightest bands afterwards measured with the same apparatus in Mr. GORE's *Nova Orionis*, of which the brightest parts were at wave-lengths 542·8, 516·2, and 494·4.* The trace of a condensation of light at W.L. 471·6, seen on September 20, agrees well with the bright line in Nova Orionis at W.L. 472·2; while the maximum of light in Nova Andromedæ at 544·4, on September 10, is closely in accord with that for the star in Orion at 542·8. The only really discordant item is the point laid down at 482·2 on September 10, which does not correspond to any known bright band in the spectra of variable stars; it was, however, entered in the note book . . . as 'a suspicion

* 'Monthly Notices, R. A. S.,' vol. 46, p. 110.

of a bright line, but hardly more.' But if it does roughly represent the position of a 'bright' line actually visible in the spectroscope, one would feel inclined to regard it as a trace of the F line. On October 2, the spectrum presented the same appearance as on the preceding day. On October 19, it could be still noted as continuous, but not uniform, with the Vogel spectroscope.

"Although the foregoing results differ widely from those obtained at Greenwich and also at Yale College" [which will be given subsequently], "as regards the three chief lines the observer cannot doubt as to their general correctness. . . . . Besides, in the case of an object losing so rapidly its power of emitting light, it is quite possible that the spectrum may have slightly varied from day to day.

"In conclusion, it seems worthy of remark that the spectrum described above is the same as that given by any ordinary hydrocarbon flame, burning so feebly that the spectrum of the blue base of the flame is just beginning to show through the continuous spectrum afforded by the white part of the flame."

On September 1, Professor VOGEL observed the spectrum and noted that the red and yellow were bright and the green faint. He also found dark lines or bands, one between the yellow and green, and another between F and G.*

Dr. HUGGINS communicated his observations of Nova Andromedæ to the 'Observatory for October, 1885. They are as follows :—"The star was observed here first on the night of the 3rd instant. It presented the appearance of an orange-coloured star of from the 8th to the 9th magnitude. With a spectroscope of low dispersive power, a continuous spectrum was seen from about C in the red to a little beyond F. There was an apparent condensation of light from about D to *b*, which might be due to bright lines in that part of the spectrum. This supposition was strengthened by the employment of a more powerful spectroscope, but I was not able to be certain on this point.

"On the night of the 9th instant, the star, which was then distinctly on one side of the principal point of condensation in the nebula, appeared to me to have a less decided orange tint. It presented a similar appearance in the spectroscope, with the exception that the light was less strong about D. I was, so far, confirmed in my suspicion of bright lines, that I have little doubt that from three to five bright lines were present between D and *b*."

On September 4 Mr. MAUNDER noted that† "the star gave a perfectly continuous

* "Die Beobachtungen am 1. und 2. September über den Stern in Nebel ergeben, dass derselbe auch bei stärkerer Vergrösserung (550 f.) vollkommen sternartig bleibt, und dass das Spectrum continuirlich ist. Die Intensitäten der Farben im Spectrum scheinen etwas abweichend von den gewöhnlichen Sternspectren zu sein, indem Roth und Gelb besonders stark hervortreten, Grün aber verhältnissmässig schwach ist. An der Grenze des Gelb und Grün habe ich eine dunkle verwaschene Bande vermuthet, eine zweite ebensolche im Blau zwischen F und G. Im Fall der Stern noch heller werden sollte, denke ich Sicherheit über weiteres Detail, welches ich im Spectrum vermuthete, zu erlangen. Ich bemerke noch dass der Andromeda-Nebel continuirliches Spectrum giebt und dass der Kern des Nebels h. 51 ein Spectrum zeigt, was mit dem des neu entstandenen Sterns in h. 50 übereinzustimmen scheint." ('Astr. Nachr.,' 2681.)

† 'Monthly Notices, R. A. S.,' vol. 46, p. 19.

spectrum, in which no lines, either bright or dark, could be detected. The red, orange, and violet were very faint, or altogether wanting, the spectrum being traceable from about D to F, but being scarcely discernible beyond those limits. The spectrum of the star resembled that of the nebula precisely, except, of course, that it was brighter, and, probably in consequence of this greater brightness, was traceable a little further in both directions.

"On the evening of September 11 the single prism spectroscope was employed, and the lower dispersion of this instrument proved more suitable for the examination of so difficult an object. The spectrum of the star was now traced from about λ 6600 to λ 4300; it was very faint in the red and range, and began to brighten at a definite point, either D or close to it. Then followed a space relatively very bright, the centre of which was not far from E, but the brightest part of which was further towards the red, at a point for which two readings were obtained, viz.:—λ 5457 and λ 5508, mean λ 5482, and where at times the brightening appeared so definite as to suggest the presence of a bright line. The same general region was also much the brightest in the region of the nebula. The spectrum of the star declined rapidly in brightness beyond F, practically it might be considered as confined between the two lines, D and F, the outer portions of the spectrum were so much fainter than the central district. From time to time bright lines were suspected in this district, the brightest being the line already mentioned, a second being measured as at λ 5327. The region of the *b* lines was specially examined, and also the neighbourhoods of both D and F, but no bright lines were seen in either locality.

"The spectrum of the star was examined with a small prism held behind the eye-piece, on September 15, and the presence of the two bright lines was again suspected And again, on September 30, they were more satisfactorily observed, although the star had then lost much of its brightness, and its spectrum was but little brighter than that of the nucleus of the nebula. A third line was also suspected about as far beyond the magnesium line, λ 5527, towards the red, as the brightest of the three bright lines, viz., that at λ 5482, was towards the blue. This would give its position as λ 5575. It could not be decided whether the lines were brighter than on September 11. They were recognised more easily than on that occasion; but, on the one hand, it was known where they should be looked for, and, on the other, the instrument now employed (the half prism spectroscope in the reversed position) was the same as that used on September 4 when no trace of a bright line could be perceived. It was believed that the brightest line, that of λ 5482, could be traced over or near the nucleus of the nebula, but with so feeble a spectrum it was most difficult to make perfectly sure. The slit also was narrow, and it was found most difficult to keep either star or nebula upon it, their spectra being so faint that they were continually being lost."

The following are some measures of the limits of visibility of the spectra of two stars, the Nova and the nebula, made by Mr. MAUNDER on September 11:—

	Limit towards the red.	Limit towards the blue.
	mm.	mm.
Arg. Z + 38° No. 90 . . .	7380	3948
Arg. Z + 38° No. 88 . . .	6608	4095
Nova	6615	4319
Nebula	6196	4363

Mr. MAUNDER remarked, with respect to these measures, "The spectrum of the Nova was traceable a little further in both directions than that of the nebula. Towards the violet this seemed entirely due to its greater brightness, but towards the red the difference was greater than could fairly be explained upon this ground. The spectrum of the Nova seemed slightly brighter than that of No. 88 in the central district from D to F, but was decidedly feebler in the red and violet regions. The Nova was estimated as being three-tenths of a magnitude fainter than No. 88."

KONKOLY, writing on September 5, noted that he observed the spectrum of Nova Andromedæ on the previous day.* In his words:—"Yesterday, when I examined the star, it seemed of a reddish-yellow colour and very faint. I estimated its diameter at, at least, two seconds; it appeared as a nebulous star. Its spectrum was unexpectedly faint, so that it was impossible to use a cylindrical lens. The edges of the spectrum seemed to be enveloped in a coloured mist. It gives the impression of bright fields on a dark ground, and in the red, yellow, green, and blue a broad band is seen. If this be the case, these broad bright regions would correspond to the hydrogen lines C, F, as well as D_3, and at an enormously high pressure. A similar broad field is also seen in the green, which certainly cannot belong to the group named. I should be inclined to class the spectrum with Type III*b*., and Professor RITTER coincides in this view.

"It must not be overlooked that the violet part is absent in the spectrum; and shortly behind the region of F the spectrum is just as if cut off. This observation was confirmed by Dr. v. KOVESLIGETBY."

Mr. O. T. SHERMAN, of Yale College Observatory, observed the spectrum of the Nova on the early morning of September 5, and noted†: "On the whitish-blue stripe forming the continuous spectrum were discerned one bright line crossing the spectrum, and further towards the red two bright knots of light. They were not very much brighter than the surrounding glare, but were evident. They remained even when the larger dispersion was employed." The positions of the lines were determined by micrometric measurements relative to bright lines in γ Cassiopeiæ and β Lyræ. Curves were plotted connecting the readings of the micrometer and wave-lengths; the four lines H_α, D_3, H_β, and H_δ being used for a foundation. The following are the wave-lengths thus found for two of the lines in the Nova, as estimated from each of the respective curves:—

* 'Astr. Nachr.,' 2681.

† 'Astr. Nachr.,' 2691.

From γ Cassiopeiæ.	From β Lyræ.	Mean.
536·0	527·0	531·5
564·0	551·0	557·5

The third line was identified as H_β, and Mr. SHERMAN thought that from its image it seemed to be due to the light of the whole nebula. The other two lines appeared to be due to the light of the Nova. All the bright lines in the Nova spectrum were noted by Mr. SHERMAN as occurring in γ Cassiopeiæ and β Lyræ.

It is also remarked that similar observations were made on September 7, 9, and 11, and that the lines were seen on September 16 and 21, but not measured.

Some observations of the Nova were made by Dr. LOHSE with the 15·5 inch Cooke refractor, at Mr. WIGGLESWORTH's Observatory.*

On September 3 it was noted "The spectrum is continuous, no lines could be distinguished in it with the Browning and Maclean spectroscopes. The bright C line was very well seen in γ Cassiopeiæ."

On September 9 Dr. LOHSE recorded: "The spectrum is still continuous, but the part from F to about midway between F and G is fainter than in the spectra of neighbouring stars. In consequence of that, the spectrum near G appears comparatively brighter than the corresponding part of the spectrum of B.—W.$_2$, $0^h.953$, which has about the same brightness. The position of the fainter part in the spectrum depends on estimation, there being no means for measuring the wave-length."

Observations made on September 12, led Dr. LOHSE to write: "The spectrum is continuous and as long as D.M. + 40°·145, if not longer. Much attention was paid to the red end of the spectrum, and repeated comparisons of the two spectra convinced me that this part of the spectrum is quite as much developed in the new star as in D.M. + 40° 145. The observations were made with a Maclean prism, held in front of the eyepiece, and are only estimations."

The late Rev. S. J. PERRY recorded:† "The Nova in Andromeda has been observed at Stonyhurst on every favourable occasion from September 13 to November 8. On September 13, the spectrum was found to be continuous, but the red end was absent, and there was a decided maximum in the green. A bright band in the green was suspected, but not clearly seen. The spectroscope was carefully focussed on the lines in the spectrum of β Andromedæ, which were well seen, before being turned on the nebula of Andromeda. On October 9 the spectrum was still brightest in the green."

The following table brings these observations together in a convenient way for reference. As far as I know, every observation which was made is included.

* 'Monthly Notices, R. A. S.,' vol. 46, p. 299.

† 'Monthly Notices, R. A. S.,' vol. 46, p. 22.

Date.	Observer.									Remarks.
1885.										
Sept. 1	Copeland	..	..	..	..	..	..	..	..	"Slight condensations observed in spectrum."
„ 2	Vogel	..	..	..	..	..	..	..	..	Red and yellow bright; green faint.
„ 3	Huggins	..	..	..	..	..	..	..	..	Bright lines suspected.
„ 3	Lohse	..	..	..	..	..	..	..	..	Continuous spectrum.
„ 4	Konkoly	..	F ?	..	..	..	..	D_3 ?	C ?	
„ 4	Maunder	..	..	..	..	..	..	..	..	"Perfectly continuous."
„ 5	Sherman	..	486	..	527–536	..	551–564	..	..	Wave-lengths may be anything between those stated.
„ 7	„	..	486	..	527–536	..	551–564	..	..	
„ 9	„	..	486	..	527–536	..	551–564	..	..	
„ 9	Lohse	?	..	..	..	..	..	..	..	"Faint part midway between."
„ 9	Huggins		..	..	..	..	..	..	..	Bright lines between D and *b*.
„ 10	Copeland	..	482	..	..	544	..	..	..	Light less strong about D.
„ 11	„	..	..	..	..	..	..	..	..	"Quite continuous."
„ 11	Sherman	..	486	..	527–536	..	551–564	..	..	Spectrum sharply terminated about D.
„ 11	Maunder	..	..	..	5327	548	..	?	..	
„ 12	Lohse	..	..	..	..	..	..	..	..	Continuous.
„ 13	Perry	..	..	..	..	..	..	..	..	"A maximum in the green."
„ 15	Maunder	..	..	..	5327	548	..	..	..	
„ 16	Sherman	..	486	..	527–536	..	551–564	..	..	
„ 20	Copeland	471·6	..	..	..	..	..	..	..	
„ 21	Sherman	..	486	..	527–536	..	551–564	..	..	
„ 30	Copeland	..	..	517	..	546	..	..	..	
„ 30	Maunder	..	..	..	5327	546	558	..	..	
Oct. 1	Copeland	..	489	517	..	546	..	..	..	
„ 2	„	..	489	517	..	546	..	..	..	
„ 9	Perry	..	..	..	..	..	..	..	..	"Brightest in the green."
„ 19	Copeland	..	..	..	..	..	..	..	..	"Continuous, but not uniform."
Suggested origins		C 468–474	H 486 (F)	C 517	Fe 527	Pb 546	Mn 558			

From the foregoing it will be evident that the observations were extremely difficult throughout, owing partly to the dimness of the star and partly to the character of the spectrum. It must also be pointed out that the spectrum observed was the integrated effect of the spectrum of the Nova and the spectrum of the nebula itself. Even the observation of the spectrum of the nebula is one of considerable difficulty, especially when the sky is at all hazy. It then appears simply as a faint continuous band of light, but when the sky is clear it is seen to have at least three maxima. It is evident from the observations of the Nova, that on some occasions the bands in the spectrum of the nebula were observed in addition to those special to the Nova, whilst on others they were not. This appears to have been especially the case in the later observations when the Nova had become dim, and this is, of course, what would be expected. I referred to the spectrum of the nebula and the connection of the Nova with it in a paper which I communicated to the Society in November, 1888.*

I then wrote:—"We have seen that some planetary nebulæ give the same spectrum as a comet at aphelion. It appeared that if the Nebula of Andromeda were further advanced than a planetary nebula in condensation, it should give a spectrum approximating to one of the more advanced cometary stages which have been already discussed.

"The spectrum of this nebula has hitherto been regarded as a perfectly continuous one, but the observations referred to show that there are some parts brighter than others. The spectrum is almost entirely wanting in red and yellow light.

"In the green there are two maxima, the brightest of which is at wave-length 517 as near as could be determined with the wide slit which it was necessary to employ, the other maximum is near 546. One of the observers, Mr. Fowler, made six independent measures of the maxima on November 20, and got very nearly the same result each time, comparison being made with the spectrum of a Bunsen, and the spectrum of chloride of lead at the temperature of the Bunsen. The measurements were repeated on November 27 with the same result, and on this occasion they were confirmed by another observer, Mr. Coppen. Another brightness near 474, as determined by comparison with the Bunsen burner, was also suspected, but it was not so easy to measure as the others.

"My suggestion as to the origin of this spectrum is that it is the integration of very slight continuous spectrum, carbon fluting radiation, and the absorption of manganese (558) and lead (546). The citron band of carbon masks, and is masked, by the manganese fluting, and the absorption fluting of lead causes, by contrast, the apparent brightness at 546. The brightest maximum is no doubt the brightest fluting of carbon at 517, and the one in the blue, which was suspected, is probably the blue carbon group 468–474.

"If these observations are confirmed, this nebula is at present at the same stage of condensation as Comet I., 1868, on April 29 (P. P., April 20), which must be regarded

* 'Roy. Soc. Proc.,' vol. 45, p. 215.

as a pretty advanced cometary stage, seeing that it was observed so near perihelion and that the perihelion distance was small."

"The discussion of the observations of Nova Andromedæ, which is not yet completed, shows that there were bright lines in exactly the same positions as the brightnesses which have now been determined in the nucleus of the nebula. The appearance of the Nova was, therefore, probably due to increased temperature, due to collisions taking place between the sparser outliers of the swarm composing the nebula and an external swarm which came in contact with them. The view of the Nova's connection with the nebula is, therefore, greatly strengthened by this inquiry."

These observations have since been confirmed by Mr. TAYLOR,* who observed two brightnesses at 517·4 and 547·3, and suspected one in the blue.

The only lines seen in the Nova which are not seen in the nebula, are F, 531, and 558, with, perhaps, D_3 and C. The spectrum of the nebula shows Mn 558 and Pb 546 absorption, masking the carbon at 564 and making it appear at 546. The Nova spectrum added manganese radiation 558. The line near 530 I have taken as E (527), the lower temperature line of iron at 579 not being visible on account of the greater brightness of the continuous spectrum in that region. The hydrogen lines C and F were probably due to collisions of the outliers of the two swarms. Mr. SHERMAN† recorded F as a line in the nebula, whilst Mr. MAUNDER noted that the line at 548 could be traced over or near the nucleus of the nebula, thus, to a certain extent, confirming the Kensington observations. There is also evidence in the observations to show that the continuous spectrum fluctuated in relative brightness during the visibility of the Nova. The effect of a brightening of the continuous spectrum would be to mask the faint lines in the green or yellow, whilst affecting but little any that might occur in the blue. LOHSE's observation of a "faint part, midway between F and G," in the absence of other lines, is a case in point. It is not unlikely that this was really the blue carbon band which is seen in the nebula itself, although its position was only roughly estimated.

Again, on September 20, Dr. COPELAND observed traces of a condensation of light at wave-length 471·6, and this also was probably the blue carbon band of the nebula, other lines being masked by the brighter continuous spectrum. On other occasions, not even the faint blue band was seen, the spectrum being recorded as quite continuous.

As the brighter continuous spectrum due to the Nova gradually dimmed, the bands of the nebula in the green became more prominent.

It is thus seen that although there does not appear to have been a regular sequence of events in the spectrum of Nova Andromedæ, as far as the actual observations go, it is probably due to difficulties of observation, and to the fact that the spectrum of the Nova was superposed upon that of the nebula. Local atmospheric conditions also had a marked effect upon the observations. Thus, the spectrum

* 'Monthly Notices, R. A. S.,' vol. 49, p. 126.

† 'Ast. Nachr.,' No. 2691.

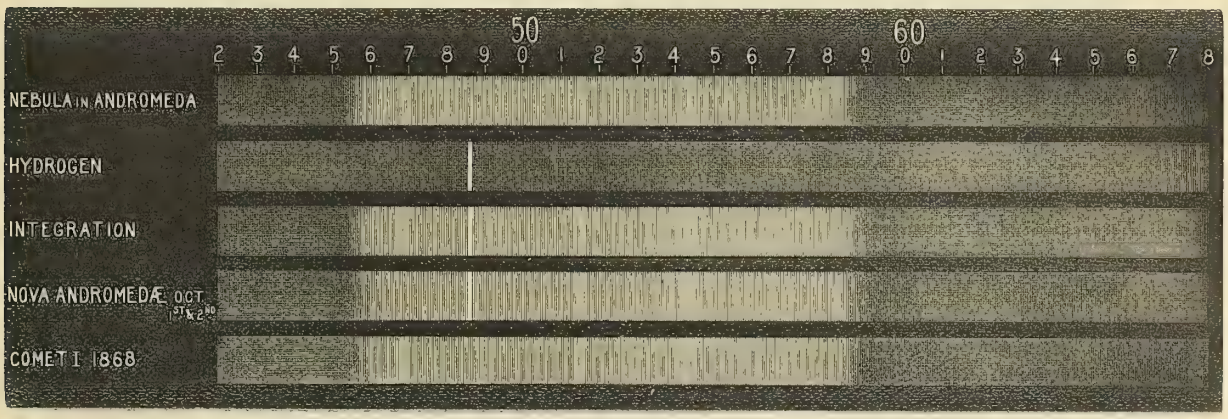
was observed on September 11, by Copeland, Sherman, and Maunder; Copeland recorded a "quite continuous" spectrum, whilst Sherman and Maunder agreed only with regard to one line.

The apparent variations from day to day are, therefore, not all real, and it is hopeless to attempt to explain them all by reference to the effects of a gradual fall of temperature. The star only diminished about two magnitudes during the period in which spectroscopic observations were made, and hence the change of temperature would not be so great as in Nova Cygni, and consequently the variations of spectrum would not be so evident.

The differences of spectra have also been examined with reference to a possible periodicity of some of the lines, but it has not been found possible to explain the variations in this way.

No lines or bands, however, were on any occasion recorded in the spectrum, with which we are not familiar in other bodies, which there is evidence to show are meteoritic swarms.

Fig. 8.



Integration of spectra of the nebula in Andromeda, and of hydrogen, compared with that of Nova Andromedæ.

The line at λ 546, which was seen in the spectrum of the Nova, was undoubtedly the maximum of light, probably due to lead, which is seen in the nebula itself. This is also the probable explanation of the other lines near λ 517 and λ 473, the latter being only observed on one occasion. The additional lines due to the collision, which produced the Nova, were, therefore, F, 5327, and 558, if we neglect Konkoly's doubtful observation of C and D_3. The appearance of the hydrogen line F is exactly what would be expected from what we know of its appearance in such stars as Mira, when, by additional collisions due to the periastron passage of a revolving swarm, the star reaches a maximum. The line near λ 558, seen by Maunder and Sherman, was in all probability due to the brightest manganese fluting at the same wave-length, which is very frequently recorded in cometary and other spectra. This, and the line 5327, which was most likely the iron line E (λ 5268) were probably produced by local collisions in the denser parts of the swarm.

Fig. 8 will show how the spectrum of the Nova, as seen by Copeland, on October 1 and October 2, can be reproduced by adding the spectrum of hydrogen to that of the Andromeda nebula. The third band of the nebula was not seen, but this is always most difficult to observe. At this time, then, the only line special to the Nova was F, and this, it will be remembered, was seen last in Nova Coronæ.

The spectrum of Comet I., 1868, which differed but slightly from that of the nebula in Andromeda, is added for completeness. In the comet the bright flutings were more prominent than in the nebula, because the continuous spectrum was relatively not so bright.

The star was first seen on August 19 at Belfast by Mr. Ward, and on the 27th, by Mr. Ludovic Gully of Rouen. The first public announcement of the change was made by Dr. Hartwig, who observed it on August 31. The following are the observed changes in magnitude :—

Date.	Magnitude.	Date.	Magnitude.
1885.		1885.	
August 19	9·0	October 4	10·39
September 1	8·47	,, 5	10·47
,, 3	8·55	,, 6	10·47
,, 5	8·63	,, 7	10·47
,, 9	8·71	,, 19	10·91
,, 10	8·71	November 4	10·95
,, 11	8·85	,, 5	10·95
,, 12	8·97	,, 7	10·95
,, 13	8·97	,, 30	11·9
,, 15	9·17	December 2	12·1
,, 17	9·49	,, 29	13·0
,, 19	9·41	1886.	
,, 29	9·75	January 2	13·5
,, 30	9·83	February 2	Not a trace of the Nova with 15 inches aperture.
October 2	9·91		

VI. The Sequence of Phenomena in the Spectra of Novæ.

Sequence of Spectra.

If the appearance of a new star be due to the collision of two meteor swarms as I have suggested, it is obvious that the spectroscopic changes should follow the same order as those observed in the spectrum of a comet during its passage from perihelion to aphelion, when differences of observing conditions, and the relative physical conditions of the two swarms which produce a Nova, are duly allowed for.

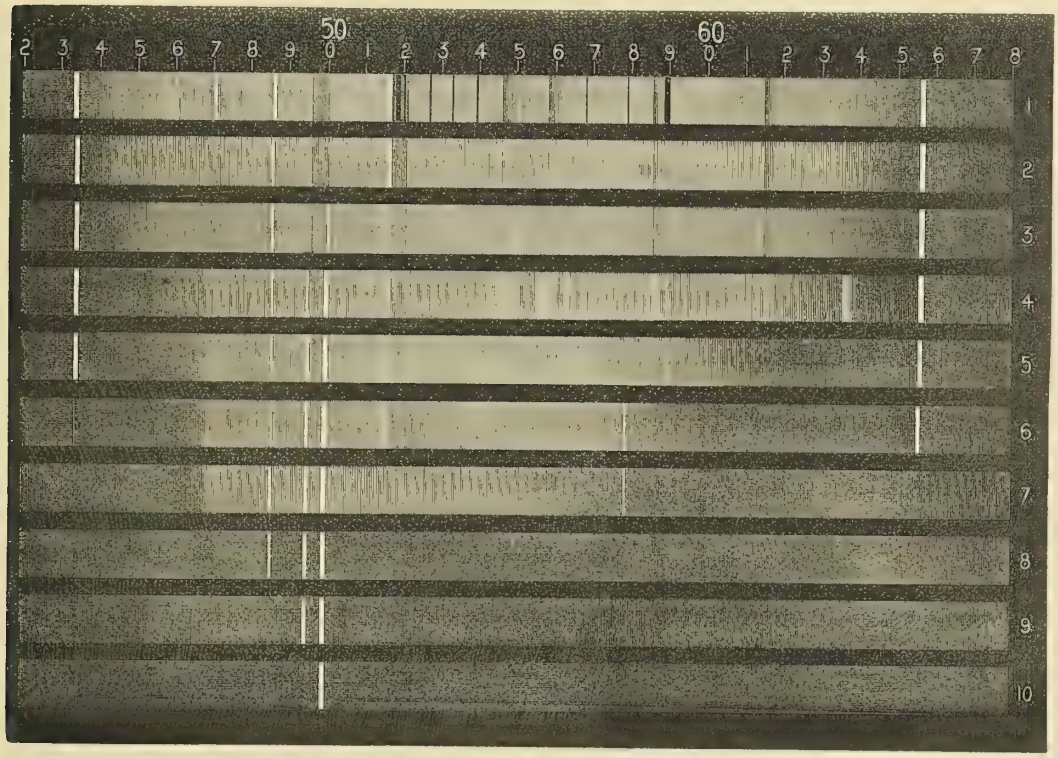
The following map (fig. 9) shows the theoretical sequence from this point of view commencing with one swarm sufficiently sparse to give only the bright lines of hydrogen and flutings of carbon, and the other sufficiently dense to give dark D and *b* and the flutings of lead, manganese, and iron, with bright carbon flutings. The first horizon accordingly shows the integration of these two spectra, and subsequent ones the effects of cooling of the mixed swarm.

None of the four Novæ, which have been spectroscopically examined, have shown the complete sequence of changes indicated in the map, but Nova Cygni passed through most of them.

The first horizon gives the typical spectrum of the light due to the collision of two swarms of meteorites; it consists of metallic absorption lines and flutings as in α Orionis, together with hydrogen and hot carbon radiation.

The spectrum of Nova Coronæ on May 16, was very similar to this.

Fig. 9.



Sequence of spectra in Novæ.

As the Nova decreases in temperature, and, therefore, in light, the line absorption disappears and the metallic fluting absorption decreases in intensity. Carbon radiation, and the hydrogen lines, C, F, and G, remain about the same as before (2).

The next stage (3) brings us to a condition represented by a swarm of Species 3, Group II.* The manganese and lead absorption at 558 and 546 respectively, are now overpowered by the carbon radiation at 564, whilst the absorption of the second manganese fluting at 586 is still apparent. The hydrogen lines, C, F, and G, remain almost as before.

On the following horizon (4) is represented the radiation of lines and flutings. The manganese and lead radiation are visible. The brightest iron and magnesium

* Bakerian Lecture, 1888, p. 66.

lines are seen; also sodium, D, and the line at 495, whilst the brightest edge of the magnesium fluting at 500 is just visible. This was a condition observed by VOGEL in Nova Cygni, on December 8, 1876. It was also observed in the Great Comet of 1882, when near perihelion, and is a condition apparently very similar to that of γ Cassiopeiæ.

The condition following this (5) is that in which lead and manganese radiation have disappeared; the three carbon flutings are well seen, C, F, and G are rather fainter, but the fluting at 500 appears brighter. The 495 line and the iron line 579 are still seen, the other iron lines in the green being masked by the brightness of the continuous spectrum. This condition was approached in Nova Cygni on February 2.

On the following horizon (6) the 564 carbon fluting is not visible, and the 517 fluting appears almost as a line. The iron lines, 579 and E, are now visible in consequence of the fading of the continuous spectrum. The hydrogen lines are still visible, whilst D has disappeared. The two magnesium flutings, 5210 and 500, are each represented, the latter being the brighter. This stage in the sequence was observed in Nova Cygni on February 2, and the brightest of the lines in Nova Coronæ, May 19. All the lines have been recorded in the spectrum of the nebula in Orion.

Both the carbon flutings, 517 and 564, now disappear (7), having become so pale that they are masked by the continuous spectrum, whilst the 468 maximum of the carbon fluting is visible, because the continuous spectrum does not reach it. F is the only remaining hydrogen line, and 579 the remaining iron line, this being the line visible at the lowest temperature. 495 is rather fainter, whilst 500 has increased in brightness. This stage was observed in Nova Cygni on March 2, 1877. The 474 carbon is the next to disappear, and three lines only are left:—F, 495, and 500 (8). The nebula G. C. 4373 and many others give this spectrum. No observations of Nova Cygni were made between March and September, 1877, and it is between these dates that this stage would have occurred.

With regard to horizon (9), the nebula G. C. 2343 and many others give a spectrum consisting of two lines, 495 and 500, and Lord LINDSAY observed in Nova Cygni, on October 10, 1877, a line at 492 (as well as the line at 500), which was most probably the nebula line at 495.

The last stage in the sequence is when the 500 fluting remains alone (10). This was observed in Nova Cygni by VOGEL and by COPELAND; it is the only line in the G. C. 4403, and, as I have shown in the appendix to the Bakerian Lecture, it is the characteristic line of comets when at a great distance from the Sun, and was observed by Dr. HUGGINS in 1866 and 1867.

It should be remarked that this is only one hypothetical case of a Nova, and that there may be considerable variations from it, according to the initial conditions of the two swarms which produce the Nova by collisions. Thus, in the case of Nova Andromedæ, one of the swarms already existed as a considerably condensed nebula and the collision with the other swarm only sufficed to bring out a few more lines,

including the F line of hydrogen. Here, then, the initial spectrum would be different, and the subsequent changes would not take place in exactly the same order. The integrations which have been given, however, show that it is possible in any of the Novæ which have yet been spectroscopically observed to get a good idea of the states of condensation of the two swarms after the first collision.

It is important to compare this sequence with that which I have already given for comets,* for, though the conditions are different, there will be a certain similarity, since both have to be regarded as meteor swarms. On the first three horizons of the Nova sequence we have mixed radiation and absorption phenomena; this also occurs in comets, though the difference in the brightness of the continuous spectrum does not make it so obvious. On the fourth horizon of the Nova sequence we have carbon, manganese, and lead radiations, which also occur in comets; but, in addition, there are lines of hydrogen and other substances due to the compound character of the swarm. Finally, both Novæ and comets give one line, the chief line of the nebulæ, which is probably due to magnesium. It will be seen that the spectra of Novæ are, in general, more complex than those of comets, which results from the collision of two swarms of different densities.

Variations in Magnitudes.

In each case where the spectra of new stars have been observed the evidence tends to show that the star was hottest at the first observation; the absorption lines giving way to bright lines in Nova Coronæ, the brightest lines fading away one by one in Nova Cygni, and the carbon becoming more manifest in Nova Andromedæ, all go to show a diminution in the temperature of the star after the first observation. Indeed, in only one case, that of Nova Andromedæ, have I been able to find any evidence of a new star increasing in brightness after its discovery. Assuming that the Nova was physically connected with the nebula, this increase of brightness is exactly what would be expected from a consideration of the beautiful photograph obtained by Mr. Roberts, which shows that the nebula is really a spiral system. We have only to suppose the incoming swarm to pass from the outside to the inside of one of the rings—a region of gradually increasing density—to give the required explanation. After a time the rotation of the nebula would bring so many more meteorites to the point of disturbance that their motions would predominate and the "Nova" would begin to fade.

It is very probable that a Nova would be overlooked until transcendently bright, and the observations of the magnitudes of Novæ show that such has been the case. It is, however, essential to my theory that the increase in temperature and in luminosity shall be much more sudden than the decrease.

The first observations of Nova Coronæ, which showed the same absorption lines as

* 'Roy. Soc. Proc.,' vol. 45, p. 190.

α Orionis, indicate a comparatively high temperature, and it also was a Nova that flashed out very suddenly. M. Courbebaisse and Mr. Baxendell* held that the Nova could not have been conspicuous two days before its discovery, and were confident it was not visible three days before. Many other observers support this statement. Hence it is evident that in these cases we are dealing with the collisions of two rather condensed swarms of meteorites, the consequent temperature being high and the increase in light sudden. In Nova Andromedæ, where the increase in luminosity was not so sudden, the temperature was not nearly so high. In this case we had most probably to deal with the collision of two swarms not nearly so condensed as in Nova Coronæ; perhaps a slightly condensed swarm passing through the Andromeda nebula, in which case the increase in temperature would be more gradual and comparatively small. When my paper of November, 1887, was read I was unaware that comets would give evidence of absorption in their spectra, but the more recent work has shown that in comets at or near perihelion the absorption of Mn and Pb have been indicated very frequently. Hence, no difference exists between the spectra of new stars and of comets. I remarked in that paper† that the spectra of stars of Class IIIa. and some "Novæ," notably Nova Orionis, were strikingly similar, and I wrote regarding this fact: "We have then, in these bodies, a spectrum integrating the *radiation* of carbon and the absorption of Mn and Pb vapour, and I have shown that comets give at some time during their revolution, providing the perihelion distance is not too great, the same spectrum"

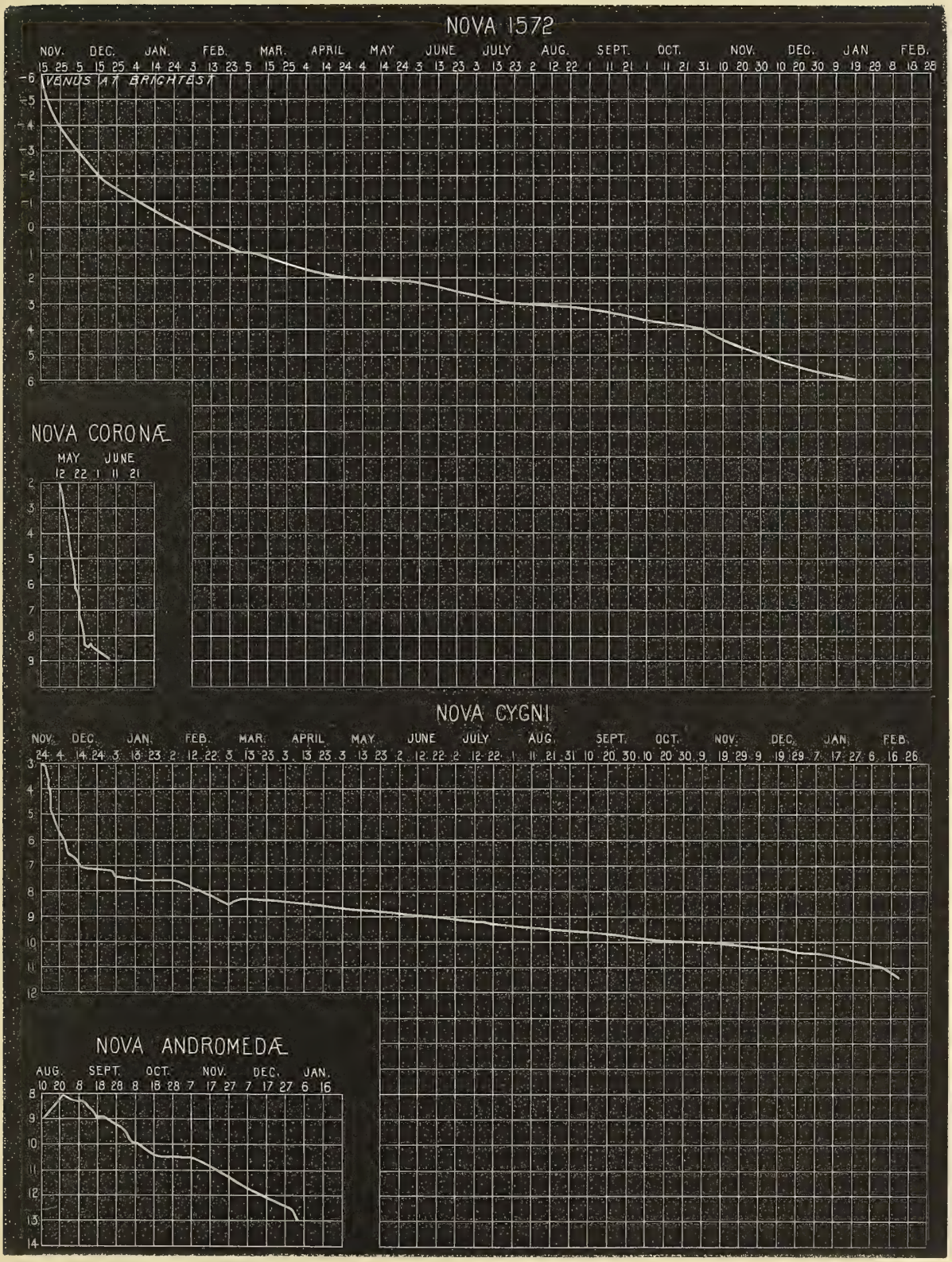
Nova Cygni decreased in magnitude in a very similar manner to Tycho Brahe's Nova, dimming very suddenly at first, and more slowly later on. In three months the Nova fell from the 3rd to the 8th magnitude, and then the fall from 8 to 11 took twelve months. The brightness of this Nova, as well as its long period of visibility, afforded the opportunity for many observations of its spectrum, the result being that the sequence of phenomena, as far as it goes, is much more complete in it than in any other Nova, which brings us to the conclusion that other Novæ would have given a more complete sequence if more spectroscopic observations could have been made.

All the changes in the magnitudes of Novæ which have been recorded, so far I know, are shown in fig. 10.

* 'Monthly Notices, R. A. S.,' vol. 26, p. 293.

† 'Roy. Soc. Proc.,' November, 1887, p. 130.

Fig. 10.


NOVA 1572
NOV. DEC. JAN. FEB. MAR. APRIL MAY JUNE JULY AUG. SEPT. OCT. NOV. DEC. JAN FEB.
VENUS AT BRIGHTEST
NOVA CORONÆ
MAY JUNE
NOVA CYGNI
NOV. DEC. JAN. FEB. MAR. APRIL MAY JUNE JULY AUG. SEPT. OCT. NOV. DEC. JAN. FEB.
NOVA ANDROMEDÆ
AUG. SEPT. OCT. NOV. DEC. JAN.

Light curves of Novæ.

VII. Colour Phenomena.

The Changes of Colour in Condensing Swarms.

During the condensation of a nebulous swarm to a star of Group IV. the changes of colour occur in the following order, so far as we know at present :—

Group IV.	Highest Temperature	Bluish-white.
Group III.	Intermediate Temperatures	White.
		Yellowish-white.
		Yellow.
		Orange.
		Reddish-yellow.
		Yellowish-red.
		Red.
		Yellowish-red.
Group II.		Reddish-yellow.
Group I.	Lowest Temperature	Dull white, grey, or pale bluish-green.

I have shown in a former paper* that the reddest stars on the ascending side of the temperature curve fall in the middle species of Group II., as represented in the above table. Hence as the cooler swarms of Group I. are of a dull white, grey, or pale bluish-green, it follows that there must be a gradual increase of redness in passing from Group I. to the middle species of Group II., and a decrease of redness in passing from there to higher points of the temperature curve. It is probable therefore that the succession of colours takes place very nearly in the order stated.

Mixed Colours in Novæ.

It must, however, be remembered that the sudden increase of temperature which determines the appearance of a New Star is of quite a different character to the increase of temperature due to the condensation of a single meteoritic swarm. The phenomena accompanying each would therefore be different.

In the case of New Stars, we have to begin with two meteoritic swarms, possibly in different stages of condensation. If no star or nebula were visible before, the sudden increase of light would be due to the collision of two dark swarms or streams. If one of the swarms engaged already existed as a nebula, the collision of another with it would cause an outburst similar to that of Nova Andromedæ. If one swarm existed as a star, and was therefore rather more condensed, the collision of another swarm with it would produce a higher temperature; this was the case with Nova Coronæ. But after the disturbance due to the collision had subsided, the tem-

* 'Roy. Soc. Proc.,' vol. 46, p. 418.

perature must begin to fall, as the mixed swarm is not in a condition to keep it up. *Novæ, therefore, run back along the temperature curve,** and their colour changes will in general take place in the opposite order to that followed by a condensing nebulous swarm. Hence the colour of New Stars will be generally of a compound nature, and made up of the luminosity of the pre-existing nebula or star, together with the added light brought about by collisions.

The changes of colour will depend upon the relative amounts of light received from the two sources of luminosity of the Nova. This again may depend either upon the relative volumes occupied by the two swarms or upon their temperatures. If, for instance, the Nova consists of a large sparse swarm combined with a smaller and more condensed one, the light at first would be mainly that of the condensed part, the feeble blue or green light of the sparser swarm being overpowered. With rapid cooling in such a case, the light from the condensation would diminish in greater proportion than that from the larger mass, and the blue colour would then begin to assert itself. If the light from the condensation were reddish, this addition of blue light would tend to produce a purple tint, as in Nova Cygni, or a leaden one as in the star of 1572.

I have compiled the following colour observations, and a compound origin, especially in the case of Nova Coronæ, comes out very strongly :—

The Nova of 173 A.D.

The first recorded observation of colour changes of new stars that I have been able to find is given by Biot in the 'Connaissance des Temps,' 1846, and is a translation of a Chinese observation made on 10th December, 173 A.D.

"Période Chi-ping 2^{e}. 10^{e} lune jour Kouli-lai. Une étoile extraordinaire parut au milieu des tran-men (α, β, pied oriental du Centaure); elle était grande comme une matte de bambou et présenta *successivement* les cinq couleurs; elle diminua peu-à-peu jusqu'à la 6^{e} lune de l'année suivante (vers juillet), où elle disparut."

I have not, however, been able to discover what colours are here referred to, so that it cannot be said whether the changes are such as occur in condensing swarms of meteorites or not.

Colour Changes in Nova Cassiopeiæ, 1572.

The treatise written by Tycho Brahe on the Nova which appeared in 1572, has already been referred to. The Nova was observed to go through regular changes of colour as it diminished in magnitude. These changes are recorded in the introductory part of this paper, and show a close similarity to those observed in other Novæ. At first the star was a dazzling white, then it became yellow, then reddish, and latterly it was hardly definable, but has been described as "leaden-coloured." The most

* 'Roy. Soc. Proc.,' vol. 43, p. 143.

manifest changes in colour took place during the first third of the period of visibility, when the star was rapidly decreasing in brightness.

To sum up then, the changes in Nova Cassiopeiæ may be taken as follows:—

White.
Yellow.
Red.
Leaden-coloured.

It will be seen that this sequence in the early stages only differs from that which takes place in condensing swarms in point of detail, the more minute changes not having been recorded. The leaden colour in the last stage was no doubt due to the blue light from the sparser regions, as I have already pointed out.

Nova Coronæ.

In this star the compounding of colours is manifest.

12th May . .	White, with a bluish look. (Birmingham, 'Monthly Notices,' vol. 26, p. 310.)
15th May . .	White, with a bluish look. (Baxendell, 'Monthly Notices,' vol. 27, p. 5.)
16th May . .	Cream-coloured, *yellow seen through blue film.* ('Monthly Notices,' vol. 27, p. 5.)
19th May . .	Buff-coloured. ('Monthly Notices,' vol. 27, p. 5.)
21st May . .	Leaden, slight orange tinge. ('Monthly Notices,' vol. 27, p. 5.)
22nd May . .	No yellow or red. ('Monthly Notices,' vol. 27, p. 5.)
23rd May . .	Dull grey. ('Monthly Notices,' vol. 27, p. 5.)
24th May . .	Dull white, tinge of orange. ('Monthly Notices,' vol. 27, p. 5.)
25th May . .	Slightly orange-white. ('Monthly Notices,' vol. 27, p. 5.)
26th May . .	Dull orange-white. ('Monthly Notices,' vol. 27, p. 5.)
29th May . .	Dull orange-yellow. ('Monthly Notices,' vol. 27, p. 5.)
2nd June . .	Orange, no longer striking. ('Monthly Notices,' vol. 26, p. 298.)
25th June . .	Orange-yellow. ('Monthly Notices,' vol. 27, p. 5.)
26th June . .	Orange. ('Monthly Notices,' vol. 27, p. 5.)
11th July . .	Dull yellow. ('Monthly Notices,' vol. 27, p. 5.)

The star had a yellowish tint to November 6, 1866. It appears, therefore, that when first visible the colour of this star was compounded of the yellowish-white colour of a swarm in an advanced stage of condensation and the bluish colour of a very early star, the blue colour in this case being due to the carbon radiation in the blue. This condition probably existed from May 12 to May 21. Afterwards, as the sparser swarm became very faint, the blue colour gradually died out, leaving a yellow tint preponderating.

The compound of colours is, perhaps, more obvious in this case than in any other, and the reason is not far to seek, since the two swarms were of such very different degrees of condensation, one being comparatively far advanced along the temperature

curve, whilst the other was only a very sparse swarm, as indicated also by the compound spectrum.

Nova Cygni.

The observations of the colours of this star by Dr. LOHSE* show that the changes were very similar to those observed by TYCHO BRAHE in the Nova of 1572. The colours observed on various dates were as follows :—

24th November, 1876 . .	Golden-yellow, almost red. (SCHMIDT, 'Astr. Nachr.,' 2113.)
5th December „ . .	Orange colour. (VOGEL, 'Berlin Monatsber.,' 1877.)
8th „ „ . .	Colour scarcely different. (VOGEL, 'Berlin Monatsber.,' 1877.)
26th „ „ . .	Of a reddish colour. (VOGEL, 'Berlin Monatsber.', 1877.)
2nd January, 1877 . .	Decided red colour. (COPELAND, 'COPERNICUS,' vol. 2, p. 102.)
8th „ „ . .	Red, *with tinge of purple!!* (COPELAND, 'COPERNICUS,' vol. 2, p. 102.)
19th „ „ . .	Much less red, hardly full orange. (COPELAND, 'COPERNICUS,' vol. 2, p. 102.)
7th February „ . .	Deep orange. (COPELAND, 'COPERNICUS,' vol. 2, p. 102.)

It will be seen that the changes in colour of Nova Cygni are in exact conformity with those given at the beginning of this note, and it will also be seen that Nova Cygni did not reach so high a temperature as Nova Coronæ ; for, whereas the latter star was bluish-white at its brightest and hottest stage, Nova Cygni was of a golden yellow. This fact is also borne out by the spectroscopic observations of the two stars.

The purple tinge noted by COPELAND on January 8 was, no doubt, produced by the addition of the blue or bluish-green light of the sparser regions to the reddish light of the denser parts of the swarm.

Referring to these observations of colour, Dr. LOHSE remarks : "The spectroscope gives the desired explanation. It showed that the continuous spectrum, which was at first exceedingly bright, gradually became pale, until at last scarcely anything but bright lines could be seen. This succession of appearances, as seen in both TYCHO'S and SCHMIDT'S stars, and which will probably occur in all stars of a similar kind, shows distinctly that a series of materials came into action requiring varying degrees of heat to set them aglow, and thus causing a dying-out of certain colours in the spectrum."

Nova Andromedæ.

The colour changes of this star observed by COPELAND, HUGGINS, and others, are also exactly what they would be in a meteor-swarm running backwards along the temperature curve. The changes have not a very great range, as the star was only visible for a short time, and never reached a high temperature. The colours observed were as follows :—

* 'Berlin Akad. Monatsber.,' 1877, p. 826.

1st Sept.	Yellowish. (Copeland, 'M. N.,' vol. 47, p. 53.)
1st Sept.	Reddish-yellow. (Konkoly, 'A. N.,' 268.)
3rd Sept.	Orange colour. (Huggins, 'Brit. Assoc. Rep.,' 1885, p. 935.)
3rd Sept.	Full yellow. (Copeland, 'M. N.,' vol. 47, p. 53.)
6th Sept.	Yellow. (Lohse, 'M. N.,' vol. 46, p. 300.)
9th Sept.	Reddish. (Stone, 'M. N.,' vol. 46, p. 56.)
9th Sept.	Slightly redder than formerly. (Lohse, 'M. N.,' vol. 46, p. 300.)
9th Sept.	Less strongly orange. (Huggins, 'Brit. Assoc. Rep.,' 1885, p. 935.)
17th Sept.	Yellowish-red. (Stone, 'M. N.,' vol. 46, p. 56.)
30th Sept.	Slight orange flush. (Maunder, 'M. N.,' vol. 46, p. 21.)

The general sequence of colours may therefore be taken as reddish-yellow, orange, reddish, and yellowish-red, which closely agree with the portion of the colour sequence reddish-yellow, yellowish-red, red, yellowish-red. The compounding of the colours of the two different swarms in this star is not very obvious, as both swarms were at a low temperature.

General Conclusions with regard to Colour.

It will be seen from the preceding that the changes of colour observed during the cooling of Novæ are perfectly in accordance with the sequence to which reference has been made. In one case, the temperature is increasing, while, in the other, it is decreasing, and the changes accordingly take place in inverse order. The Nova observed by Tycho Brahe, appears to have reached an exceptionally high temperature, as indicated by its colour and brightness, and the changes of colour observed are exactly what they would have been in a cooling swarm of meteorites. This also was the case in the other Novæ of which colour observations have been recorded. Nova Cygni passed from a golden yellow to red, and then to orange, which agrees with the portion of the general colour sequence—reddish-yellow, yellow, red, yellowish-red.

Nova Andromedæ was first reddish-yellow, then orange coloured, reddish, and yellowish-red. Many observations of variations in the colour of Nova Coronæ have been made, and these show that from bluish-white it ran down to a dull yellow. From these observations of colour it is evident that this was the hottest of the new stars. The "white, with a bluish look," recorded by Baxendell, is at the top of our colour stages, and the spectroscopic examination of the star indicated a high temperature. Baxendell noted that, after the first observations, no blue tinge was seen.

The discussion of colour observations therefore strengthens the view that new stars are complex bodies, probably produced by the collision of two swarms of different densities.

VIII. Carbon Radiation in Novæ.

In a former paper* I showed that the record of the presence of bright carbon flutings was unbroken from a planetary nebula through stars with bright-line spectra to those resembling α Herculis; that is entirely through Groups I. and II. of my classification. In comets, also, carbon is one of the chief features of the spectrum, and here there can be little doubt that we are dealing with swarms of meteorites.

Carbon is thus one of the chief characteristics of the spectra of uncondensed meteor swarms. We have already seen that there is evidence of its existence in Novæ, but to emphasise this point, it may be convenient to summarise the observations which demonstrate it.

In Nova Coronæ the evidence depends upon two lines in the blue at approximately λ 467 and 473. The more refrangible was stated by Dr. Huggins to be either double or ill defined, and this was no doubt the maximum of luminosity of the compound carbon fluting at 468. The line at 473 was probably the least refrangible maximum of the same group. This double maximum of the carbon fluting has frequently been recorded in cometary spectra. The green flutings of carbon in Nova Coronæ were masked by the continuous spectrum, which, however, did not extend far enough into the blue to mask the other.

This double maximum was also seen by Vogel in Nova Cygni soon after its appearance, but after a time the sharp termination at 474 ceased to be visible, and the more refrangible one remained alone. Both these conditions have frequently been recorded in comets, and there is no doubt that they were due to carbon. In Nova Cygni, however, there was other evidence of carbon in the appearance of the brightest fluting at 517. Both this and the one at 468 faded away as the star gradually assumed the spectrum of a planetary nebula.

For Nova Andromedæ we have Copeland's statement that the spectrum was "the same as that given by any ordinary hydrocarbon flame." Although the flutings seen exist also in the nebula, it is probable that they were slightly intensified in the Nova, because the same observers did not note them in the nebula itself. Some of the observers, however, noted that the spectrum of the Nova was that of the nebula intensified, although they did not recognise the true character of the nebula spectrum.

Again, a characteristic feature of the spectra of Novæ at some period is the apparent breaking up of the blue end of the spectrum in two parts. Thus, speaking of Nova Cygni, Vogel† says: "It must be also mentioned as characteristic of this spectrum that the blue and violet were very distinct compared with what they are in other stars possessing a band spectrum;" and with respect to an observation made on January 1, he further remarks: "After F followed a broad dark band which divided the spectrum into two parts," as behind it the continuous spectrum, although very

* 'Roy. Soc. Proc.,' vol. 47, p. 39.

† 'Berlin Akad. Monatsber.,' 1877, p. 241.

weak, is still further visible. A little later, he states that the dark band had a wave-length of 474–486.

Cornu* also shows the spectrum of Nova Cygni divided into two parts beyond F, and Dr. Lohse† noted what he considered to be an intensely dark absorption band beyond F.

Again, Konkoly noted regarding Nova Andromedæ,† "Shortly behind the region of F the spectrum is just as if cut off."

In all these cases, the apparent breaking-up of the spectrum was doubtless due to the existence of the bright blue fluting of carbon standing out beyond the end of the continuous spectrum, exactly as is seen in the spectra of bright-line "stars" and some of the condensing swarms of Group II.

In the discussions of the spectra of condensing swarms‡ I remarked :—"When in these stars the spectrum is seen far in the blue, the luminosity really proceeds first from the carbon fluting, and, in the hotter stars, from the hydrocarbon one, which is still more refrangible, in addition. In the stars which have been examined so far, the dark parts of the spectrum, which at first sight appear due to absorption, are shown to be most likely caused by the defect of radiation in that part of the spectrum between the blue end of the continuous spectrum from the meteorites and the bright band of carbon."

In such cases as these just described the carbon fluting at 474 appears broken off from the remainder of the spectrum.

Speaking of Lalande 13412, I remarked§ :—"The bright part of the spectrum extending from 473 towards the blue with its maximum at 468 is, I would again suggest, the carbon band appearing beyond the continuous spectrum"; and, again referring to 2nd Cygnus, "The bright band in the blue at 473 is most probably the carbon band appearing bright upon a faint continuous spectrum, this producing the apparent absorption from 486 to 473." In 3rd Cygnus the same thing occurs, a dark band appearing from about 488 to 473, which is doubtless nothing more than the dark space between the end of the continuous spectrum and the carbon fluting at 474. I also remarked, concerning the origin of the discontinuous spectrum‖ :—"I have already shown that when the meteorites are wide apart, though not at their widest, and there is no marked condensation, the spectrum will extend farther into the blue, and therefore the flutings in the blue will be quite bright; in fact, under this condition the chief light in this part of the spectrum, almost indeed the only light, will come from the bright carbon. Under this same condition the temperature of the meteorites will

* 'Comptes Rendus,' vol. 83, p. 172.

† 'Astr. Nachr.,' No. 2681.

‡ Bakerian Lecture, 1888, p. 31.

§ Bakerian Lecture, 1888, p. 35.

‖ Bakerian Lecture, 1888, p. 54.

not be very high; there will, therefore, be little continuous spectrum to be absorbed in the red and yellow."

There can be no doubt, therefore, that the spectra of Novæ are similar to the spectra of bodies of Group II., and the later species of Group I., as far as carbon is concerned. All these, again, are closely related to cometary spectra, and the acknowledged meteoritic nature of the latter strengthens the view that Novæ are produced by the collisions of meteor-swarms.

IX. General Conclusions.

In this paper I have endeavoured to bring together all the existing spectroscopic observations of Novæ. They show that the changes observed are very closely related to those observed in cometary spectra, the difference in observing conditions, and the compound character of Novæ being duly allowed for. The spectra in both cases indicate comparatively low temperatures. The temperature of a Nova depends upon the degree of condensation of the meteor-swarms which produce it. Its visibility depends also to a certain extent on its size. Hence it is that all Novæ do not attain the same maximum temperature or degree of visibility, and that some are lost to view before they descend to the same low temperature as others. A similar case is seen in comets which differ in their maximum temperature, according to their different perihelion distances.

But with Novæ, as with comets, a more or less complete sequence of spectra has been found by considering the whole of the spectroscopic observations, and joining them together on a descending scale of temperature.

The general result of the comparison of new stars with variables, is that in passing from a variable to a new star, we pass from one swarm (or many) revolving in an elliptic orbit round a central swarm, to one probably revolving in a parabolic or hyperbolic orbit.

The changes in magnitude observed in Novæ are also in strict accordance with the meteoritic theory of their origin. The sudden increase of luminosity, which denotes the creation of a new star, may well be due to the colliding of two swarms of meteorites, whilst the rapid fading away conclusively demonstrates, that small bodies and not large ones are engaged, for, as Dr. Croll notes in "Stellar Evolution," p. 33: "Suppose two bodies, each half the mass of the sun, moving directly towards each other, with a velocity of 476 miles per second; these bodies would, in virtue of that velocity, possess 4149×10^{38} foot-pounds of energy, which is equal to 100,000,000,000 foot-pounds per pound of the mass, and this converted into heat by the stoppage of their motions would suffice to maintain the present rate of the sun's radiation for a period of 50,000,000 years."

The complete discussion, therefore, tends to confirm the conclusion which I arrived at in November, 1887,* namely: "New stars, whether seen in connection with

* 'Roy. Soc. Proc.,' vol. 43, p. 154.

nebulæ or not, are produced by the clash of meteor swarms, the bright lines seen being low temperature lines of elements, the spectra of which are most brilliant at a low stage of heat."

I have finally to express my obligations to my assistants, Messrs. Fowler, Baxandall, and Gregory. Mr. Fowler has assisted chiefly in the discussion of the observations, Mr. Baxandall in the preparation of the maps, and Mr. Gregory in looking up the records on the subject.

XII. *On* Stokes's *Current Function.*

By R. A. Sampson, *B.A., Fellow of St. John's College, Cambridge.*

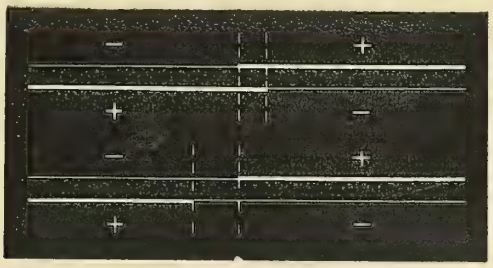
Communicated by Professor Greenhill, *F.R.S.*

Received November 24,—Read December 11, 1890.

In Maxwell's 'Electricity and Magnetism'* a view is put forward, in accordance with which we may regard any irrotational motion in a perfect liquid, for which the velocity potential is a solid zonal harmonic, as due to the juxtaposition at the origin, and upon the axis of symmetry, of sinks and sources.

But, in a liquid, any irrotational motion which is symmetrical with respect to an axis gives a velocity potential which may be expressed as a sum of a series of solid zonal harmonics, their common axis being the axis of symmetry, and their origin arbitrary, provided it is excluded from the region to which the expressions apply. The position of the origin upon the axis is arbitrary, since, by a transference formula, we may pass from one origin to another.

Let us now consider the system formed by a line source and a line sink, of equal strengths, extending along the axis from an arbitrary origin to infinity in opposite directions. Such a system I shall call an *extended doublet*, of strength m, where m is the strength per unit length of that part which lies on the positive side of the origin.

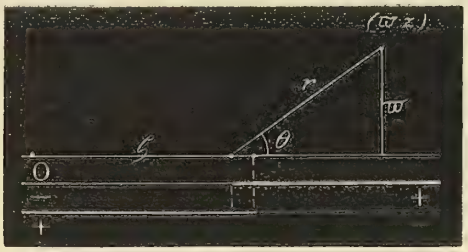


By the superposition of two extended doublets of equal but opposite strengths we can produce a sink or a source upon the axis. Hence, in a liquid, any irrotational motion which is symmetrical with respect to an axis may be produced by superposition of extended doublets, whose origins depart but little from an arbitrary point on the axis of symmetry.

Now, for an extended doublet of strength m, Stokes's Current Function ψ, for

* Vol. 1, chapter ix.

any point distant r from the origin, is $-2mr$. For let ζ be the distance of the origin of the doublet from the origin of coordinates, and let $\psi(m, \zeta)$ be the value of Stokes's Current Function for any point (ϖ, z).



Then if $\delta\psi$ be the Current Function for a source of strength $2m\,\delta\zeta$ at the point ζ of the axis we get

$$\frac{1}{\varpi}\frac{d}{dr}\delta\psi = 0,$$

$$\frac{1}{\varpi}\frac{d}{rd\theta}\delta\psi = \frac{2m\delta\zeta}{r^2}.$$

Therefore

$$\frac{d}{d\theta}\delta\psi = 2m\frac{\varpi}{r}\delta\zeta,$$

$$= 2m\sin\theta\,\delta\zeta,$$

whence

$$\delta\psi = -2m\delta\zeta\cos\theta,$$

the constant of integration being zero.

But

$$\delta\psi = \psi(m, \zeta) + \psi(-m, \zeta + \delta\zeta),$$

and clearly,

$$\psi(-m, \zeta + \delta\zeta) = -\psi(m, \zeta + \delta\zeta).$$

Hence

$$\psi(m, \zeta) - \left[\psi(m, \zeta) + \frac{d\psi}{d\zeta}\delta\zeta\right],$$

$$= -\frac{d\psi(m, \zeta)}{d\zeta}\delta\zeta,$$

$$= -2m\delta\zeta\cos\theta,$$

$$= -2m\delta\zeta\frac{z - \zeta}{\sqrt{\{\varpi^2 + (z - \zeta)^2\}}}.$$

Therefore

$$\frac{d\psi}{d\zeta} = 2m\frac{z - \zeta}{\sqrt{\{\varpi^2 + (z - \zeta)^2\}}},$$

and

$$\psi = -2mr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

$$r = \sqrt{\{\varpi^2 + (z - \zeta)^2\}},$$

disregarding a constant.

Thus, if

$$m = f(\zeta)\, d\zeta,$$

we may, by properly choosing the function f, write

$$\psi = \int_{-\infty}^{+\infty} f(\zeta) \sqrt{\{\varpi^2 + (z - \zeta)^2\}}\, d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where ψ is the current function for any irrotational motion in a liquid, symmetrical about the axis of z.

Again, if

$$r = \sqrt{\{\varpi^2 + (z - \zeta)^2\}},$$

$$\frac{dr}{d\varpi} = \frac{\varpi}{r}, \qquad \frac{dr}{dz} = \frac{z - \zeta}{r},$$

$$\frac{d^2r}{d\varpi^2} = \frac{1}{r} - \frac{\varpi^2}{r^3}, \qquad \frac{d^2r}{dz^2} = \frac{1}{r} - \frac{(z - \zeta)^2}{r^3}.$$

Therefore

$$\frac{d^2r}{d\varpi^2} + \frac{d^2r}{dz^2} = \frac{2}{r} - \frac{\varpi^2 + (z - \zeta)^2}{r^3},$$

$$= \frac{1}{r},$$

$$= \frac{1}{\varpi}\frac{dr}{d\varpi},$$

and the expression (1), and consequently also (2) satisfies the differential equation

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

or, as I shall write it, $D\psi = 0$.

When the motion is rotational, (3) no longer holds. In fact, as is well known, we have under all circumstances

$$\frac{1}{\varpi} D\psi = -2\omega,$$

where ω is the resultant spin at the point (ϖ, z).

Thus, if there is spin in the fluid, (3) is replaced by

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = -2\varpi\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a).$$

Again, if ∇^2 stand for the operator

$$\frac{d^2}{d\varpi^2}+\frac{d^2}{dz^2}+\frac{1}{\varpi}\frac{d}{d\varpi}+\frac{1}{\varpi^2}\frac{d^2}{d\phi^2},$$

ϕ being the azimuthal angle about the axis of symmetry, it may be seen at once that

$$\mathrm{D}\psi=\frac{\varpi}{\sin\phi}\nabla^2\frac{\sin\phi}{\varpi}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Consequently, (3a) may be written

$$\nabla^2\frac{\psi\sin\phi}{\varpi}=-2\omega\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a').$$

Consequently,

$$\psi=\psi_o+\frac{\varpi}{2\pi\sin\phi}\iiint\frac{\sin\phi'\omega'\,dx'\,dy'\,dz'}{r},$$

where ψ_o is a solution of (3), or, ψ consists of a solution of (3), together with $\frac{\varpi}{2\pi\sin\phi}\times$ the potential at the point considered of a distribution of mass of density at any point, $\sin\phi\times$ the spin at that point. This result is given by Basset, 'Hydrodynamics,' vol. 2, § 306.

I give one other general result. Since

$$\omega=-\frac{1}{2\varpi}\mathrm{D}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a),$$

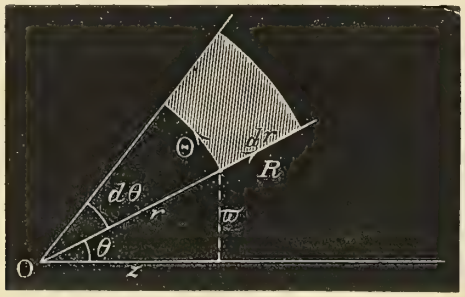
the circulation in any evanescible circuit drawn in a meridional plane is

$$-\iint\frac{1}{\varpi}\mathrm{D}\psi\,d\varpi\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the integration extends over the area embraced by the circuit.

This result enables us to transform $\mathrm{D}\psi$ readily from cylindrical to other systems of coordinates. For instance, consider polar coordinates, r, θ, and let us find the circulation in a small rectangle bounded by r, $r+dr$, θ, $\theta+d\theta$.

Let the velocities in the direction of r, and perpendicular to it, be R, Θ.



Then the circulation in this circuit is

$$\mathrm{R}\,dr + \left[\Theta r + \frac{d}{dr}(\Theta r)\,dr\right]d\theta - \left[\mathrm{R} + \frac{d\mathrm{R}}{d\theta}\,d\theta\right]dr - \Theta r\,d\theta$$
$$= r\,dr\,d\theta\left[\frac{d\Theta}{dr} + \frac{\Theta}{r} - \frac{d\mathrm{R}}{rd\theta}\right].$$

Now

$$\Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr}$$
$$\mathrm{R} = \frac{1}{r^2\sin\theta}\frac{d\psi}{d\theta}.$$

Thus the expression in square brackets is

$$-\frac{1}{r\sin\theta}\left[\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\left(\frac{d^2\psi}{d\theta^2} - \cot\theta\frac{d\psi}{d\theta}\right)\right],$$

or,

$$\mathrm{D}\psi = \frac{d^2\psi}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right)$$
$$= \frac{d^2\psi}{dr^2} + \frac{1-\mu^2}{r}\frac{d^2\psi}{d\mu^2} \quad \ldots\ldots\ldots\ldots \quad (6),$$

if μ stands for $\cos\theta$.

Other applications will be found later.

Reverting now to the expression (1), it will be seen that the direct distance of any point from a point on the axis of symmetry, plays the same part in the theory of Stokes's Current Function that is played by its reciprocal in the theory of the potential function belonging to symmetrical distributions of matter.

Thus, if r_o, 0, r, θ be the coordinates of a point upon the axis, and of any other point, the distance between these points, $\sqrt{(r_o^2 - 2r_o r\cos\theta + r^2)}$, may be developed in a converged series, say

$$\sum_{n=0}^{n=\infty} -\frac{r^n}{r_o^{n-1}}\mathrm{I}_n(\cos\theta) \quad \text{or} \quad \sum_{n=0}^{n=\infty} -\frac{r_o^n}{r^{n-1}}\mathrm{I}_n(\cos\theta)$$

according as r_o is greater or less than r, $\mathrm{I}_n(\cos\theta)$ being a certain function of θ, and we see from (6) that

$$(1-\mu^2)\frac{d^2\mathrm{I}_n(\mu)}{d\mu^2} + n(n-1)\,\mathrm{I}_n(\mu) = 0 \quad \ldots\ldots \quad (7).$$

Now it is evident from the analogue of Zonal Harmonics, that it is proper to discuss the function $\mathrm{I}_n(\cos\theta)$, and other solutions of (7) before considering the applications of Stokes's Current Function to the motion of liquids. It is with this discussion that

the first three chapters are occupied, and, as might be expected, the theory closely resembles that of Spherical Harmonics. I have accordingly made free use of the order and methods adopted by HEINE, in his 'Handbuch d. Kugelfunctionen,' more especially in Chapters I. and II., where the necessary changes were slight. Moreover, the functions I deal with have themselves been discussed by HEINE, on a different method, and most of the expressions which I find in the following pages are given by him. Full references to these are given on p. 461.

The idea of developing the solutions of $D\psi = 0$ in a manner more or less analogous to that employed with regard to LAPLACE'S equation, appears to have been first used by O. E. MEYER,* who obtains the equation (7), shows that the functions contain $1-\mu^2$ as a factor, and that they obey (22A), Chapter II. An expression which shows the relation of the functions to Zonal Harmonics was given by Mr. BUTCHER,† and functions of fractional order have been used by Mr. HICKS,‡ in connection with his researches on the theory of the motion of vortex rings. The fuller account of such functions, which is found in the following pages, may be of interest in relation to these; for example, I would refer to p. 44.

The applications to Hydrodynamics, which I here give, are of mathematical interest rather than physical. They are chiefly in connection with the motion of viscous liquids. In 'Crelle-Borchardt,' vol. 81, OBERBECK has given the velocities produced in an infinite viscous liquid by the steady motion of an ellipsoid through it, in the direction of one of its axes, and from these Mr. HERMAN§ has found the equation of a family of surfaces containing the stream-lines relative to the ellipsoid. In Chapter VI., STOKES'S current function is obtained by a direct process for the flux of a viscous liquid past a spheroid, and it is shown that the result differs only by a constant multiple from the particular case of Mr. HERMAN'S integral.

Some minor applications are also given; namely, the solutions are obtained for flux past an approximate sphere, and past an approximate spheroid. The solution is also obtained for flux through a hyperboloid of one sheet, where it appears that the stream surfaces are hyperboloids of the confocal system. A particular case is that of flux through a circular hole in a wall, and this is interesting, because we see that by supposing internal friction to take place in the liquid, we find an expression which gives zero velocity at the sharp edge, and thus avoids the difficulty which is always present in the solution of such problems, on the supposition that the liquid is perfect. A comparison may be instituted between this problem and that of the effect of a disturbing periodic force upon a dynamical system capable of vibrating alone with a period equal to that of the force. It is well known that the amplitude of the vibration induced appears infinite, if we totally disregard friction, and this difficulty

* 'CRELLE,' vol. 73.

† 'London Math. Soc. Proc.,' vol. 8, see p. 49.

‡ 'Phil. Trans.,' 1884; 1885.

§ 'Quart. Journ. Math.,' 1889 (No. 92).

is met by the fact that the damping effect of even slight friction is increased considerably by high velocities. Now a viscous liquid can move irrotationally, and if there were no friction at the boundaries this is the class of motion it would take in cases of flux past or through obstacles. But if the obstacle terminated in a sharp edge, this would make the velocity there infinite, and the friction, however inconsiderable elsewhere, would here become of account. The boundary conditions which were necessary for the existence of irrotational motion throughout the liquid would no longer apply, and the whole character of the solution would be changed.

This would, at any rate, seem to apply to cases in which the whole motion is slow, and when, consequently, the boundary conditions which must hold are pretty well understood.

The paper concludes with an attempt to discuss the flux past a spheroid or through a hyperboloid at whose boundary there may be slipping. The current function is not obtained, all that appears being that it probably differs from the parallel case of the sphere in being far more complicated than when there is no slipping. From this we except the case of flux through a circular hole in a plane wall, when the solution for no slipping satisfies the new conditions.

Chapter I.—Various Forms of $I_n(x)$.

Where the method of development in the following pages does not necessarily differ from that of Heine, I confine myself to a statement of result, with an indication of the method, and a reference to the corresponding section in his 'Handbuch,' vol. 1. In one place the reference is to Ferrers's 'Spherical Harmonics.'

Let R denote $\sqrt{(1 - 2\alpha x + x^2)}$.

Then writing

$$\mathrm{R} = -\Sigma \alpha^n \mathrm{I}_n(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we find

$$\mathrm{I}_n(r) = \frac{1.3 \ldots 2n-3}{1.2 \ldots n}\left[x^n - \frac{n(n-1)}{2(2n-3)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-3)(2n-5)}x^{n-4} \ldots\right]$$

the series ending with x^1 or x^0,

$$\left.\begin{aligned} &= 2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}x^n \mathrm{F}\left(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, -n+\frac{3}{2}, x^{-2}\right) \\ &= 2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}x^{n-2}(x^2-1)\mathrm{F}\left(\frac{n}{2}+\frac{3}{2}, 1-\frac{n}{2}, \frac{3}{2}-n, x^{-2}\right) \end{aligned}\right\} \quad . \quad . \quad (9)$$

n even]

$$\left.\begin{aligned} &= (-1)^{n/2} \frac{\Pi\left(\frac{n}{2} - \frac{3}{2}\right)}{2\Pi\left(\frac{n}{2}\right)\Pi\left(-\frac{1}{2}\right)} \mathrm{F}\left(-\frac{n}{2}, \frac{n}{2} - \frac{1}{2}, \frac{1}{2}, x^2\right) \\ &= (-1)^{n/2} \frac{\Pi\left(\frac{n}{2} - \frac{3}{2}\right)}{2\Pi\left(\frac{n}{2}\right)\Pi\left(-\frac{1}{2}\right)} (1 - x^2) \mathrm{F}\left(\frac{n}{2} + \frac{1}{2}, -\frac{n}{2} + 1, \frac{1}{2}, x^2\right) \end{aligned}\right\} \quad . \quad . \quad (9\text{A})$$

n odd]

$$\left.\begin{aligned} &= (-1)^{(n-1)/2} \frac{\Pi\left(\frac{n}{2} - 1\right)}{\Pi\left(\frac{n}{2} - \frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)} x \mathrm{F}\left(-\frac{n}{2} + \frac{1}{2}, \frac{n}{2}, \frac{3}{2}, x^2\right) \\ &= (-1)^{(n-1)/2} \frac{\Pi\left(\frac{n}{2} - 1\right)}{\Pi\left(\frac{n}{2} - \frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)} x(1 - x^2) \mathrm{F}\left(\frac{n}{2} + 1, -\frac{n}{2} + \frac{3}{2}, \frac{3}{2}, x^2\right) \end{aligned}\right\} \quad . \quad . \quad (9\text{B})$$

There are two exceptional cases, viz.,

$$\mathrm{I}_1(x) = x; \qquad \mathrm{I}_0(x) = -1.$$

The first eleven functions are given in Chapter IV., Table II. [*Cf.* HEINE, § 4.]
If

$$x = \cos\theta, \quad \text{and} \quad \xi = x - \sqrt{(x^2 - 1)} = e^{i\theta},$$

and therefore

$$2\cos n\theta = \xi^n + \xi^{-n},$$

such a sign being given to the square root as shall make mod. $\xi > 1$, and therefore such a sign to $i\theta$ as shall make its real part, if not zero, negative, we have

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \frac{\Pi\left(n - \frac{3}{2}\right)}{2\Pi(n)\Pi\left(-\frac{1}{2}\right)} \xi^{-n} \mathrm{F}\left(-\frac{1}{2}, -n, -n + \frac{3}{2}, \xi^2\right) \\ \mathrm{I}_n(\cos\theta) &= \frac{\Pi\left(n - \frac{3}{2}\right)}{2\Pi(n)\Pi\left(-\frac{1}{2}\right)} \xi^{-n} (1 - \xi^2)^2 \mathrm{F}\left(-n + 2, \frac{3}{2}, -n + \frac{3}{2}, \xi^2\right) \end{aligned}\right\} \quad (10)$$

$$\begin{aligned} = \frac{\Pi\left(n - \frac{3}{2}\right)}{\Pi(n)\Pi\left(-\frac{1}{2}\right)} \Bigg[& 2\cos n\theta - \frac{2n}{2(2n-3)} \cdot 2\cos(n-2)\theta \\ & - \frac{2n(2n-2).1}{2.4.(2n-3)(2n-5)} \cdot 2\cos(n-4)\theta \\ & - \frac{2n(2n-2)(2n-4).1.3}{2.4.6(2n-3)(2n-5)(2n-1)} \cdot 2\cos(n-6)\theta \ldots \Bigg] \end{aligned} \quad (10\text{A})$$

n being in each case a positive integer. [HEINE, § 5.]

We shall show (p. 471)

$$\mathrm{I}_n(x) = \frac{\xi^{-n}}{\pi}\int_{\xi^2}^{1} u^{n-2}(1-u)^{\frac{1}{2}}(u-\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This form is equivalent to (10) when n is a positive integer.

Putting

$$\alpha y = 1 - \sqrt{(1 - 2\alpha x + \alpha^2)},$$

so that

$$y = x + \frac{\alpha}{2}(y^2 - 1);$$

whence, by the help of LAGRANGE'S theorem,

$$\mathrm{I}_n(x) = \frac{1}{\lfloor n-1}\left(\frac{d}{dx}\right)^{n-2}\left(\frac{x^2-1}{2}\right)^{n-1}. \quad [\text{HEINE, § 7}] \quad . \quad (12).$$

The distance between two points whose coordinates are ϖ, z; o, ζ; is

$$\begin{aligned}&\sqrt{\{\varpi^2 + (z-\zeta)^2\}}\\ &= r\sqrt{(1 - 2xh + h^2)},\end{aligned}$$

where

$$x = z/r, \qquad h = \zeta/r, \qquad r = \sqrt{(\varpi^2 + z^2)};$$

whence, by TAYLOR'S theorem,

$$\mathrm{I}_n(x) = (-1)^{n-1}\frac{r^{n-1}}{\lfloor n}\cdot\frac{d^n r}{dz^n} \quad [\text{FERRERS, § 5}] \quad . \quad . \quad (13).$$

HEINE shews, §§ 8, 9, that

$$\int_0^{\pi}\frac{d\eta}{\alpha x - 1 - \alpha\cos\eta\sqrt{(x^2-1)}} = -\frac{\pi}{\sqrt{(1-2\alpha x+\alpha^2)}},$$

provided mod. $\alpha\sqrt{x^2-1} <$ mod. $(\alpha x - 1)$, a result which is secured by taking α small enough.

Hence

$$\begin{aligned}&1 - \alpha x - \sqrt{(1 - 2\alpha x + \alpha^2)}\\ &= \frac{1}{\pi}\int_0^{\pi} d\eta\left[1 - \alpha x + \frac{1 - 2\alpha x + \alpha^2}{\alpha x - 1 - \alpha\sqrt{x^2-1}\,.\cos\eta}\right]\\ &= \frac{1}{\pi}\int_0^{\pi} d\eta\,.\,\frac{\alpha^2(1 - x^2 + x\sqrt{x^2-1}\,.\cos\eta) - \alpha\cos\eta\sqrt{x^2-1}}{\alpha(x - \cos\eta\sqrt{x^2-1}) - 1}.\end{aligned}$$

Let each expression be developed in ascending powers of α, and equate coefficients of α^n.

Then

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_0^{\pi}(x^2-1)\sin^2\eta\,(x-\cos\eta\sqrt{x^2-1})^{n-2}\,d\eta \quad . \quad . \quad . \quad (14),$$

$$= \frac{1}{\pi}\int_0^{\pi}(x^2-1)\sin^2\phi\,(x+\cos\phi\sqrt{x^2-1})^{-n-1}\,d\phi \quad . \quad . \quad (14\text{A}),$$

where

$$1 = (x-\cos\eta\sqrt{x^2-1})\,(x+\cos\phi\sqrt{x^2-1}) \quad [\text{HEINE}, \S\ 9].$$

The form (14A) is inadmissible when x is a pure imaginary; for suppose the path of integration of η to include real values only; then ϕ is complex and discontinuous, passing at a step from $0 + i\infty$ to $\pi + i\infty$.

The forms (14) and (14A) may be derived from (11) by substituting

$$u/\xi = x - \cos\eta\,\sqrt{x^2-1}.$$

From (14) we find a form analogous to one of MEHLER'S forms for the Zonal Harmonic.

Write

$$x - \cos\eta\,\sqrt{(x^2-1)} = e^{\theta},$$

where θ will in general be complex, and we shall suppose its imaginary part to lie between $\pm i\pi$.

Then

$$\sin\eta\,\sqrt{(x^2-1)} = \sqrt{(-1+2xe^{\theta}-e^{2\theta})}$$

if such a sign is given to the square root at the right as shall make its real part of like sign into $\sqrt{(x^2-1)}$.

Thus

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_{\theta_0}^{\theta_1} d\theta\,\sqrt{\{2\,(x-\cosh\theta)\}}\;e^{(n-\frac{1}{2})\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where

$$x - \sqrt{(x^2-1)} = e^{\theta_0}$$
$$x + \sqrt{(x^2-1)} = e^{\theta_1}.$$

Thus if $x = \cosh\vartheta$, and the imaginary part of ϑ lie between $\pm i\pi$,

$$\theta_0 = -\vartheta, \qquad \theta_1 = +\vartheta,$$

and

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_{-\vartheta}^{\vartheta} d\theta\,\sqrt{\{2\,(x-\cosh\theta)\}}\,[\cosh(n-\tfrac{1}{2})\,\theta + \sinh(n-\tfrac{1}{2})\,\theta]$$

$$= -\frac{2}{\pi}\int_0^{\cosh^{-1}x} d\theta\,\sqrt{\{2\,(x-\cosh\theta)\}}\,\cosh(n-\tfrac{1}{2})\,\theta \quad . \quad . \quad . \quad . \quad (15\text{A}),$$

where $\sinh(\cosh^{-1}x) = -\sqrt{(x^2-1)}$, and the imaginary part of $\cosh^{-1}x$ lies between $\pm i\pi$.

Write $i\phi$ for θ

$$I_n(x) = -\frac{2}{\pi}\int_0^{\cos^{-1}x} d\phi\,\sqrt{\{2(\cos\phi - x)\}}\cos(n-\tfrac{1}{2})\phi \quad . \quad . \quad . \quad (15\text{B}),$$

where $\sin(\cos^{-1}x) = \sqrt{(1-x^2)}$, and the real part of $\cos^{-1}x$ lies between $\pm\pi$.

Another form may be obtained when x is real and lies between ± 1, and n is a positive integer, provided we make the same assumption that underlies the ordinary process of finding DIRICHLET'S forms for the Zonal Harmonic, viz. :—

$$I_n(x) = \frac{2}{\pi}\int_{\cos^{-1}x}^{\pi} \sin(n-\tfrac{1}{2})\phi\,\sqrt{\{2(x-\cos\phi)\}}\,d\phi. \quad [\text{HEINE, §11}] \quad . \quad (15\text{C}).$$

We have (p. 453),

$$(1-x^2)\frac{d^2I_n(x)}{dx^2} + n(n-1)I_n(x) = 0.$$

From this equation we obtain the forms (9), (9A), (9B), (10), directly.

1st. Put

$$I_n(x) = x^m z, \quad x^2 = \frac{1}{t};$$

then

$$t(1-t)\frac{d^2z}{dt^2} + [\tfrac{3}{2} - m - (\tfrac{3}{2}-m)t]\frac{dz}{dt} - \frac{m(m-1)}{4}z = 0 \quad . \quad . \quad . \quad (7\text{A}),$$

if

$$(m-n)(m+n-1) = 0;$$

of this

$$F\left(-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}, \frac{3}{2}-m, t\right)$$

is a solution.

2nd. Put

$$x^2 = t;$$

$$t(1-t)\frac{d^2I_n}{dt^2} + \left(\frac{1}{2}-\frac{t}{2}\right)\frac{dI_n}{dt} + \frac{n(n-1)}{4}I_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (7\text{B}),$$

of which

$$F\left(-\frac{n}{2}, \frac{n}{2}-\frac{1}{2}, \frac{1}{2}, t\right)$$

is a solution.

3rd. Put

$$I_n = xz, \quad x^2 = t,$$

$$t(1-t)\frac{d^2z}{dt^2} + \left(\frac{3}{2}-\frac{3t}{2}\right)\frac{dz}{dt} + \frac{n(n-1)}{4}z = 0 \quad . \quad . \quad . \quad . \quad . \quad (7\text{C}),$$

of which

$$\mathrm{F}\left(-\tfrac{n}{2}+\tfrac{1}{2},\ \tfrac{n}{2},\ \tfrac{3}{2},\ t\right)$$

is a solution.

4th. Put

$$\xi = x - \sqrt{(x^2-1)}, \quad \mathrm{I}_n = \xi^m z, \quad t = \xi^2.$$

Then

$$t(1-t)\frac{d^2z}{dt^2} + [m+\tfrac{3}{2} - (m+\tfrac{1}{2})t]\frac{dz}{dt} + \frac{m}{2}z = 0 \quad . \quad . \quad . \quad . \quad (7\text{D})$$

if

$$(m+n)(m-n+1) = 0.$$

Of this

$$\mathrm{F}(-\tfrac{1}{2},\quad m,\quad m+\tfrac{3}{2},\quad t)$$

is a solution.

From the differential equation (7), we may form a conception of $\mathrm{I}_n(x)$, where n is not restricted to integral values; the discussion is found in Chapter III., where reasons are given for choosing (11), or its derived and equally general forms (14), (15), (15A), or (15B) as the definition of $\mathrm{I}_n(x)$. These are identical with (9), (10), when n is a positive integer, and not otherwise.

$\mathrm{I}_n(\pm 1) = 0$. This is obvious from (11). When n is a positive integer all the roots of $\mathrm{I}_n(x) = 0$ are real, different, and lie between ± 1, and excepting zero, occur in pairs of the form $\pm\alpha$. This may be seen from (12). [HEINE, § 7.]

The sequence equation

$$(n+1)\mathrm{I}_{n+1}(x) - (2n-1)x\mathrm{I}_n(x) + (n-2)\mathrm{I}_{n-1}(x) = 0 \quad . \quad . \quad (16)$$

connects three consecutive functions; this follows readily from (15B); n is not necessarily an integer.

We also find—a result required hereafter—

$$x^2\mathrm{I}_n(x) = \delta_n \mathrm{I}_{n+2}(x) + \epsilon_n \mathrm{I}_n(x) + \zeta_n \mathrm{I}_{n-2}(x) \quad . \quad . \quad . \quad . \quad (17),$$

where

$$\delta_n = \frac{(n+1)(n+2)}{(2n-1)(2n+1)},$$

$$\epsilon_n = \frac{2n^2-2n-3}{(2n+1)(2n-3)},$$

$$\zeta_n = \frac{(n-2)(n-3)}{(2n-1)(2n-3)};$$

exceptions,

$$\zeta_1 = 0, \qquad \zeta_0 = 0.$$

We have for arbitrary n

$$\mathrm{P}_n(x) = \frac{2}{\pi}\int_0^{\cos^{-1}x} d\phi \cdot \frac{\cos(n+\tfrac{1}{2})\phi}{\sqrt{\{2(\cos\phi - x)\}}}.$$

Whence, comparing with (15B),

$$\frac{dI_n(x)}{dx} = P_{n-1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Other relations, of use in transformations, may be introduced here

$$(2n-1)\,I_n = P_n - P_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$I_n = -P_n + 2xP_{n-1} - P_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$(1-x^2)\frac{dI_n}{dx} = (n-1)\,x\,I_n - (n+1)\,I_{n+1}$$

$$= -nx\,I_n + (n-2)\,I_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

These hold without restriction upon n.

We see from (18) that

$$I_n(x) = \frac{1.3\ldots 2n-3}{1.2\ldots n}\cdot \mathfrak{P}_1^{(n-1)}(x),$$

where $\mathfrak{P}_1^{(n-1)}(x)$ is defined and discussed by HEINE (Chapter III., §§ 30–33, and Chapter IV.), who obtains equivalents of the following results:—(9), (10), in § 51; (12) in § 31; (14), (14A) in § 50. Also the following, which are proved below:—(28), (32) in § 51; (33) and equivalents of (34), (35) in § 52; (22), (23) in § 62.

These might, therefore, have been assumed and the theory based upon them, but it seemed preferable to indicate briefly the direct development.

CHAPTER II.—DEVELOPMENTS IN TERMS OF $I(x)$.

From equation (7) we find

$$(m-n)(m+n-1)\int_{-1}^{+1}\frac{I_m(x)\,I_n(x)}{1-x^2}dx = \left[I_m(x)\frac{dI_n(x)}{dx} - I_n(x)\frac{dI_m(x)}{dx}\right]_{-1}^{+1} \quad (22).$$

Whence

$$\int_{-1}^{+1}\frac{I_m(x)\,I_n(x)}{1-x^2}dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22A),$$

when $m \neq n$, and neither of them is 0 or 1.

The limiting value of this integral when $m = n$ is

$$\int_{-1}^{+1}\frac{I_n(x)\,I_n(x)}{1-x^2}dx = \frac{1}{2n-1}\left[\frac{dI_n}{dn}\cdot P_{n-1} - I_n\frac{dP_{n-1}}{dn}\right]_{-1}^{+1} = \frac{1}{2n-1}\cdot\left[\frac{dI_n}{dn}\cdot P_{n-1}\right]_{-1}^{+1}.$$

Using (15B),

$$\frac{dI_n}{dn} = \frac{2}{\pi}\int_0^{\cos^{-1}x}\phi\,.\sin\left(n-\tfrac{1}{2}\right)\phi\,\sqrt{\{2(\cos\phi - x)\}}\,d\phi.$$

This vanishes at the upper limit, and at the lower gives us

$$\frac{2}{\pi}\int_0^{\pi}\phi\,.\,\sin\,(n-\tfrac{1}{2})\phi\,.\,2\cos\frac{\phi}{2}\,.\,d\phi.$$

Integrating, we get the required expression

$$-\frac{\mathrm{P}_{n-1}(-1)}{2n-1}\cdot\left\{\frac{2\cos n\pi}{n\,(n-1)}-\frac{2\sin n\pi\,.\,(2n-1)}{n^2\,(n-1)^2}\right\}$$

and when n is a positive integer

$$\int_{-1}^{-1}\frac{\mathrm{I}_n(x)\,\mathrm{I}_n(x)}{1-x^2}\,dx=\frac{2}{n\,(n-1)(2n-1)}\quad\ldots\ldots\quad(23).$$

Every member of the set of functions, $\mathrm{I}_2(x)$, $\mathrm{I}_3(x)$, . . . $\mathrm{I}_n(x)$ . . . , vanishes with $x+1$ and $x-1$; besides, $\mathrm{I}_n(x)$ has $n-2$ real factors lying between these two; and any two members of the set obey (22A). Now, if we have another function, $\phi(x)$, also vanishing with $x+1$ and $x-1$, it has been shown* that we can find a linear function of the I's, $\sum_2^{\infty}\mathrm{A}_n\mathrm{I}_n(x)$, which shall be equal to $\phi(x)$ for all values of x between $+1$ and -1, and

$$\mathrm{A}_n=\int_{-1}^{+1}dx\,.\,\phi(x)\,.\,\mathrm{I}_n(x)/(1-x^2)\Big/\int_{-1}^{+1}dx\,.\,\mathrm{I}_n(x)\,\mathrm{I}_n(x)/(1-x^2).$$

Or,

$$\phi(x)=\sum_{n=2}^{n=\infty}\frac{n\,(n-1)\,(2n-1)}{2}\cdot\int_{-1}^{+1}dy\;\phi(y)\frac{\mathrm{I}_n(y)\,\mathrm{I}_n(x)}{1-y^2},\quad\ldots\quad(24)$$

where

$$\phi(\pm 1)=0,\qquad\text{and}\qquad 1>x>-1$$

It is easy to prove that this development is unique; I shall now show when it is convergent.

From the sequence equation, (16), we find

$$\begin{aligned}(n+1)\,n\,(n-1)\,[\mathrm{I}_{n+1}(x)\,\mathrm{I}_n(y)-\mathrm{I}_n(x)\,\mathrm{I}_{n+1}(y)]\\-n\,(n-1)\,(n-2)\,[\mathrm{I}_n(x)\,\mathrm{I}_{n-1}(y)-\mathrm{I}_{n-1}(x)\,\mathrm{I}_n(y)]\\=(x-y)\,.\,n\,(n-1)\,(2n-1)\,\mathrm{I}_n(x)\,\mathrm{I}_n(y)\end{aligned}$$

from $n=2$, onwards; and, therefore, the sum of the series (24) to n terms is

$$\tfrac{1}{2}\int_{-1}^{+1}dy\,.\,\phi(y)\,.\,(n+1)\,n\,(n-1)\,.\,\frac{\mathrm{I}_{n+1}(x)\,\mathrm{I}_n(y)-\mathrm{I}_n(x)\,\mathrm{I}_{n+1}(y)}{(x-y)\,(1-y^2)}\,.$$

* 'Liouville,' vol. 1. The method is quoted, substantially, by Lord RAYLEIGH, 'Theory of Sound,' vol. 1, ch. 6.

Similarly, it may be shown that the sum of n terms of the development of $f(x)$ in Zonal Harmonics is

$$\tfrac{1}{2}\int_{-1}^{+1} dy\, f(y)\,(n+1)\,\frac{\mathrm{P}_{n+1}(x)\,\mathrm{P}_n(y) - \mathrm{P}_n(x)\,\mathrm{P}_{n+1}(y)}{x-y}.$$

And it is shown by HEINE (vol. 1, § 119), that the limit of this, when n is infinite is the mean $\frac{1}{2}\{f(x+0)+f(x-0)\}$, provided $f(x)$ is not a function with an infinite number of maxima and minima.

Moreover, (HEINE, vol. 1, § 40), when n is very great

$$\mathrm{P}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}} n^{\frac{1}{2}}},$$

and it will be shown, (p. 476), when n is very great

$$\mathrm{I}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}} n^{\frac{3}{2}}},$$

where $\xi = x - \sqrt{(x^2-1)}$, as on p. 456.

Therefore, the two integrals above tend respectively to the same limits, as

$$\frac{1}{2\pi}\int_{-1}^{+1} \phi(y)\,\frac{(\xi\eta)^{-n}}{\xi\eta - 1}\,\frac{(1-\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\,d\eta,$$

$$\frac{1}{2\pi}\int_{-1}^{+1} f(y)\cdot\frac{(\xi\eta)^{-n-1}}{\xi\eta - 1}\,\frac{\xi}{\eta}\,\frac{(1-\eta^2)^{\frac{1}{2}}}{(1-\xi^2)^{\frac{1}{2}}}\,d\eta,$$

where $\eta = y - \sqrt{(y^2-1)}$.

The limit of the second we know, and if we write

$$f(y) = \phi(y)\,\frac{\eta}{\xi}\,\frac{1-\xi^2}{1-\eta^2} = \phi(y)\,\frac{\sqrt{(1-x^2)}}{\sqrt{(1-y^2)}},$$

the second reduces to the first. Therefore the limit of the first is the mean value of $\phi(x)$ at the point x, provided $\phi(x)/\sqrt{(1-x^2)}$ is a finite function of x, which does not possess an infinite number of maxima and minima.

We find the function

$$\tfrac{1}{2} + \frac{x}{2} + \sum_{r=1}^{r=\infty} \mathrm{I}_{2r}(x)\,\frac{4r-1}{2}\cdot\frac{n(n-2)\ldots(n-2r+2)}{(n+2r-1)(n+2r-3)\ldots(n+1)}$$
$$+ \sum_{r=1}^{r=\infty} \mathrm{I}_{2r+1}(x)\,\frac{4r+1}{2}\cdot\frac{(n-1)(n-3)\ldots(n-2r+1)}{(n+2r)(n+2r-2)\ldots(n+2)} \quad . \quad . \quad (25)$$

is one which is equal to x^n when x is positive and to zero when x is negative, for all values of n.

When n is a positive integer

$$x^n = \frac{1.2\ldots n}{1.3\ldots 2n-1}\cdot\Big[(2n-1)\,\mathrm{I}_n(x) + (2n-5)\frac{2n-1}{2}\cdot\mathrm{I}_{n-2}(x) + (2n-9)\frac{(2n-1)(2n-3)}{2.4}\cdot\mathrm{I}_{n-4} + \ldots\Big] \quad (25\text{A}),$$

whether x is positive or negative; this equation is in fact an identity.

And if

$$\mathrm{F}(x) = c_0 + c_1x + c_2x^2 + \ldots$$

we may write

$$\mathrm{F}(x) = b_0\mathrm{I}_0(x) + b_1\mathrm{I}_1(x) + c_2\mathrm{I}_2(x) + \ldots,$$

where

$$b_n = \frac{1.2\ldots n}{1.3\ldots 2n-3}\Big[c_n + \frac{(n+1)(n+2)}{2(2n+1)}c_{n+2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+1)(2n+3)}c_{n+4}\cdots\Big] \quad (25\text{B})$$

from $n = 2$, onwards, while

$$-b_0 + b_1 = \mathrm{F}(1)\ ; \quad -b_0 - b_1 = \mathrm{F}(-1). \quad [\text{Heine, § 16.}]$$

We have

$$\cos m\theta = \cos\frac{m\pi}{2}\cos m\Big(\frac{\pi}{2}-\theta\Big) + \sin\frac{m\pi}{2}\sin m\Big(\frac{\pi}{2}-\theta\Big),$$

$$\sin m\theta = \sin\frac{m\pi}{2}\cos m\Big(\frac{\pi}{2}-\theta\Big) - \cos\frac{m\pi}{2}\sin m\Big(\frac{\pi}{2}-\theta\Big);$$

and if $\cos\theta = x$, we find

$$\cos m\Big(\frac{\pi}{2}-\theta\Big) = \cos\frac{m\pi}{2}\Big[-\mathrm{I}_0(x) - 3\cdot\frac{m^2}{m^2-1^2}\cdot\mathrm{I}_2(x) + 7\cdot\frac{m^2.m^2-2^2}{m^2-1^2.m^2-3^2}\cdot\mathrm{I}_4(x) - \ldots\Big]$$

$$\sin m\Big(\frac{\pi}{2}-\theta\Big) = \sin\frac{m\pi}{2}\Big[\mathrm{I}_1(x) - 5\cdot\frac{m^2-1^2}{m^2-2^2}\cdot\mathrm{I}_3(x) + 9\cdot\frac{m^2-1^2.m^2-3^2}{m^2-2^2.m^2-4^2}\cdot\mathrm{I}_5(x) - \ldots\Big] \quad (26).$$

When m is a positive integer

$$\cos m\theta = \frac{2.4\ldots 2m-2}{1.3\ldots 2m-1}\Big[(2m-1)\,\mathrm{I}_m(\cos\theta) + \frac{1.2m-1}{2.2m-2}(2m-5).\,\mathrm{I}_{m-2}(\cos\theta) + \frac{1.3.2m-1.2m-3}{2.4.2m-2.2m-4}(2m-9)\,\mathrm{I}_{m-4}(\cos\theta)+\ldots\Big] \quad . \quad (26\text{A}).$$

Where m is even, m^2 occurs in the numerator; where m is odd, m is absent from the denominator. [Heine, § 19.]

We have, by (19)

$$\mathrm{P}_n(x) = (2n-1)\,\mathrm{I}_n(x) + \mathrm{P}_{n-2}(x) = (2n-1)\,\mathrm{I}_n(x) + (2n-5)\,\mathrm{I}_{n-2}(x) + \ldots$$

Hence, if we can effect the development

$$\mathrm{F}(x) = c_0\mathrm{P}_0(x) + c_1\mathrm{P}_1(x) + c_2\mathrm{P}_2(x) + \ldots.$$

we can write

$$\mathrm{F}(x) = b_0\mathrm{I}_0(x) + b_1\mathrm{I}_1(x) + b_2\mathrm{I}_2(x) + \ldots,$$

where

$$b_n = (2n-1)(c_n + c_{n+2} + \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

If we develop $(1-yx)/(y-x)$ in ascending powers of x, and then transform by (25B), we get

$$\frac{1-yx}{y-x} = -x + \sum_{n=2}^{n=\infty} n(n-1)(2n-1)\,\mathrm{H}_n(y)\,\mathrm{I}_n(x),$$

where

$$\mathrm{H}_n(y) = -\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{2^n\Pi(n-\frac{1}{2})}\,y^{-n+1}\,\mathrm{F}\left(\frac{n-1}{2},\ \frac{n}{2},\ n+\tfrac{1}{2},\ y^{-2}\right) \quad . \quad (28).$$

The convergency of this development is discussed on p. 476.

In future references to the expression $\mathrm{H}_n(y)$ we shall not in general suppose n to be a positive integer. On comparing it with the expression (9) for $\mathrm{I}_n(x)$, it is seen that the former may be derived from the latter by changing x into y, n into $-n+1$, and multiplying by a constant factor.

Since a similar change transforms

$$(1-x^2)\frac{d^2z}{dx^2} + n(n-1)z = 0 \quad \text{into} \quad (1-y^2)\frac{d^2z}{dy^2} + n(n-1)z = 0,$$

we see that $\mathrm{H}_n(x)$ is another solution of (7), a result at which we have already arrived on p. 459.

There are two independent solutions to the equation (7), whatever the value of the

independent variable may be, but (28) ceases to be a permissible expression for one of them when mod. $y^2 < 1$.

For mod. $y^2 = 1$, the series is convergent, since $(n-1)/2 + n/2 - (n + \frac{1}{2})$ is negative, and we have, by the well-known expression for the sum of a hypergeometric series of which the fourth element is unity,

$$\mathrm{H}_n(1) = -\frac{1}{n(n-1)} \quad \ldots\ldots\ldots\ldots \quad (29).$$

The function $\mathrm{H}_n(x)$ is of odd degree in x when n is even, and of even degree when n is odd; it vanishes when x is infinite. For $n = 0$ or 1 the given expression fails, but if H is still a solution of (7), different from the corresponding I, we may put

$$\mathrm{H}_0(x) = x; \qquad \mathrm{H}_1(x) = 1 \quad \ldots\ldots\ldots\ldots \quad (30).$$

Certain functions which are not finite for all values of x between ± 1, may be developed in a series involving $\mathrm{H}(x)$ in place of $\mathrm{I}(x)$.

Thus

$$\frac{1}{x-y} = \sum_2^\infty n(n-1)(2n-1)\,\mathrm{H}_n(x)\,\frac{\mathrm{I}_n(y)}{1-y^2} = \sum_2^\infty (2n-1)\,\mathrm{H}_n(x)\,\frac{d^2\mathrm{I}_n(y)}{dy^2};$$

differentiate $n-2$ times with respect to y, and then put $y = 0$, and we find

$$x^{-n+1} = -\frac{1.3\ldots 2n-3}{1.2\ldots n-2}\left[(2n-1)\,\mathrm{H}_n(x) - (2n+3)\,\frac{2n-1}{2}\,\mathrm{H}_{n+2}(x) + (2n+7)\,\frac{2n-1.2n+1}{2.4}\,\mathrm{H}_{n+4}(x) - \ldots\right] \quad (31),$$

and if

$$\mathrm{F}(x) = c_1x^{-1} + c_2x^{-2} + c_3x^{-3} + \ldots,$$

we may re-write it

$$= b_2\mathrm{H}_2(x) + b_3\mathrm{H}_3(x) + \ldots,$$

where

$$b_n = -\frac{1.3\ldots 2n-6}{1.2\ldots n-2}\cdot\left[c_{n-1} - \frac{n-2.n-3}{2.2n-3}\,c_{n-3} + \frac{n-2.n-3.n-4.n-5}{2.4.2n-3.2n-5}\,c_{n-5} - \ldots\right] \quad (31\text{A}),$$

provided the expansion should prove convergent. [Heine, § 21.]

A thorough discussion of the function H is given in the next chapter.

Chapter III.—On the Function $H_n(x)$.

The function $H_n(x)$ which, when mod. x is not less than 1, may be written

$$H_n(x) = -2^{-n}\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{\Pi(n-\frac{1}{2})}x^{-n+1}\,F\left(\frac{n-1}{2}, \frac{n}{2}, n+\tfrac{1}{2}, x^{-2}\right) \quad . \quad . \quad (28),$$

may be identified in any expression, in which mod. x may be as great as we please, as that solution of (7), which vanishes when mod. x is infinite, and is equal to $-\frac{1}{n(n-1)}$ when $x = 1$.

Thus, reverting to the expressions of p. 460, we see that, introducing a constant multiplier,

$$H_n(x) = -\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}\xi^{n-1}\,F(-\tfrac{1}{2}, n-1, n+\tfrac{1}{2}, \xi^2). \quad . \quad . \quad (32).$$

Again, by a well-known theorem

$$\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}F(-\tfrac{1}{2}, n-1, n+\tfrac{1}{2}, \xi^2) = \int_0^1 u^{n-2}(1-u)^{\frac{1}{2}}(1-u\xi^2)^{\frac{1}{2}}\,du.$$

Hence

$$H_n(x) = -\xi^{n-1}\int_0^1 u^{n-2}(1-u)^{\frac{1}{2}}(1-u\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad (33),$$

a form which may be compared with (11).

In (32) and (33) no restriction is placed upon the value of x, and n is supposed to be a real positive quantity, but not necessarily an integer.

From (33) we get a form analogous to (14).

For write $u\xi = x - \sqrt{(x^2-1)}\cosh\phi$; this gives, on substitution,

$$H_n(x) = -\int_0^{\phi_0} d\phi\,(x^2-1)\sinh^2\phi\,\{x - \sqrt{(x^2-1)}\cosh\phi\}^{n-2} \quad . \quad . \quad (34),$$

where

$$\phi_0 = \tfrac{1}{2}\log\frac{x+1}{x-1}.$$

The sign which we attach to the logarithm is immaterial; I shall take that sign which makes the real part, if any, positive; or, as we may write it,

$$\phi_0 = \tfrac{1}{2}\log\frac{x+1}{x-1} = \log\rho - i\sigma,$$

$$\rho > 1, \quad \text{and} \quad -\tfrac{1}{2}\pi < \sigma < \tfrac{1}{2}\pi.$$

In (34) write

$$(x - \sqrt{(x^2 - 1)}\cosh\phi)\,(x + \sqrt{(x^2 - 1)}\cosh\theta) = 1\,.$$

Substitute, and we get

$$\mathrm{H}_n(x) = -\int_0^\infty \frac{(x^2 - 1)\sinh^2\theta\, d\theta}{(x + \sqrt{x^2 - 1}\,.\cosh\theta)^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In this expression, x may not be negative and greater than unity.

The expression (34) may be made to furnish us with another, on the model of MEHLER'S form for $\mathrm{P}_n(x)$.

For, write

$$x - \sqrt{(x^2 - 1)}\cosh\phi = e^\chi.$$

Then

$$\sqrt{(x^2 - 1)}\sinh\phi = \sqrt{(e^{2\chi} - 2xe^\chi + 1)},$$

where the square root on the right has such a sign that its real part is of like sign with the real part of $\sqrt{(x^2 - 1)}\sinh\phi$.

Thus

$$\mathrm{H}_n(x) = -\int_{-\infty}^{\cosh^{-1}x} d\chi\,\sqrt{\{2\,(\cosh\chi - x)\}}\,e^{(n-\frac{1}{2})\chi} \quad . \quad . \quad . \quad . \quad (36),$$

where

$$\sinh(\cosh^{-1}x) = -\sqrt{(x^2 - 1)}.$$

This expression corresponds to (15), p. 458; (15) and (36) may be differentiated once with respect to x, but not twice.

We have, by p. 465, where n is a positive integer,

$$\sum_{n=2}^{n=\infty} n\,(n-1)\,(2n-1)\,\mathrm{H}_n(y)\,\mathrm{I}_n(x) = \frac{1 - yx}{y - x} + x = \frac{1 - x^2}{y - x} \quad . \quad . \quad (37).$$

Hence, multiplying by $y - x$,

$$-2\mathrm{I}_2(x)$$
$$= \sum_{n=2}^{n=\infty} n\,(n-1)\Big[(2n-1)y\,\mathrm{H}_n(y)\mathrm{I}_n(x) - \mathrm{H}_n(y)\,\{(n+1)\,\mathrm{I}_{n+1}(x) - (n-2)\,\mathrm{I}_{n-1}(x)\}\Big],$$

since

$$(2n-1)\,x\mathrm{I}_n(x) = (n+1)\,\mathrm{I}_{n+1}(x) + (n-2)\,\mathrm{I}_{n-1}(x) \quad . \quad . \quad . \quad (16).$$

Hence we get

$$(n+1)\,\mathrm{H}_{n+1}(y) - (2n-1)\,y\mathrm{H}_n(y) + (n-2)\,\mathrm{H}_{n-1}(y) = 0 \quad . \quad . \quad (38)$$

$$3\mathrm{H}_3(y) - 3y\mathrm{H}_2(y) + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38\text{A}).$$

A proof of (38) which is valid when n is not a positive integer, may be obtained from (36).

From (38) it follows that

$$x^2 H_n(x) = \delta_n H_{n+2}(x) + \epsilon_n H_n(x) + \zeta_n H_{n-2}(x) \quad . \quad . \quad . \quad . \quad (39),$$

where δ_n, ϵ_n, ζ_n are the quantities so called on p. 460; with the following exceptions

$$\left.\begin{aligned} x^2 H_3(x) &= \delta_3 H_5(x) + \epsilon_3 H_3(x) - \tfrac{1}{15} \\ x^2 H_2(x) &= \delta_2 H_4(x) + \epsilon_2 H_2(x) - \frac{x}{3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (39\text{A}).$$

If $Q_n(x)$ be the Zonal Harmonic of the second kind

$$Q_n(x) = \int_0^{\phi_0} d\phi \, \{x - \sqrt{(x^2-1)}\cosh\phi\}^n \, d\phi,$$

where

$$\phi_0 = \tfrac{1}{2}\log\frac{x+1}{x-1}. \quad [\text{Heine, § 36.}]$$

From this may be derived, as on p. 458 for $I_n(x)$, the expression

$$Q_n(x) = \int_{-\infty}^{\cosh^{-1}x} \frac{e^{(n+\frac{1}{2})\chi} d\chi}{\sqrt{\{2(\cosh\chi - x)\}}}$$

where $\sinh(\cosh^{-1}x) = -\sqrt{(x^2-1)}$.

Hence

$$\frac{d\,H_n(x)}{dx} = Q_{n-1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

and also

$$(2n-1)\,H_n(x) = Q_n(x) - Q_{n-2}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

Again, as on p. 461, we find

$$H_n(x) = -Q_n(x) + 2xQ_{n-1}(x) - Q_{n-2}(x) \quad . \quad . \quad . \quad . \quad . \quad (42),$$

and

$$H_n(x) = -\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{2^n \Pi(n-\frac{1}{2})} \mathfrak{D}_1^{(n)}(x),$$

where $\mathfrak{D}_1^{(n)}(x)$ is defined by Heine, § 31.

By means of the sequence equations

$$(n+1)\,I_{n+1}(x) - (2n-1)\,xI_n(x) + (n-2)\,I_{n-1}(x) = 0 \quad . \quad . \quad . \quad (16)$$

$$(n+1)\,H_{n+1}(x) - (2n-1)\,xH_n(x) + (n-2)\,H_{n-1}(x) = 0 \quad . \quad . \quad (38)$$

we see that

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \alpha_r \mathrm{I}_{n-r}(x) - \beta_r \mathrm{I}_{n-r-1}(x) \\ \mathrm{H}_n(x) &= \alpha_r \mathrm{H}_{n-r}(x) - \beta_r \mathrm{H}_{n-r-1}(x) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

where α_r, β_r are rational functions of degrees r, $r-1$ respectively in x. This holds whether n is integral or not. If n is a positive integer, we have, taking regard to the equations

$$\begin{aligned} 3\mathrm{I}_3(x) - 3x\mathrm{I}_2(x) &= 0 \\ 3\mathrm{H}_3(x) - 3x\mathrm{H}_2(x) + 1 &= 0 \end{aligned}$$

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \alpha_{n-2} \mathrm{I}_2(x) \\ \mathrm{H}_n(x) &= \alpha_{n-2} \mathrm{H}_2(x) + \beta_{n-2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43\text{A}).$$

But by (34)

$$\mathrm{H}_2(x) = -\int_0^{\frac{1}{2}\log(x+1)/(x-1)} (x^2-1)\sinh^2\phi \,.\, d\phi = \tfrac{1}{2}\mathrm{I}_2(x)\log\frac{x+1}{x-1} - \frac{x}{2} \quad . \quad (44\text{A}).$$

Hence we get the expression, n being a positive integer,

$$\mathrm{H}_n(x) = \tfrac{1}{2}\mathrm{I}_n(x)\log\frac{x+1}{x-1} - \mathrm{K}_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

where $\mathrm{K}_n(x)$ is a rational integral function of x, of degree $n-1$.

By substituting (44) in the equation (7)

$$(1-x^2)\frac{d^2\mathrm{H}_n(x)}{dx^2} + n(n-1)\mathrm{H}_n(x) - 0$$

we can obtain an expression for $\mathrm{K}_n(x)$ in terms of the I's, viz.,

$$\mathrm{K}_n(x) = \sum^{r=k} \frac{2(2n-4r+1)}{(2r-1)(n-r)}\left[1 - \frac{(2r-1)(n-r)}{n(n-1)}\right]\mathrm{I}_{n-2r+1} \quad . \quad . \quad (44\text{B}),$$

the series beginning with I_0 or I_1. [HEINE, § 26.]

The expression (44), where the logarithm is defined on p. 467, might be taken as the definition of $\mathrm{H}_n(x)$ when n is a positive integer, without restriction upon x; but when x is real and less than unity, it is advisable to modify it; viz., from (44) we get

$$\tfrac{1}{2}\mathrm{I}_n(x)\log\frac{1+x}{1-x} - \mathrm{K}_n(x) \pm \frac{i\pi}{2}\,.\,\mathrm{I}_n(x).$$

In this case, I take

$$H_n(x) = \tfrac{1}{2} I_n(x) \log \frac{1+x}{1-x} - K_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (44C),$$

a form which, along with $I_n(x)$, is capable of expressing any solution of (7).

The form (44) is easily derived from

$$H_n(x) = \tfrac{1}{2} \int_{-1}^{+1} \frac{I_n(y)}{x-y}\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44D),$$

which itself follows directly from the development (37). [HEINE, § 28.]

The two forms (33), (11), are simply transformations of the two solutions

$$y = \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-tu)^{-\alpha}\, du, \qquad \beta, \gamma-\beta, \text{ positive}$$

$$y = \int_1^{\frac{1}{t}} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-tu)^{-\alpha}\, du, \qquad \gamma-\beta, 1-\alpha, \text{ positive}$$

which, among others, JACOBI gives ('Crelle,' vol. 56) as solutions of

$$t(1-t)\frac{d^2y}{dt^2} + [\gamma - (\alpha+\beta+1)t]\frac{dy}{dt} - \alpha\beta y = 0.$$

See (7D) p. 460 ; $m = n-1$ in the first, and $= -n$ in the second. $t = \xi^2$, and JACOBI supposes it positive, but here that is not necessary.

Let us now coordinate the various solutions that have been obtained.

The different series used are only permissible expressions for the solutions of (7) so long as they are convergent.

Now $F(\alpha, \beta, \gamma, x)$ is convergent,

when $\alpha+\beta-\gamma$ is not positive, if mod. $x < 1$
when $\alpha+\beta-\gamma$ is negative, if mod. $x \not> 1$.

Hence

(9) (28) are permissible for mod. $x \not< 1$,
(9A) (9B) are permissible for mod. $x \not> 1$,
(10) (32) whatever x may be.

Now, by direct development, we know, ($\beta, \gamma-\beta$, positive),

$$\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\, du = \frac{\Pi(\beta-1)\,\Pi(\gamma-\beta-1)}{\Pi(\gamma-1)} F(\alpha, \beta, \gamma, x).$$

Hence (33) $\equiv$ (32), and, therefore, $\equiv$ (31), when this last is a permissible expression.

But (11) $\not\equiv$ (9), or (10), unless n is a positive integer. For (11) $\equiv$ (14) and from

$$(14) \qquad \operatorname*{Lt}_{x=1} \frac{I_n(x)}{x^2-1} = \tfrac{1}{2}.$$

$$(9) \qquad \frac{I_n(x)}{x^2-1} = -2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\,x^n\,F\left(\tfrac{3}{2}-\frac{n}{2},\ 1-\frac{n}{2},\ \tfrac{3}{2}-n,\ x^{-2}\right).$$

$$(10) \qquad \frac{I_n(x)}{x^2-1} = 2\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\,\xi^{-n+2}\,F(2-n,\ \tfrac{3}{2},\ \tfrac{3}{2}-n,\ \xi^2).$$

But the series on the right are divergent for $x = 1$. Similarly (9A) and (9B) do not agree with (11).

Taking the definitions (11) and (33) for $I_n(x)$ and $H_n(x)$, we find

$$I_n(x) = e^{in\pi}\,I_n(-x) \qquad H_n(x) = e^{i(n-1)\pi}\,H_n(-x) \quad . \quad . \quad . \quad . \quad (A)$$

$$I_n(\pm 1) = 0 \qquad H_n(\pm 1) = -\frac{1}{n(n-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$$I_n(0) = \frac{2}{\pi}\frac{\Pi(\frac{1}{2}n-\frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\cdot\cos\frac{n\pi}{2} \qquad H_n(0) = \frac{\Pi(\frac{1}{2}n-\frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\left(i\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right) \quad (C)$$

$$\operatorname*{Lt}_{x=\infty} x^{-n}\,I_n(x) = 2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})} \qquad \operatorname*{Lt}_{x=\infty} x^{n-1}\,H_n(x) = -2^{-n}\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{\Pi(n-\frac{1}{2})} \quad (D)$$

Add to these, when n is a positive integer,

n odd $\qquad \operatorname*{Lt}_{x=0}\dfrac{I_n(x)}{x} = P_{n-1}(0) = (-1)^{\frac{1}{2}(n-1)}\dfrac{\Pi(\frac{1}{2}n-1)}{\Pi(\frac{1}{2}n-\frac{1}{2})\,\Pi(-\frac{1}{2})}$

n even $\qquad \operatorname*{Lt}_{x=0}\dfrac{H_n(x)}{x} = Q_{n-1}(0) = (-1)^{\frac{1}{2}n}\dfrac{\Pi(\frac{1}{2}n-1)\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n-\frac{1}{2})}$ $\Bigg\}$ . . . (E)

[Heine, §§ 4, 25.]

These results will be quoted as . . . . . (45)

We can now give the expression for $H_n(x)$ in ascending powers of x. For the expressions (9A) and (9B) satisfy the equation (7), whether n is odd or even; and where $I_n(x)$ can be developed in even powers of x, $H_n(x)$ can be developed in odd powers, and conversely. The constant factor is determined by (45); hence

n even

$$H_n(x) = (-1)^{\frac{1}{2}n} \frac{\Pi(\frac{1}{2}n - 1)\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n - \frac{1}{2})}\, xF\left(-\frac{n}{2} + \tfrac{1}{2}, \frac{n}{2}, \tfrac{3}{2}, x^2\right) \quad . \quad . \quad . \quad (28A),$$

n odd

$$H_n(x) = (-1)^{\frac{1}{2}(n+1)} \frac{\Pi(\frac{1}{2}n - \frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\, F\left(-\frac{n}{2}, \frac{n}{2} - \tfrac{1}{2}, \tfrac{1}{2}, x^2\right) \quad . \quad . \quad (28B).$$

These apply when mod. $x \not> 1$; and only when n is integral; for we see from (45A) that neither $I_n(x)$ nor $H_n(x)$ can be expanded in a series of integral powers of x when n is not an integer.

Two functions, R_n, T_n, have been used by Mr. Hicks* in connection with his researches on the motion of circular vortices, which are such that

$$R_n = \frac{4n^2 - 1}{4} I_{n+\frac{1}{2}}(\cosh u),$$

$$T_n = -\frac{4n^2 - 1}{4} H_{n+\frac{1}{2}}(\cosh u).$$

Mr. Basset† employs a function L_n, where

$$L_n(\xi) = (-1)^n \frac{2}{\pi} . \xi^{\frac{1}{2}} H_{n+\frac{1}{2}}(x),$$

and finds the sequence equation corresponding to (38), and the values of L_0, L_1 in elliptic integrals.

Mr. Hicks gives equations which are equivalent to (18) and (40); approximate expressions for the first five R's and first four T's in terms of $k = e^{-u}$ (or, if we write $x = \cosh u$, k is the ξ of p. 456); and an expression for T_n, which leads to

$$H_{n+\frac{1}{2}}(x) = \int_0^\pi \cos n\theta \sqrt{\{2(x - \cos\theta)\}}\, d\theta \quad . \quad . \quad . \quad . \quad . \quad (46).$$

A proof of (46) will be given in the next chapter (p. 483); but I will here show that it applies only when n is a positive integer.

For

$$H_{n+\frac{3}{2}}(x) + H_{n-\frac{1}{2}}(x) = 2xH_{n+\frac{1}{2}}(x) - 2\sqrt{2}\int_0^\pi \cos n\theta\,(x - \cos\theta)^{\frac{1}{2}}\, d\theta.$$

* 'Phil. Trans.,' 1884, 1885.

† 'Hydrodynamics,' vol. 1, § 115.

Now

$$2\sqrt{2}\int_0^\pi \cos n\theta\,(x-\cos\theta)^{\frac{3}{2}}\,d\theta = \left[\frac{2\sqrt{2}}{n}\sin n\theta\,(x-\cos\theta)^{\frac{3}{2}}\right]_0^\pi + \frac{3}{2n}\{H_{n+\frac{3}{2}}(x) - H_{n-\frac{1}{2}}(x)\}.$$

Now, *if n is a positive integer*, the expression in [ ] vanishes at both limits, and we find the sequence equation

$$(n+\tfrac{3}{2})\,H_{n+\frac{3}{2}}(x) - 2nxH_{n+\frac{1}{2}}(x) + (n-\tfrac{3}{2}).H_{n-\frac{1}{2}}(x) = 0 \quad . \quad . \quad . \quad (38).$$

We see from p. 470 that we may write

$$I_{n+\frac{1}{2}}(x) = \alpha_{n-1}I_{\frac{3}{2}}(x) - \beta_{n-1}I_{\frac{1}{2}}(x)$$
$$H_{n+\frac{1}{2}}(x) = \alpha_{n-1}H_{\frac{3}{2}}(x) - \beta_{n-1}H_{\frac{1}{2}}(x),$$

where, n being a positive integer, α_{n-1}, β_{n-1} are rational integral functions of x of degrees $n-1$, $n-2$, respectively; and $I_{\frac{3}{2}}$, $I_{\frac{1}{2}}$, $H_{\frac{3}{2}}$, $H_{\frac{1}{2}}$ may be expressed in terms of complete elliptic integrals of the first and second kinds.

For we have, by (14A)

$$I_{\frac{1}{2}}(x) = \frac{1}{\pi}\int_0^\pi \frac{(x^2-1)\sin^2\theta\,d\theta}{(x+\sqrt{x^2-1}\,.\cos\theta)^{\frac{3}{2}}}$$
$$= -\frac{2}{\pi}\int_0^\pi \frac{\sqrt{x^2-1}\,.\cos\theta\,d\theta}{(x+\sqrt{x^2-1}\cos\theta)^{\frac{3}{2}}}$$
$$= -\frac{2}{\pi}\int_0^\pi \left\{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}} - \frac{x}{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}}\right\}d\theta,$$

and

$$x+\cos\theta\sqrt{(x^2-1)} = \frac{1}{\xi}\left\{1-k'^2\sin^2\frac{\theta}{2}\right\}$$

where

$$k'^2 = 1-\xi^2,$$

$$\therefore\ I_{\frac{1}{2}}(x) = -\frac{2}{\pi}\left[\xi^{-\frac{1}{2}}\,E\left(k',\frac{\pi}{2}\right) - x\xi^{\frac{1}{2}}F\left(k',\frac{\pi}{2}\right)\right] \quad . \quad . \quad . \quad (47).$$

Also, by (14)

$$I_{\frac{3}{2}}(x) = \frac{1}{\pi}\int_0^\pi \frac{(x^2-1)\sin^2\theta\,d\theta}{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}}$$
$$= \frac{2}{\pi}\int_0^\pi \sqrt{x^2-1}\cos\theta\,(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}\,d\theta,$$

and

$$(x^2-1)\sin^2\theta = -\sqrt{x^2-1}\cos\theta\,(x+\sqrt{x^2-1}\,.\cos\theta)$$
$$+ x(x+\sqrt{x^2-1}\,.\cos\theta) - 1$$

$$\therefore \quad I_{\frac{3}{2}}(x) = -\tfrac{1}{2}I_{\frac{3}{2}}(x) + \frac{x}{\pi}\int_0^{\pi}(x + \cos\theta\sqrt{x_2 - 1})^{\frac{1}{2}}\,d\theta$$

$$- \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(x + \cos\theta\sqrt{x^2-1})^{\frac{1}{2}}},$$

or

$$I_{\frac{3}{2}}(x) = \frac{2}{3\pi}\left[x\xi^{-\frac{1}{2}}\,\mathrm{E}\left(k', \frac{\pi}{2}\right) - \xi^{\frac{1}{2}}\mathrm{F}\left(k', \frac{\pi}{2}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (47\text{A}).$$

Also, using the results given by Mr. BASSET ('Hydrodynamics,' vol. 1, p. 108), we have

$$H_{\frac{1}{2}}(x) = \quad 2\left[\xi^{-\frac{1}{2}}\,\mathrm{E}\left(\xi, \frac{\pi}{2}\right) - \xi^{\frac{1}{2}}\sqrt{(x^2-1)}\,\mathrm{F}\left(\xi, \frac{\pi}{2}\right)\right]$$

$$H_{\frac{3}{2}}(x) = -\tfrac{2}{3}\left[\xi^{-\frac{1}{2}}\,x\,\mathrm{E}\left(\xi, \frac{\pi}{2}\right) - \xi^{-\frac{1}{2}}\sqrt{(x^2-1)}\,\mathrm{F}\left(\xi, \frac{\pi}{2}\right)\right] \quad . \quad . \quad . \quad (48).$$

The expressions (48) give very little new information about the functions H, since we have already obtained in (32) the expression for $H_n(x)$ in an ascending series of powers of ξ; but (47) throws some light on the difference between the forms (10) and (11); for having regard to the expressions (CAYLEY'S 'Elliptic Functions,' Chapter III., § 77),

$$\mathrm{F}\left(\sqrt{1-\xi^2}, \frac{\pi}{2}\right) = \log\frac{4}{\xi} + \frac{1^2}{2^2}\xi^2\left[\log\frac{4}{\xi} - \frac{2}{1.2}\right] + \frac{1^2.3^2}{2^2.4^2}\xi^4\left[\log\frac{4}{\xi} - \frac{2}{1.2} - \frac{2}{3.4}\right] + \&c.,$$

$$\mathrm{E}\left(\sqrt{1-\xi^2}, \frac{\pi}{2}\right) = 1 + \tfrac{1}{2}\xi^2\left[\log\frac{4}{\xi} - \frac{1}{1.2}\right] + \frac{1^2.3}{2^2.4}\xi^4\left[\log\frac{4}{\xi} - \frac{2}{1.2} - \frac{1}{3.4}\right],$$

it is at once obvious that (11), which leads to (47) and (47A), gives for $I_n(x)$ a function of ξ, which, in general, is not identical with (10).

We shall have to deal with infinite series into which I's and H's enter; it is, therefore, requisite to know to what values these approximate when their order, n, becomes infinite. We have, whether n is integral or fractional,

$$H_n(x) = -\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,\xi^{n-1}\,\mathrm{F}(\tfrac{1}{2},\ n-1,\ n+\tfrac{1}{2},\ \xi^2) \quad . \quad . \quad . \quad (32),$$

whence

$$H_n(x) = -\tfrac{1}{2}\frac{\pi^{\frac{1}{2}}}{n^{\frac{1}{2}}}.\,\xi^{n-1}(1-\xi^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49),$$

when n is very great. [HEINE, § 40.]

Again,

$$I_n(x) = \frac{\xi^{-n}}{\pi}\int_{\xi^2}^{1} u^n(1-u)^{\frac{1}{2}}(u-\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Now, when n is very great, each element $u^n(1-u)^{\frac{1}{2}}(u-\xi^2)^{\frac{1}{2}}du$ of the integral is very small, except when u is nearly equal to 1; we may, therefore, put

$$\mathrm{I}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{\frac{1}{2}}}{\pi}\int_0^1 u^n(1-u)^{\frac{1}{2}}\,du$$

when n is very great,

$$= \frac{1}{\pi}\frac{\Pi(n)\,\Pi(\frac{1}{2})}{\Pi(n+\frac{3}{2})}\xi^{-n}(1-\xi^2)^{\frac{1}{2}} = \frac{1}{2\sqrt{(\pi n^3)}}\xi^{-n}(1-\xi^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (49\text{A}).$$

The expressions (49) and (49A) enable us to discuss the convergency of the development

$$\frac{1-x^2}{y-x} = \sum_{n=2}^{n=\infty} n(n-1)(2n-1)\,\mathrm{H}_n(y)\,\mathrm{I}_n(x) \quad . \quad . \quad . \quad . \quad . \quad (37).$$

For, from the equations (16) and (38), we find

$$\sum_2^n n(n-1)(2n-1)(y-x)\,\mathrm{I}_n(x)\,\mathrm{H}_n(y)$$
$$= 1-x^2+(n-1)\,n\,(n+1)\,[\mathrm{I}_n(x)\,\mathrm{H}_{n+1}(y) - \mathrm{H}_n(y)\,\mathrm{I}_{n+1}(x)].$$

Now the limit when n is very great of the expression in [ ] is

$$\frac{1}{4n^3}\left(\frac{\eta}{\xi}\right)^n\sqrt{(1-\xi^2\,.\,1-\eta^2)}\,.\left(1-\frac{1}{\xi\eta}\right).$$

Hence (37) is true so long as mod. η > mod. ξ. [HEINE, § 45.]

CHAPTER IV.—ON FUNCTIONS APPROPRIATE TO THE SPHEROID AND THE TORE.

We have seen in the introduction that the expression

$$\psi = (\mathrm{A}r^n + \mathrm{B}r^{-n+1})\,\{\mathrm{C}\mathrm{I}_n(\cos\theta) + \mathrm{D}\,\mathrm{H}_n(\cos\theta)\}$$

where A, B, C, D, and n are arbitrary, is a solution of

$$\mathrm{D}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and we shall find that solutions expressed in this manner are appropriate to the discussion of questions of fluid motion, when the boundary is a sphere.

In the present chapter I propose to develop solutions which shall be appropriate to the spheroid and the tore.

Consider a spheroid whose centre is at the origin, and whose major axis, and axis of symmetry coincide in the axis of z. Let the distance between the foci on the axis of z be $2h$. Then if ph, qh, be the major axes of the confocals to this spheroid which pass through the point (ϖ, z), p^2, q^2 are the roots of the equation in λ,

$$\frac{\varpi^2}{\lambda - 1} + \frac{z^2}{\lambda} = h^2.$$

So that

$$z = hpq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$
$$\varpi = ih\sqrt{(1 - p^2 \,.\, 1 - q^2)}$$

where we shall suppose p^2 algebraically greater than q^2, and, by an arbitrary convention, q to change sign with z.

From (50) we find

$$d\varpi^2 + dz^2 = h^2 (q^2 - p^2) \left(\frac{dp^2}{1 - p^2} - \frac{dq^2}{1 - q^2} \right);$$

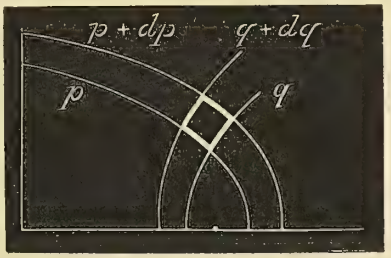
whence the normal distance between the two spheroids p, $p + dp$, is

$$h \sqrt{\left(\frac{q^2 - p^2}{1 - p^2} \right)} dp,$$

and between the hyperboloids q, $q + dq$,

$$h \sqrt{\left(\frac{p^2 - q^2}{1 - q^2} \right)} dq.$$

Now, reverting to the hydrodynamical origin of the subject, it is shown in the introduction that $-\varpi^{-1} D\psi$ is the circulation per unit area for any closed curve drawn in a meridional plane. Consider the small curve bounded by p, $p + dp$, q, $q + dq$.



The velocities in the directions of the outward-drawn normals are

to p,

$$\left.\begin{aligned} \text{P} &= \frac{1}{h\varpi}\sqrt{\frac{1-q^2}{p^2-q^2}}\frac{d\psi}{dq} \\ \text{to } q, \qquad \text{Q} &= -\frac{1}{h\varpi}\sqrt{\frac{1-p^2}{q^2-p^2}}\frac{d\psi}{dp} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (51),$$

and the circulation in the circuit considered is

$$+\frac{d}{dp}\left[\text{Q}h\sqrt{\frac{p^2-q^2}{1-q^2}}\right]dp\,dq,$$

$$-\frac{d}{dq}\left[\text{P}h\sqrt{\frac{q^2-p^2}{1-p^2}}\right]dp\,dq,$$

$$= -\left[\frac{1}{1-q^2}\frac{d^2\psi}{dp^2} - \frac{1}{1-p^2}\frac{d^2\psi}{dq^2}\right]\frac{dp\,dq}{h}$$

But this is equal to

$$-\frac{1}{\varpi}\,\text{D}\psi.h\sqrt{\frac{q^2-p^2}{1-p^2}}\,dp.h\sqrt{\frac{p^2-q^2}{1-q^2}}\,dq\,;$$

hence

$$\text{D}\psi = -\frac{1}{h^2(p^2-q^2)}\left[(1-p^2)\frac{d^2\psi}{dp^2} - (1-q^2)\frac{d^2\psi}{dq^2}\right] \quad \ldots\ldots \quad (6a),$$

and the equation

$$\text{D}\psi = 0 \quad \ldots\ldots\ldots\ldots \quad (3)$$

transforms into

$$(1-p^2)\frac{d^2\psi}{dp^2} - (1-q^2)\frac{d^2\psi}{dq^2} = 0 \quad \ldots\ldots\ldots \quad (3b).$$

In obtaining the above I have supposed the ellipsoids and hyperboloids of revolution about their transverse axes; and, in this case, p, q are the reciprocals of the eccentricities of the generating curves. Hence, for *ovary spheroid*,

p lies between 1 on axis, ∞ at infinity;
q „ 1 „ , 0 at plane of symmetry.

If we wish our equations to apply to planetary spheroids and hyperboloids of one sheet, we must suppose $-h^2$ to be the square of the radius of the focal circle in the plane of symmetry, and p, q are equal to $(e^2-1)^{\frac{1}{2}}/e$, where e is the eccentricity of the generating ellipse or hyperbola.

Hence, for *planetary spheroid*,

p lies between 0 within ring of foci, $i\infty$ at infinity, being everywhere a pure imaginary,
q lies between 0 on plane of symmetry beyond focal circle, 1 on axis.

Hence, whether we divide up space by means of ovary spheroids and hyperboloids of

two sheets, or by means of planetary spheroids and hyperboloids of one sheet, q is real and lies between 0 and 1.

Hence, by p. 462, any function of the coordinates, which remains everywhere finite, may be expressed by the series $\sum_{n=0}^{n=\infty} \mathrm{I}_n(q) f_n(p)$ where n is a positive integer. If this is a solution of (3) we find on substituting

$$\sum_{n=0}^{n=\infty} \mathrm{I}_n(q) \cdot \left[(1-p^2)\frac{d^2 f_n(p)}{dp^2} + n(n-1) f_n(p)\right] = 0,$$

whence,

$$(1-p^2)\frac{d^2 f_n(p)}{dp^2} + n(n-1) f_n(p) = 0,$$

which is the equation (7).

Hence, any solution of (3) may be written

$$\Sigma\, \mathrm{I}_n(q)\left[\mathrm{A}_n \mathrm{I}_n(p) + \mathrm{B}_n \mathrm{H}_n(p)\right]$$

where n is a positive integer.

When n is a positive integer $\mathrm{I}_n(p)\,\mathrm{I}_n(q)$ is a rational integral function of the coordinates ϖ, z, of degree n.

In the first place, let n be even. The product may be resolved into a series of products, $(p^2-\alpha^2)(q^2-\alpha^2)$ where α is any root of $\mathrm{I}_n(x) = 0$.

And by (50),

$$(p^2-\alpha^2)(q^2-\alpha^2) = -\frac{\varpi^2}{h},$$

when $\alpha = 1$;

$$= \frac{\alpha^2(1-\alpha^2)}{h^2}\left[\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} - h^2\right],$$

when $\alpha \neq 1$.

But by (9) we see that, if $\Pi(\alpha^2)$ denotes the product of all the α^2 s,

$$\Pi(\alpha^2) = \frac{1.2\ldots n}{2.4\ldots n.n-1.n+1\ldots 2n-3},$$

and

$$\Pi(1-\alpha^2) = \operatorname*{Lt}_{x=1} \frac{\mathrm{I}_n(x)}{x^2-1} = \frac{\mathrm{P}_{n-1}(1)}{2} = \tfrac{1}{2}.$$

Hence, when n is even,

$$\mathrm{I}_n(p)\,\mathrm{I}_n(q) = -\frac{1.2\ldots n}{2.4\ldots n.n-1\ldots 2n-3}\cdot\frac{\varpi^2}{2h^{2n}}\,\Pi\left(\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} \quad h^2\right) \quad . \quad . \quad (52),$$

where all the roots occur in the product excepting $\alpha^2 = 1$.

Similarly, when n is odd,

$$\mathrm{I}_n(p)\,\mathrm{I}_n(q) = -\frac{1.2\ldots n}{2.4\ldots n-1.n.n+2\ldots 2n-3}\cdot\frac{z\varpi^2}{2h^{2n+1}}\,\Pi\left(\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} - h^2\right),$$

where, under the product, all the roots are included, excepting $\alpha = \pm 1$ and $\alpha = 0$, the latter of which gives the term pq, or z/h.

Now it is easy to see that, except for a constant factor, there is only one homo

geneous rational function of the coordinates of positive degree r which satisfies (3). For such a function in its most general form involves $r + 1$ constants, and, when we have operated upon it with $\varpi\left(\frac{d^2}{d\varpi^2} + \frac{d^2}{dz^2}\right) - \frac{d}{d\varpi}$, we are left with a similar function of degree $r - 1$, each of whose terms must vanish. This gives r linear equations between the $r + 1$ constants, showing that one, and one only, is independent, and that appears as an arbitrary factor.

Hence the r functions $\mathrm{I}_r(p)\,\mathrm{I}_r(q)$, $\mathrm{I}_{r-1}(p)\,\mathrm{I}_{r-1}(q)$, ... must be equivalent to these r homogeneous solutions of (3), and are therefore sufficient for expressing any rational integral function of the coordinates of positive degree r, which satisfies (3).

We shall require hereafter the solution of

$$(1 - p^2)\frac{d^2\psi}{dp^2} - (1 - q^2)\frac{d^2\psi}{dq^2} = f_m(p)\,\phi_n(q) \quad . \quad . \quad . \quad . \quad . \quad (3c),$$

where $f_m(p)$, $\phi_n(q)$ are linear functions of $\mathrm{I}_m(p)$ and $\mathrm{H}_m(p)$, $\mathrm{I}_n(q)$ and $\mathrm{H}_n(q)$, and $m \neq n$.

A particular integral is

$$\psi = \frac{f_m(p)\ \phi_n(q)}{(n - m)(n + m - 1)}.$$

I shall now consider the functions which are appropriate to the tore.

Write

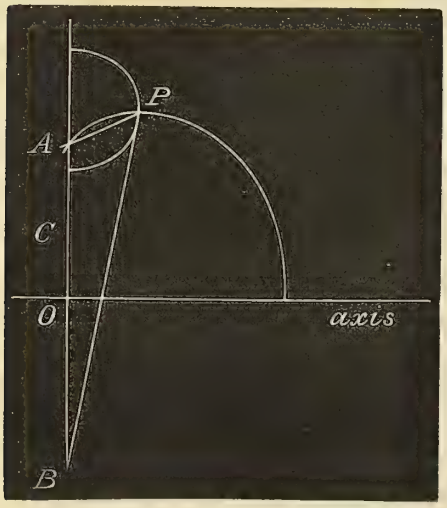
$$\varpi + iz = c \coth \frac{\eta - i\xi}{2},$$

$$= c\,\frac{\sinh\eta + i\sin\xi}{\cosh\eta - \cos\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

Then

$$\varpi^2 + z^2 - 2zc\cot\xi - c^2 = 0,$$

$$\varpi^2 + z^2 - 2\varpi c\coth\eta + c^2 = 0.$$



And if P be any point in a meridional plane, A, B two points in that plane symmetrically placed at distance c on opposite sides of the axis of symmetry,

$$e^{\eta} = \frac{\mathrm{BP}}{\mathrm{AP}},$$

$$\xi = \mathrm{APB}.$$

Hence the surfaces represented by $\eta =$ const. are tores, and those represented by $\xi =$ const., the surfaces given by the intersection of two equal spheres.

We also find

$$d\varpi^2 + dz^2 = c^2 \frac{d\eta^2 + d\xi^2}{(\cosh\eta - \cos\xi)^2}.$$

Hence the normal distance between the surfaces η, $\eta + d\eta$ is

$$\frac{c\,d\eta}{\cosh\eta - \cos\xi},$$

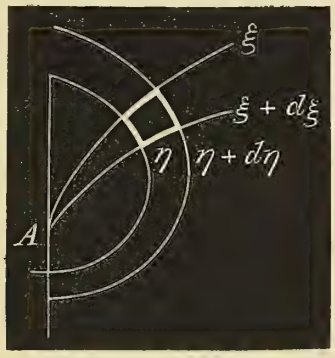
and between ξ, $\xi + d\xi$,

$$\frac{c\,d\xi}{\cosh\eta - \cos\xi}.$$

Thus the velocities in the directions of the outward-drawn normals to the surfaces η, ξ are

$$\left.\begin{aligned} \mathrm{H} &= -\frac{\cosh\eta - \cos\xi}{c\varpi}\frac{d\psi}{d\xi} \\ \Xi &= -\frac{\cosh\eta - \cos\xi}{c\varpi}\frac{d\psi}{d\eta} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (54),$$

and hence the circulation in the circuit formed by η, $\eta + d\eta$, ξ, $\xi + d\xi$,



$$-\frac{d}{d\xi}\left[\mathrm{H}\frac{c}{\cosh\eta - \cos\xi}\right]d\xi\,d\eta + \frac{d}{d\eta}\left[-\Xi\frac{c}{\cosh\eta - \cos\xi}\right]d\xi\,d\eta,$$

$$= \left\{\frac{d}{d\xi}\left[\frac{1}{\varpi}\frac{d\psi}{d\xi}\right] + \frac{d}{d\eta}\left[\frac{1}{\varpi}\frac{d\psi}{d\eta}\right]\right\}d\xi\,d\eta,$$

and we get

$$\mathrm{D}\psi = -(\cosh\eta - \cos\xi)^2\varpi\left[\frac{d}{d\xi}\left(\frac{1}{\varpi}\frac{d\psi}{d\xi}\right) + \frac{d}{d\eta}\left(\frac{1}{\varpi}\frac{d\psi}{d\eta}\right)\right] \quad . \quad . \quad . \quad (6b)$$

where

$$\varpi = \frac{c\sinh\eta}{\cosh\eta - \cos\xi}.$$

If we write

$$\psi = \chi/(\cosh\eta - \cos\xi)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$\left.\begin{aligned} \frac{1}{\varpi}\frac{d\psi}{d\xi} &= \frac{1}{c\sinh\eta}\left[(\cosh\eta - \cos\xi)^{\frac{1}{2}}\frac{d\chi}{d\xi} - \frac{\chi\sin\xi}{2(\cosh\eta - \cos\xi)^{\frac{1}{2}}}\right] \\ \frac{1}{\varpi}\frac{d\psi}{d\eta} &= \frac{1}{c\sinh\eta}\left[(\cosh\eta - \cos\xi)^{\frac{1}{2}}\frac{d\chi}{d\eta} - \frac{\chi\sinh\eta}{2(\cosh\eta - \cos\xi)^{\frac{1}{2}}}\right] \end{aligned}\right\} \quad . \quad . \quad (56),$$

and we find

$$\begin{aligned} \mathrm{D}\psi &= -\frac{(\cosh\eta - \cos\xi)^{\frac{3}{2}}}{c^2}\left[\frac{d^2\chi}{d\xi^2} + \frac{d^2\chi}{d\eta^2} - \coth\eta\frac{d\chi}{d\eta} + \tfrac{1}{4}\right], \\ &= -\frac{(x - \cos\xi)^{\frac{3}{2}}}{c^2}\left[\frac{d^2}{d\xi^2} - (1 - x^2)\frac{d^2}{dx^2} + \tfrac{1}{4}\right](x - \cos\xi)^{\frac{1}{2}}.\psi \quad . \quad . \quad (6c), \end{aligned}$$

where

$$x = \cosh\eta,$$

and

$$\mathrm{D}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

transforms into

$$\frac{d^2\psi}{d\xi^2} - (1 - x^2)\frac{d^2\chi}{dx^2} + \frac{\chi}{4} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c),$$

where

$$\chi = \psi(\cosh\eta - \cos\xi)^{\frac{1}{2}}.$$

Now ξ, as it occurs in the equation (3c), is a real angle lying between π and 0. Hence, any function, χ, which satisfies (3c), may be expressed, by FOURIER'S theorem, in a series

$$\Sigma f_n(x)\cos(n\xi + \alpha_n),$$

where n is a positive integer.

Substitute in (3c), and we find

$$\Sigma - \cos(n\xi + \alpha_n)\left[(1 - x^2)\frac{d^2 f_n(x)}{dx^2} + (n^2 - \tfrac{1}{4})f_n(x)\right] = 0,$$

whence

$$(1 - x^2)\frac{d^2 f_n(x)}{dx^2} + (n + \tfrac{1}{2})(n - \tfrac{1}{2})f_n(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad (7),$$

which shows that

$$f_n(x) = \mathrm{A}_n\mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n\mathrm{H}_{n+\frac{1}{2}}(x),$$

n being a positive integer, and any solution of (3) may be written

$$\psi = \frac{1}{(x - \cos\xi)^{\frac{1}{2}}}\Sigma\cos(n\xi + \alpha_n)\left[\mathrm{A}_n\mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n\mathrm{H}_{n+\frac{1}{2}}(x)\right] \quad . \quad . \quad . \quad (57),$$

an expression identical, except as regards notation, with that given by Mr. HICKS. ('Phil. Trans.,' 1884.)

We are now in a position to show how the expression given on p. 473 for $\mathrm{H}_{n+\frac{1}{2}}(x)$ may be derived. [*Cf.* HICKS, 'Phil. Trans.,' 1884.]

For $\psi = \sqrt{(\varpi^2 + z^2)}$ is a solution of (3); and

$$\varpi^2 + z^2 = \frac{\cosh\eta + \cos\xi}{\cosh\eta - \cos\xi}$$

Hence we may write

$$(x + \cos\xi)^{\frac{1}{2}} = \sum_{n=0}^{n=\infty} \cos(n\xi + \alpha_n)[A_n I_{n+\frac{1}{2}}(x) + B_n H_{n+\frac{1}{2}}(x)].$$

But it is clear that the left-hand member may be developed in cosines of multiples of ξ, and, therefore, $\alpha_n = 0$.

Hence

$$A_n I_{n+\frac{1}{2}}(x) + B_n H_{n+\frac{1}{2}}(x) = \frac{2}{\pi}\int_0^\pi \cos n\xi \sqrt{(x + \cos\xi)}\, d\xi.$$

Now, when $x = \infty$, the right-hand member vanishes. In fact, since

$$\int_0^\pi \cos n\xi \cos^r \xi\, d\xi = 0,$$

when n, r are positive integers and $r < n$, it is clear that the highest power occurring on the right when the expression is developed in descending powers of x, is $x^{-n+\frac{1}{2}}$. Hence $A_n = 0$. To find B_n, put $x = 1$.

$$\int_0^\pi \cos n\xi \sqrt{2}\cos\frac{\xi}{2}\, d\xi = \frac{(-1)^{n-1}}{\sqrt{2}} \frac{1}{(n+\frac{1}{2})(n-\frac{1}{2})}.$$

But

$$H_{n+\frac{1}{2}}(1) = -\frac{1}{(n+\frac{1}{2})(n-\frac{1}{2})},$$

therefore

$$B_n = \frac{(-1)^n}{\pi}$$

and

$$H_{n+\frac{1}{2}}(x) = (-1)^n \int_0^\pi \cos n\xi \sqrt{\{2(x + \cos\xi)\}}\, d\xi$$

$$= \int_0^\pi \cos n\xi \sqrt{\{2(x - \cos\xi)\}}\, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

where n is a positive integer, and x is real and > 1.

These functions have been employed by Mr. Hicks in the papers already referred to; and, except for purely mathematical interest, it is chiefly in connection with the theory of fluid-motion about a tore that functions of fractional order claim our notice.

Mr. Basset* has, however, employed the function $L_n = \frac{2}{\pi}(-1)^n e^{-n/2} H_{n+\frac{1}{2}}(\cosh\eta)$, to find the current function for motion in an infinite liquid due to the rotation of an infinite cylinder whose cross-section is a lemniscate.

* 'Hydrodynamics,' vol. 1, § 115.

Any solution of (3), which consists exclusively of positive integral powers of the coordinates ϖ, z, can be written in the form

$$\frac{1}{(x-\cos\xi)^{\frac{1}{2}}}\sum_{n=o}^{n=\infty} B_n H_{n+\frac{1}{2}}(x)\cos(n\xi+\alpha_n).$$

To arrive at this theorem we notice that

$$2\frac{d}{d\xi}(x-\cos\xi)^{\frac{1}{2}} = z(x-\cos\xi)^{\frac{1}{2}}.$$

But

$$2\frac{d}{d\xi} = -\varpi z\frac{d}{d\varpi} + (\varpi^2 - z^2 - 1)\frac{d}{dz};$$

hence we see that

$$2^r\left(\frac{d}{d\xi}\right)^r(x-\cos\xi)^{\frac{1}{2}} = U_r(x-\cos\xi)^{\frac{1}{2}},$$

where U_r is a rational integral function of ϖ, z, of degree r, and consecutive functions U_r, U_{r+1} are connected by the relation

$$U_{r+1} = -2\varpi z\frac{dU_r}{d\varpi} + (\varpi^2 - z^2 - 1)\frac{dU_r}{dz} + zU_r,$$

and $U_0 = 1$.

Forming the first five functions, we have

$$\begin{aligned}
U_1 &= z,\\
U_2 &= \varpi^2 - 1,\\
U_3 &= z(-3\varpi^2 - 1),\\
U_4 &= -3\varpi^4 + 12\varpi^2 z^2 + 2\varpi^2 + 1,\\
U_5 &= z(45\varpi^4 - 60\varpi^2 z^2 - 30\varpi^2 - 1).
\end{aligned}$$

From these it is clear that when r is odd, z is a factor of U_r; but that, otherwise, U_r is an even function of the coordinates, ϖ, z, $-\varpi$ occurring in higher power than z.

Now

$$(x-\cos\xi)^{\frac{1}{2}} = \sum_{n=0}^{n=\infty}\frac{1}{\pi}H_{n+\frac{1}{2}}(x)\cos n\xi.$$

Therefore,

$$U_r = \frac{1}{(x-\cos\xi)^{\frac{1}{2}}}\sum_{n=0}^{n=\infty}\cdot\frac{(2n)^r}{\pi}H_{n+\frac{1}{2}}(x)\cos\left(n\xi+\frac{r\pi}{2}\right),$$

and therefore $\psi = U_r$ is a solution of (3). But there are only r rational functions of the coordinates of positive integral degree not greater than r, which satisfy (3), and which are independent. Hence, in terms of the functions, U_r, U_{r-1}, &c., we can express any such solution of (3); whence we obtain the theorem stated at the beginning of this section, and we see that B_n is a rational integral function of n, of the same degree in n as the solution of (3) is in the coordinates, ϖ, z.

TABLE I.—Functions of ϖ and z of Positive Degree satisfying

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

(1) z.

(2) ϖ^2.

(3) $z\varpi^2$.

(4) $\varpi^2(4z^2 - \varpi^2)$.

(5) $z\varpi^2(4z^2 - 3\varpi^2)$.

(6) $\varpi^2(8z^4 - 12\varpi^2z^2 + \varpi^4)$.

(7) $z\varpi^2(8z^4 - 20z^2\varpi^2 + 5\varpi^4)$.

(8) $\varpi^2(64z^6 - 240z^4\varpi^2 + 120z^2\varpi^4 - 5\varpi^6)$.

(9) $z\varpi^2(64z^6 - 336z^4\varpi^2 + 280z^2\varpi^4 - 35\varpi^6)$.

(10) $\varpi^2(128z^8 - 896z^6\varpi^2 + 1120z^4\varpi^4 - 280z^2\varpi^6 + 7\varpi^8)$.

The above were calculated from the formula

$$U_n = (2n-3)\,zU_{n-1} - (\varpi^2 + z^2)\frac{dU_{n-1}}{dz},$$

which may be readily deduced from (13).

TABLE II.—$I_n(x)$ in powers of x.

$n = 0.$ $\quad -1.$

1. $\quad x.$

2. $\quad \dfrac{x^2-1}{2}.$

3. $\quad \dfrac{x^3-x}{2}.$

4. $\quad \dfrac{5x^4-6x^2+1}{8}.$

5. $\quad \dfrac{7x^5-10x^3+3x}{8}$

6. $\quad \dfrac{21x^6-35x^4+15x^2-1}{16}.$

7 $\quad \dfrac{33x^7-63x^5+35x^3-5x}{16}.$

8 $\quad \dfrac{429x^8-924x^6+630x^4-140x^2+5}{128}.$

9. $\quad \dfrac{715x^9-1716x^7+1386x^5-420x^3+35x}{128}.$

10. $\quad \dfrac{2431x^{10}-6435x^8+6006x^6-2310x^4+315x^2-7}{256}.$

TABLE III.—$\mathrm{K}_n(x)$ in powers of x.

$n=$ 2 ; $\frac{x}{2}$.

3 ; $\frac{x^2}{2}-\frac{1}{3}$.

4 ; $\frac{5}{8}x^3-\frac{13}{24}x$.

5 ; $\frac{7}{8}x^4-\frac{23}{24}x^2+\frac{2}{15}$.

6 ; $\frac{21}{16}x^5-\frac{7}{4}x^3+\frac{113}{240}x$.

7 ; $\frac{33}{16}x^6-\frac{13}{4}x^4+\frac{103}{80}x^2-\frac{8}{105}$.

8 ; $\frac{429}{128}x^7-\frac{781}{128}x^5+\frac{2039}{640}x^3-\frac{1873}{4480}x$.

9 ; $\frac{715}{128}x^8-\frac{4433}{384}x^6+\frac{957}{128}x^4-\frac{6967}{4480}x^2+\frac{16}{315}$.

10 ; $\frac{2431}{256}x^9-\frac{8437}{384}x^7+\frac{2717}{160}x^5-\frac{4367}{896}x^3+\frac{30563}{80460}x$.

These were calculated from the sequence equations (38), (38A).

$$n\mathrm{K}_n(x)=(2n-3)\,x\mathrm{K}_{n-1}(x)-(n-3)\,\mathrm{K}_{n-2}(x),$$

$$3\mathrm{K}_3(x)=3x\mathrm{K}_2(x)-1.$$

TABLE IV.—x^n in terms of $\mathrm{I}(x)$'s.

$n=$ 0 ; $-\mathrm{I}_0$.

1 ; I_1.

2 ; $2\mathrm{I}_2-\mathrm{I}_0$.

3 ; $2\mathrm{I}_3+\mathrm{I}_1$.

4 ; $\frac{8}{5}\mathrm{I}_4+\frac{12}{5}\mathrm{I}_2-\mathrm{I}_0$.

5 ; $\frac{8}{7}\mathrm{I}_5+\frac{20}{7}\mathrm{I}_3+\mathrm{I}_1$.

6 ; $\frac{16}{21}\mathrm{I}_6+\frac{8}{3}\mathrm{I}_4+\frac{18}{7}\mathrm{I}_2-\mathrm{I}_0$.

7 ; $\frac{16}{33}\mathrm{I}_7+\frac{24}{11}\mathrm{I}_5+\frac{10}{3}\mathrm{I}_3+\mathrm{I}_1$.

8 ; $\frac{128}{429}\mathrm{I}_8+\frac{64}{39}\mathrm{I}_6+\frac{112}{33}\mathrm{I}_4+\frac{8}{3}\mathrm{I}_2-\mathrm{I}_0$.

9 ; $\frac{128}{715}\mathrm{I}_9+\frac{64}{55}\mathrm{I}_7+\frac{432}{143}\mathrm{I}_5+\frac{40}{11}\mathrm{I}_3+\mathrm{I}_1$.

10 ; $\frac{256}{2431}\mathrm{I}_{10}+\frac{1920}{2431}\mathrm{I}_8+\frac{32}{13}\mathrm{I}_6+\frac{560}{143}\mathrm{I}_4+\frac{30}{11}\mathrm{I}_2-\mathrm{I}_0$.

TABLE V.—Cos $(n\cos^{-1}x)$ in terms of $\mathrm{I}(x)$'s.

$n=$ 0 ; $-\mathrm{I}_0$.

1 ; I_1.

2 ; $4\mathrm{I}_2-\mathrm{I}_0$.

3 ; $8\mathrm{I}_3+\mathrm{I}_1$.

4 ; $\frac{64}{5}\mathrm{I}_4+\frac{16}{5}\mathrm{I}_2-\mathrm{I}_0$.

5 ; $\frac{128}{7}\mathrm{I}_5+\frac{40}{7}\mathrm{I}_3+\mathrm{I}_1$.

6 ; $\frac{512}{21}\mathrm{I}_6+\frac{128}{15}\mathrm{I}_4+\frac{108}{35}\mathrm{I}_2-\mathrm{I}_0$.

7 ; $\frac{1024}{33}\mathrm{I}_7+\frac{128}{11}\mathrm{I}_5+\frac{16}{3}\mathrm{I}_3+\mathrm{I}_1$.

8 ; $\frac{16384}{429}\mathrm{I}_8+\frac{4096}{273}\mathrm{I}_6+\frac{256}{33}\mathrm{I}_4+\frac{64}{21}\mathrm{I}_2-\mathrm{I}_0$.

9 ; $\frac{32768}{715}\mathrm{I}_9+\frac{1024}{55}\mathrm{I}_7+\frac{10368}{1001}\mathrm{I}_5+\frac{400}{77}\mathrm{I}_3+\mathrm{I}_1$.

10 ; $\frac{1310720}{2431}\mathrm{I}_{10}+\frac{163840}{7293}\mathrm{I}_8+\frac{512}{39}\mathrm{I}_6+\frac{3200}{429}\mathrm{I}_4+\frac{100}{33}\mathrm{I}_2-\mathrm{I}_0$.

TABLE VI.—Various Forms of $I_n(x)$.

$$2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}x^n\,\mathrm{F}\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},-n+\frac{3}{2},x^{-2}\right) \quad . \quad . \quad . \quad (9)$$

$$(-1)^{n/2}\frac{\Pi\left(\frac{n}{2}-\frac{3}{2}\right)}{2\Pi\left(\frac{n}{2}\right)\Pi(-\frac{1}{2})}\,\mathrm{F}\left(-\frac{n}{2},\frac{n}{2}-\frac{1}{2},\frac{1}{2},x^2\right) \quad . \quad . \quad . \quad . \quad (9\text{A})$$

$$(-1)^{\frac{1}{2}(n-1)}\frac{\Pi\left(\frac{n}{2}-1\right)}{\Pi\left(\frac{n}{2}-\frac{1}{2}\right)\Pi(-\frac{1}{2})}x\mathrm{F}\left(-\frac{n}{2}+\frac{1}{2},\frac{n}{2},\frac{3}{2},x^2\right) \quad . \quad . \quad . \quad (9\text{B})$$

$$\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\cdot\left[\cos(n\cos^{-1}x)-\frac{2n}{2(2n-3)}\cos(\overline{n-2}\,\cos^{-1}x)-\ldots\right]. \quad (10\text{A})$$

$$\frac{\Pi(n-\frac{3}{2})}{2\Pi(n)\,\Pi(-\frac{1}{2})}\xi^{-n}\,\mathrm{F}(-\tfrac{1}{2},-n,-n+\tfrac{3}{2},\xi^2) \quad . \quad . \quad . \quad (10)$$

$$\frac{\xi^{-n}}{\pi}\int_{\xi^2}^{1}u^{n-2}(1-u)^{\frac{1}{2}}(u-\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{1}{\underline{n-1}}\left(\frac{d}{dx}\right)^{n-2}\left(\frac{x^2-1}{2}\right)^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$(-1)^{n-1}\frac{r^{n-1}}{\underline{n}}\frac{d^nr}{dz^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{1}{\pi}\int_0^{\pi}\sin^2\eta\,(x^2-1)\,(x-\cos\eta\,\sqrt{x^2-1})^{n-2}\,d\eta \quad . \quad (14)$$

$$\frac{1}{\pi}\int_0^{\pi}\sin^2\phi\,(x^2-1)\,(x+\cos\phi\,\sqrt{x^2-1})^{-n-1}\,d\phi \quad . \quad . \quad (14\text{A})$$

$$-\frac{1}{\pi}\int_{-\cosh^{-1}x}^{\cosh^{-1}x}d\phi\,\sqrt{\{2(x-\cosh\phi)\}}\,e^{(n-\frac{1}{2})\phi} \quad . \quad . \quad . \quad . \quad (15)$$

$$-\frac{2}{\pi}\int_0^{\cos^{-1}x}d\phi\,\sqrt{\{2(\cos\phi-x)\}}\,\cos(n-\tfrac{1}{2})\,\phi \quad . \quad . \quad . \quad (15\text{B})$$

$$\frac{2}{\pi}\int_{\cos^{-1}x}^{\pi}d\phi\,\sqrt{\{2(x-\cos\phi)\}}\,\sin(n-\tfrac{1}{2})\,\phi \quad . \quad . \quad . \quad (15\text{C})$$

TABLE VII.—Various Forms of $H_n(x)$.

$$-2^{-n}\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,x^{-n+1}\,\mathrm{F}\left(\frac{n-1}{2},\frac{n}{2},n-\tfrac{1}{2},x^{-2}\right) \quad . \quad . \quad (28)$$

$$(-1)^{n/2}\frac{\Pi\left(\frac{n}{2}-1\right)\Pi(\frac{1}{2})}{\Pi\left(\frac{n}{2}-\frac{1}{2}\right)}\,x\,\mathrm{F}\left(-\frac{n}{2}+\tfrac{1}{2},\frac{n}{2},\tfrac{3}{2},x^{2}\right) \quad . \quad . \quad . \quad . \quad (28\text{A})$$

$$(-1)^{\frac{1}{2}(n+1)}\frac{\Pi\left(\frac{n}{2}-\frac{3}{2}\right)\Pi(\frac{1}{2})}{\Pi\left(\frac{n}{2}\right)}\,\mathrm{F}\left(-\frac{n}{2},\frac{n}{2}-\tfrac{1}{2},\tfrac{1}{2},x^{2}\right) \quad . \quad . \quad (28\text{B})$$

$$-\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,\xi^{n-1}\,\mathrm{F}(-\tfrac{1}{2},n-1,n+\tfrac{1}{2},\xi^{2}) \quad . \quad . \quad . \quad (32)$$

$$-\xi^{n-1}\int_0^1 u^{n-2}(1-u)^{\frac{1}{2}}(1-u\xi^{2})^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$-\int_0^{\log(x+1)/(x-1)} d\phi\,(x^{2}-1)\sinh^{2}\phi\,(x-\sqrt{x^{2}-1}\cosh\phi)^{n-2} \quad . \quad (34)$$

$$-\int_0^{\infty} d\theta\,(x^{2}-1)\sinh^{2}\theta\,(x+\sqrt{x^{2}-1}\cosh\theta)^{-n-1} \quad . \quad . \quad . \quad (35)$$

$$-\int_{-\infty}^{\cosh^{-1}x} d\phi\,\sqrt{\{2(\cosh\phi-x)\}}\,e^{(n-\frac{1}{2})\phi} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\begin{cases} \frac{1}{2}\,\mathrm{I}_n(x)\log\dfrac{x+1}{x-1}-\mathrm{K}_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44) \\ *\,\frac{1}{2}\,\mathrm{I}_n(x)\log\dfrac{1+x}{1-x}-\mathrm{K}_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44\text{C}) \\ \frac{1}{2}\displaystyle\int_{-1}^{+1}\frac{\mathrm{I}_n(y)}{x-y}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44\text{D}) \\ \displaystyle\int_0^{\pi}\cos(n-\tfrac{1}{2})\theta\,\sqrt{\{2(x-\cos\theta)\}}\,d\theta \quad . \quad . \quad . \quad . \quad . \quad (46) \end{cases}$$

(9), (9A), (9B), (10), (10A), (12), (13), (15C), (28A), (28B), (44), (44C), and (44D) require n to be positive integer. (46) requires n of form *integer* $+\frac{1}{2}$. (9A) and (28A) n even; (9B) and (28B) n odd. (28A), (28B), mod. $x \not> 1$. (15C) and (44C) x real and < 1. (28) mod. $x \not< 1$. (46) x real and > 1. (14A) x not a pure imaginary. (35) x not real, negative, and > 1. Subject to these limitations, the above forms give the same values, with the exception of (44C) among the functions H.

Chapter V.—Irrotational Motion in Hydrodynamics.

The chief interest attaching to Stokes's Current Function lies in its application where the motion is rotational, and, consequently, where the Velocity Potential fails us. But it is never without value, since the equation $\psi = \text{const.}$ gives the stream surfaces. Moreover, methods are best illustrated by their simpler applications, and the results serve for comparison with more complicated cases. Accordingly, in this chapter I find the value of ψ for certain known cases of liquid motion.

The problems we can attack with Stokes's Current Function are those of motion symmetrical with respect to an axis, and this requires that the boundaries shall be symmetrical. The boundaries may, therefore, be such surfaces as a sphere, a tore, a hyperboloid of revolution of one sheet. We may suppose the boundary and the liquid at infinity, either at rest or in motion, symmetrically. When the liquid at infinity is in motion, I shall suppose its motion uniform; and we may then reduce any of the cases within this range to the problem of flux past a fixed obstacle, of liquid moving uniformly at infinity.

The motion being irrotational, we have $D\psi = 0$, and consequently ψ consists of a series of terms such as we have found as solutions of (3) in the foregoing chapters. If these are expressed as functions of the coordinates ϖ, z, it is easy to see that the only positive power of the coordinates that can occur is ϖ^2; for any higher power would make the velocity infinite at infinity, and the term z would make it infinite at the axis. This immediately strikes out various terms. For example,

$r\cos\theta$; $r^3 \mathrm{I}_3'(\cos\theta)$; $r^4 \mathrm{I}_4(\cos\theta)$, &c., and $\mathrm{I}_1(p)\,\mathrm{I}_1(q)$, $\mathrm{I}_3(p)\,\mathrm{I}_3(q)$ . . . by p. 479.

Our process is, then, to assume a value for ψ which contains a complete series of terms, omitting these impossible ones. The mode of expression that is appropriate differs with the form of boundary. We then use the principle that $\psi = \text{const.}$ is a stream surface, and may, therefore, be made to represent the surface of the obstacle; while at infinity $\psi = -\frac{1}{2}\mathrm{V}\varpi^2$, where V is the velocity there from right to left.

1. *Fixed Spherical Obstacle.*

We may assume

$$\psi = \mathrm{A} + \mathrm{B}\cos\theta + \left(\mathrm{C}r^2 + \frac{\mathrm{D}}{r}\right)\mathrm{I}_2(\cos\theta) + \frac{\mathrm{E}}{r^2}\mathrm{I}_3(\cos\theta) +, \text{ &c.},$$

the centre being the origin.

Then, if a is the radius, $\mathrm{B} = \mathrm{E} = \ldots = 0$, and

$$\mathrm{C}a^2 + \frac{\mathrm{D}}{a} = 0.$$

But

$$r^2\mathrm{I}_2(\cos\theta) = -\frac{\varpi^2}{2}.$$

Hence

$$\psi = \mathrm{V}\left(r^2 - \frac{a^3}{r}\right)\mathrm{I}_2(\cos\theta)$$
$$= -\tfrac{1}{2}\,\mathrm{V}\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta \quad \ldots\ldots\ldots \quad (58).$$

2. *Spheroidal Obstacle.*

Assume

$$\begin{aligned}\psi = \mathrm{A} &+ [\mathrm{BI}_0(p) + \mathrm{CH}_0(p)]\,\mathrm{I}_0(q)\\ &+ [\mathrm{DI}_1(p) + \mathrm{EH}_1(p)]\,\mathrm{I}_1(q)\\ &+ [\mathrm{FI}_2(p) + \mathrm{GH}_2(p)]\,\mathrm{I}_2(q)\\ &+ [\mathrm{KI}_3(p) + \mathrm{LH}_3(p)]\,\mathrm{I}_3(q) + \ldots \quad \ldots\ldots \quad (59).\end{aligned}$$

Putting D, K, &c., zero, in accordance with p. 489, the first terms become

$$\mathrm{B} - \mathrm{C}p + \mathrm{E}q,$$

since

$$\mathrm{H}_0(p) = p. \qquad \mathrm{H}_1(p) = 1.$$

Of these the term $\mathrm{C}p$ must be excluded; for referring to Chapter IV., p. 478, we see that the velocities normal to the surfaces p, q, are

$$\mathrm{P} = \frac{1}{h\varpi}\sqrt{\frac{1-}{p^2-q^2}}\frac{d\psi}{dq}$$
$$\mathrm{Q} = -\frac{1}{h\varpi}\sqrt{\frac{1-p^2}{q^2-p^2}}\frac{d\psi}{dp} \quad \ldots\ldots\ldots \quad (51).$$

Hence the presence of this term would make Q infinite at the axis, where

$$0 = \varpi = ih\sqrt{(1-p^2 \,.\, 1-q^2)}.$$

These expressions, in fact, tell us that where we may have $1 - q^2 = 0$, the terms of ψ that involve p, must contain the factor $1 - q^2$, and *vice versâ*. The loci to which these values correspond are given on p. 478. They are, namely, for

Ovary spheroid and hyperboloid of two sheets:—

$p = 1$ on axis; ∞ at infinity.
$q = 1$ on axis; 0 at plane of symmetry.

Planetary spheroid and hyperboloid of one sheet:—

$p = 0$ within ring of foci; $i\infty$ at infinity.
$q = 1$ on axis; 0 on plane of symmetry beyond focal circle.

From these we see, too, that p is the member that becomes infinite, and hence, whether it occurs in the term $I_n(p)\,I_n(q)$ or not, or whether ψ is a solution of $D\psi = 0$ or not, no higher positive power of p than the second can occur in ψ.

Thus, in the region outside a spheroidal obstacle, we may have

$$\psi = A + Eq + [FI_2(p) + GH_2(p)]\,I_2(q) + LH_3(p)\,I_3(q) + \ldots$$

giving

$$E = L = \ldots = 0,$$

$$\psi = 2Vh^2\,I_2(q)\,.\,I_2(p_0)\left[\frac{I_2(p)}{I_2(p_0)} - \frac{H_2(p)}{H_2(p_0)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (60),$$

where p_0 is the parameter of the spheroidal obstacle.

When p_0 is very small, that is to say, when the obstacle approximates to a disc,

$$\psi = 2Vh^2\,I_2(q)\left[I_2(p) + \frac{H_2(p)}{2p_0}\right],$$

a form which shows that the velocity becomes infinite at the edge of a disc.

A solution will be given hereafter, namely, on the supposition that the liquid is viscous, and the motion slow, which makes the velocity zero at the edge. See p. 504.

3. *Within a Hyperboloid.*

Let us consider the motion within a hyperboloid of one sheet, q_0, let us say. The velocity cannot, of course, be finite at infinity, and V will here denote its value at the centre of the hyperboloid.

We may assume

$$\psi = A + Eq + GH_2(p)\,I_2(q) + LH_3(p)\,I_3(q)\ldots,$$

and we find that all the coefficients must vanish except A and E; giving

$$\psi = Vh^2q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

It is easy to see that if we attempt to proceed to the case of flux through a circular hole in a wall, we find an infinite velocity at the edge. See p. 509.

4. *The Obstacle a Tore.*

The solution in this case, and in terms of the functions which I have been discussing in this paper, has been given by Mr. Hicks ('Phil. Trans.,' 1884), who also

considers the cyclic motion which can take place. No useful purpose would be served by quoting here Mr. HICKS's results. I shall, therefore, confine myself to proving rigorously a point which he assumes, namely, that in the expression of the current function due to the motion of a tore through a liquid at rest at infinity, in the form (57), the functions H do not enter.

This is deserving of some notice, since it is no longer true when the liquid at infinity is in motion.

Assuming (p. 482)

$$\psi = \frac{1}{(x - \cos\xi)^{\frac{1}{2}}} \sum_{n=0}^{n=\infty} \{A_n I_{n+\frac{1}{2}}(x) + B_n H_{n+\frac{1}{2}}(x)\} \cos(n\xi + \alpha_n) \quad . \quad . \quad (57),$$

the condition that the velocity should be continuous requires $d\psi/d\xi = 0$ at the axis, *i.e.*, where $x = 1$.

Now $I_{n+\frac{1}{2}}(1) = 0$. This condition, therefore, gives us no information as to how the functions I enter the expression. But it shows that if

$$\chi = \Sigma B_n H_{n+\frac{1}{2}}(x) \cos(n\xi + \alpha_n),$$

then

$$2(1 - \cos\xi)\frac{d\chi}{d\xi} - \chi \sin\xi = 0,$$

when

$$x = 1,$$

for all values of ξ.

In the first place, omit α_n.

We find

$$\begin{aligned} &2(1 - \cos\xi) \cdot 0 - \sin\xi B_0 H_{\frac{1}{2}} \\ &- 2(1 - \cos\xi) B_1 H_{\frac{3}{2}} \sin\xi - \sin\xi B_1 H_{\frac{3}{2}} \cos\xi \\ &+ \ldots \\ &- 2(1 - \cos\xi) B_n H_{n+\frac{1}{2}} n \sin n\xi - \sin\xi B_n H_{n+\frac{1}{2}} \cos n\xi \\ &+ \&c., \textit{ad. inf.} \quad = 0, \end{aligned}$$

the argument 1 being understood in the functions H.

Hence equating to zero the coefficient of $\sin n\xi$,

$$(2n - 3) B_{n-1} H_{n-\frac{1}{2}} - 4n B_n H_{n+\frac{1}{2}} + (2n + 3) B_{n+1} H_{n+\frac{3}{2}} = 0,$$

with one exception, viz.,

$$- 2B_0 H_{\frac{1}{2}} - 4B_1 H_{\frac{3}{2}} + 5B_2 H_{\frac{5}{2}} = 0.$$

But

$$H_{n+\frac{1}{2}}(1) = -\frac{1}{n^2 - \frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45\text{B}),$$

whence

$$\frac{-B_n + B_{n+1}}{2n+1} = \frac{B_n - B_{n-1}}{2n-1} = \ldots = \frac{B_1 - 2B_0}{3}.$$

Whence $B_n = 2B_0 + \beta n^2$, where B_0, β are arbitrary.

Similarly, if we supposed χ to consist of terms $B'_n H_{n+\frac{1}{2}}(x) \sin n\xi$, we should find $B'_0 = 0$, $B'_1 = 0$, and, therefore, the whole series $B'_n = 0$.

Hence, if the functions H occur in ψ, they must be in the form of the series

$$\frac{1}{(x - \cos\xi)^{\frac{3}{2}}}\Big\{ \alpha\left[\tfrac{1}{2} H_{\frac{1}{2}}(x) + H_{\frac{3}{2}}(x)\cos\xi + H_{\frac{5}{2}}(x)\cos 2\xi + \ldots\right] + \beta\left[H_{\frac{3}{2}}(x)\cos\xi + 2^2 H_{\frac{5}{2}}(x)\cos 2\xi + \ldots\right]\Big\},$$

that is, by p. 484,

$$\alpha + \frac{\pi}{4}\beta(\varpi^2 - 1).$$

The constant term, $\alpha - \pi\beta/4$, is irrelevant; the term $\pi\beta/4\,\varpi^2$ implies a finite velocity at infinity, viz., $\pi\beta/2$, from left to right.

Chapter VI.—Rotational Motion.

Among the problems on rotational motion in liquids, those are probably of most interest in which the circumstances most closely resemble nature, those, namely, in which the spin is due to the internal friction of the liquid. Certain cases of rotational motion in perfect fluids have been discussed by Mr. Hicks ('Phil. Trans.,' 1885), in connection with the theory of the motion of vortex rings, in terms of the functions I have before alluded to (p. 473). I confine myself, in the present chapter, to the consideration of certain cases of motion in viscous liquid.

When the motion in a viscous liquid is slow, the equations may be reduced to the forms, (Basset, 'Hydrodynamics,' § 473)

$$\frac{d\xi}{dt} = \nu\nabla^2\xi; \qquad \frac{d\eta}{dt} = \nu\nabla^2\eta; \qquad \frac{d\zeta}{dt} = \nu\nabla^2\zeta;$$

where ξ, η, ζ are the components of spin parallel to fixed axes, and $\nu = \mu/\rho$, the kinematic coefficient of viscosity. If the motion is symmetrical about the axis of z, we may put $\zeta = 0$, $\xi = -\omega\sin\phi$, $\eta = \omega\cos\phi$, where $\omega = -1/2\varpi\,D\psi$ is the resultant spin, and ϕ is an azimuthal angle measured about the axis of z. But we have seen, in the introduction (p. 452), that

$$\frac{\sin\phi}{\varpi} DV = \nabla^2 \frac{\sin\phi}{\varpi} V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

write $2\varpi\omega$ for V, and we get

$$D^2\psi = \frac{1}{\nu}\frac{d}{dt}D\psi,$$

Stokes's well-known equation. ('Cambridge Phil. Trans.,' vol. 9, Pt. 2.)

Hence when the motion is steady, ψ is a solution of

$$D^2\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3d).$$

Or we may assume

where

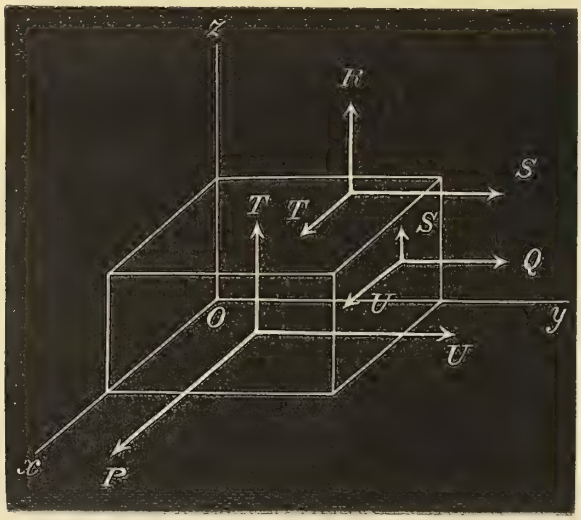
$$\left.\begin{aligned} D\psi &= V \\ DV &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3e)$$

so that ψ consists of a function of the form of V, together with a particular integral of (3*e*).

When the boundary is a symmetrical obstacle, the conditions which must be fulfilled there are :—

(1) The velocity normal to the obstacle shall be zero.

(2) The tangential velocity shall be proportional to the stress in the same direction, per unit area, across the surface of the obstacle.



Now if u, v, w are the velocities at any point parallel to fixed rectangular axes, whose origin is at the point, and if P, Q, R, S, T, U denote the stresses per unit area in the fluid, at the point, in the usual manner, as illustrated by the accompanying figure, then, it is known (see Basset, 'Hydrodynamics,' vol. 2, p. 242)

$$P = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{du}{dx} = -p - \tfrac{2}{3}\mu\theta + 2\mu e$$

$$Q = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dv}{dy} = -p - \tfrac{2}{3}\mu\theta + 2\mu f$$

$$\mathrm{R} = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dw}{dz} = -p - \tfrac{2}{3}\mu\theta + 2\mu g$$

$$\mathrm{S} = \mu\left(\frac{dw}{dy} + \frac{dv}{dz}\right) = 2\mu a$$

$$\mathrm{T} = \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right) = 2\mu b$$

$$\mathrm{U} = \mu\left(\frac{dv}{dx} + \frac{du}{dy}\right) = 2\mu c,$$

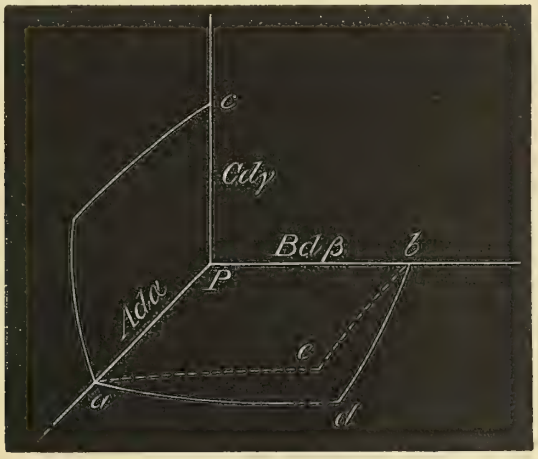
where

$$\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

I will now write these in terms of α, β, γ, the parameters of three orthogonal surfaces, whose intersection may be a convenient determination of the point.

Let

$$dx^2 + dy^2 + dz^2 = \mathrm{A}^2 d\alpha^2 + \mathrm{B}^2 d\beta^2 + \mathrm{C}^2 d\gamma^2.$$



Then, if θ_1, θ_2, θ_3 are the angles through which the normals at α, β, γ to the three surfaces must be rotated about themselves to make them parallel to the normals at $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, it is not hard to see

$$\theta_3 = \frac{\mathrm{B}}{\mathrm{A}d\alpha}d\beta - \frac{\mathrm{A}}{\mathrm{B}d\beta}d\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

with corresponding expressions for θ_1, θ_2. For, in passing from P to b, along the normal to the surface β, the normal to the surface α is rotated through the small angle ebd, or $\frac{d\mathrm{B}}{\mathrm{A}d\alpha}d\beta$, and, in passing from b to d, it is rotated through the angle ead or $\frac{d\mathrm{A}}{\mathrm{B}d\beta}d\alpha$, in the negative sense.

But the change in u, the velocity parallel to the normal at α, β, γ to the surface α, as we pass from α, β, γ to $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, is

$$\frac{du}{d\alpha}\,d\alpha + \frac{du}{d\beta}\,d\beta + \frac{du}{d\gamma}\,d\gamma - v\theta_3 + w\theta_2$$
$$= d\alpha\left[\frac{du}{d\alpha} + \frac{v}{\mathrm{B}}\frac{d\mathrm{A}}{d\beta} + \frac{w}{\mathrm{C}}\frac{d\mathrm{A}}{d\gamma}\right] + d\beta\left[\frac{du}{d\beta} - \frac{v}{\mathrm{A}}\frac{d\mathrm{B}}{d\alpha}\right] + d\gamma\left[\frac{du}{d\gamma} - \frac{w}{\mathrm{A}}\frac{d\mathrm{C}}{d\alpha}\right].$$

Hence we get

$$e = \frac{1}{\mathrm{A}}\left[\frac{du}{d\alpha} + \frac{v}{\mathrm{B}}\frac{d\mathrm{A}}{d\beta} + \frac{w}{\mathrm{C}}\frac{d\mathrm{A}}{d\gamma}\right]$$

$$2a = \frac{1}{\mathrm{C}}\frac{dv}{d\gamma} + \frac{1}{\mathrm{B}}\frac{dw}{d\beta} - \frac{w}{\mathrm{BC}}\frac{d\mathrm{C}}{d\beta} - \frac{v}{\mathrm{BC}}\frac{d\mathrm{B}}{d\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

with four others, which may be written down from symmetry.

These enable us to express the second boundary condition (p. 494) in terms of any coordinates which we may wish to employ.

Let us consider the solution of (3e) in terms of the coordinates r, θ.

We may assume, p. 489,

$$\mathrm{V} = \sum_{n=0}^{n=\infty} (\mathrm{A}_n r^n + \mathrm{B}_n r^{-n+1})\,\mathrm{I}_n(\cos\theta),$$

and

$$\mathrm{D} = \frac{d^2}{dr^2} + \frac{1-\mu^2}{r^2}\frac{d^2}{d\mu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $\mu = \cos\theta$.

Hence we require a particular integral of the equation,

$$\frac{d^2\psi_n}{dr^2} - \frac{n(n-1)}{r^2}\psi_n = \mathrm{A}_n r^n + \mathrm{B}_n r^{-n+1}.$$

Clearly

$$\psi_n = \frac{\mathrm{A}_n r^{n+2} - \mathrm{B}_n r^{-n+3}}{2(2n-1)}$$

is a solution.

Hence, a particular integral of (3e) is, changing the arbitrary constants,

$$\psi = \sum_{n=0}^{n=\infty} (\mathrm{A}_n r^{n+2} + \mathrm{B}_n r^{-n+3})\,\mathrm{I}_n(\cos\theta),$$

and, omitting terms, which, by the expressions (p. 453),

$$\mathrm{R} = \frac{1}{r^2\sin\theta}\frac{d\psi}{d\theta}$$

$$\Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr}$$

may be seen to give infinite velocities, we may write as a value of ψ, capable of expressing any slow, steady motion in a region of viscous liquid from which the origin is excluded,

$$\psi = D_1 \cos\theta + (C_2 r^2 + D_2 r^{-1} + B_2 r)\, I_2(\cos\theta) + \sum_{n=3}^{n=\infty} (D_n r^{-n+1} + B_n r^{-n+3})\, I_n(\cos\theta) \quad . \quad . \quad . \quad . \quad (65).$$

An expression equivalent to this is given by Mr. Butcher ('London Math. Soc. Proc.,' vol. 8).

In terms of the coordinates r, θ, I shall consider two cases of motion, viz., (1) when the obstacle is a sphere, (2) when it departs little from sphericity. When there is no slipping, the boundary conditions are $R = 0$, $\Theta = 0$; when there is slipping the conditions that hold at the boundary are not easily expressed in the second case; in its discussion I shall, therefore, confine myself to the supposition that the velocity at the boundary is zero.

1. *The Obstacle a Sphere.*

We find from (64)

$$2c = \frac{dR}{r\,d\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r},$$

and, consequently, if a be the radius, we must have, when $r = a$,

$$R = 0,$$

$$\mu\left(\frac{dR}{r\,d\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r}\right) = \beta\Theta,$$

or, written in terms of ψ,

$$\frac{d\psi}{d\theta} = 0$$

$$-(1-\mu^2)\frac{d^2\psi}{d\mu^2} + r^2\frac{d^2\psi}{dr^2} - 2r\frac{d\psi}{dr} = \frac{\beta}{\mu}r^2\frac{d\psi}{dr},$$

where β is a constant, which is infinite if there is no slipping, and $\mu = \cos\theta$.

But

$$-(1-\mu^2)\frac{d^2}{d\mu^2}I_n(\mu) = n(n-1)\,I_n(\mu).$$

Hence the above conditions require $D_n = B_n = 0$, except when $n = 2$, in which case we have

$$C_2 a^2 + D_2 a^{-1} + B_2 a = 0,$$

$$2C_2 a^2 - D_2 a^{-1}\left(1 + \frac{6\mu}{a\beta}\right) + B_2 a = 0.$$

These give

$$\psi = C_2\left[r^2 - a^2\,\frac{3\left(1+\frac{2\mu}{\beta a}\right)\frac{r}{a} - \frac{a}{r}}{2\left(1+\frac{3\mu}{\beta a}\right)}\right] I_2(\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (66).$$

It is clear that $C_2 = V$, the velocity from right to left at infinity.

Passing from this well-known result, which is here given for reference and illustration, let us consider the motion about an approximately spherical, symmetrical obstacle, at whose surface no slipping takes place.

Let $r = a[1 + f(\theta)]$ be the equation of the surface, and let $f(\theta)$ be developed by Chapter II. into the form $\Sigma\alpha_n I_n(\cos\theta)$.

Let us consider the surface

$$r = a[1 + \alpha_n I_n(\cos\theta)],$$

where we shall suppose α_n so small that its square may be neglected.

We must have $\frac{d\psi}{dr} = 0$, $\frac{d\psi}{d\theta} = 0$ in (65) when $r = a[1 + \alpha_n I_n(\cos\theta)]$; and since the motion will not be far different from that in the case when the surface is a perfect sphere, all the coefficients which occur are of the order of α_n, excepting C_2, D_2, B_2, and, therefore, except where these coefficients enter, we may disregard the departure from a spherical form. We find, omitting the argument cos θ, and writing D'_n, B'_n, for $D_n a^{-n+1}$, $B_n a^{-n+3}$ respectively,

$$\begin{aligned} a\frac{d\psi}{dr} &= [(2C_2a^2 - D_2a^{-1} + B_2a) + \alpha_n I_n(2C_2a^2 + 2D_2a)]\,I_2 \\ &\qquad + \Sigma[-(r-1)D'_r - (r-3)B'_r]\,I_r \\ &= 0. \end{aligned}$$

$$\begin{aligned} \frac{d\psi}{d(\cos\theta)} &= [(C_2a^2 + D_2a^{-1} + B_2a) + \alpha_n I_n(2C_2a^2 - D_2a^{-1} + B_2a)]\cos\theta \\ &\qquad + \Sigma[D'_r + B'_r]\,P_{r-1} \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned} C_2a^2 + D_2a^{-1} + B_2a &= 0, \\ 2C_2a^2 - D_2a^{-1} + B_2a &= 0. \end{aligned}$$

Making use of these results we find

$$\Sigma[D'_r + B'_r]\,P_{r-1} = 0;$$

whence

$$D'_r + B'_r = 0$$

for all values of r, and

$$-\alpha_n I_n\,2B_2a\,I_2 + \Sigma\,2B'_r\,I_r = 0.$$

But

$$\mathrm{I}_n\,\mathrm{I}_2 = \frac{\delta_n}{2}\,\mathrm{I}_{n+2} + \frac{\epsilon_n - 1}{2}\,\mathrm{I}_n + \frac{\zeta_n}{2}\,\mathrm{I}_{n-2}$$

where δ_n, ϵ_n, ζ_n are given by (17), p. 460.

Therefore

$$\mathrm{B}'_{n+2} = \mathrm{B}_2 a\,\alpha_n \frac{\delta_n}{2},$$

$$\mathrm{B}'_n = \mathrm{B}_2 a\,\alpha_n \frac{\epsilon_n - 1}{2},$$

$$\mathrm{B}'_{n-2} = \mathrm{B}_2 a\,\alpha_n \frac{\zeta_n}{2},$$

all the others vanishing.

Hence

$$\begin{aligned}\psi = &-\frac{\mathrm{V}a^2}{2}\,\mathrm{I}_2(\cos\theta)\left[\frac{3r}{a} - \frac{a}{r} - \frac{2r^2}{a^2}\right]\\ &-\frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\zeta_n}{2}\,\mathrm{I}_{n-2}(\cos\theta)\left[\frac{a^{n-5}}{r^{n-5}} - \frac{a^{n-3}}{r^{n-3}}\right]\\ &-\frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\epsilon_n - 1}{2}\,\mathrm{I}_n(\cos\theta)\left[\frac{a^{n-3}}{r^{n-3}} - \frac{a^{n-1}}{r^{n-1}}\right]\\ &-\frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\delta_n}{2}\,\mathrm{I}_{n+2}(\cos\theta)\left[\frac{a^{n-1}}{r^{n-1}} - \frac{a^{n+1}}{r^{n+1}}\right] \quad . \quad . \quad . \quad . \quad . \quad (67)\end{aligned}$$

From this formula the solution for the case $r = a\,\{1 + \Sigma\alpha_n\,\mathrm{I}_n(\cos\theta)\}$ can be built up.

2. *Spheroids.*

I will now consider the solution of (3*e*), p. 494, in terms of co-ordinates, p, q, defined on p. 477.

By p. 479, we may write

$$\mathrm{V} = \sum_{n=0}^{n=\infty}\left[\mathrm{A}_n\mathrm{I}_n(p) + \mathrm{B}_n\mathrm{H}_n(p)\right]\mathrm{I}_n(q),$$

and

$$\mathrm{D} = \frac{-1}{h^2(p^2 - q^2)}\left[(1 - p^2)\frac{d^2}{dp^2} - (1 - q^2)\frac{d^2}{dq^2}\right]. \quad . \quad . \quad . \quad . \quad (6a).$$

Hence

$$(1 - p^2)\frac{d^2\psi}{dp^2} - (1 - q^2)\frac{d^2\psi}{dq^2} = -h^2\sum_{n=0}^{n=\infty}(p^2 - q^2)\left[\mathrm{A}_n\mathrm{I}_n(p) + \mathrm{B}_n\mathrm{H}_n(p)\right]\mathrm{I}_n(q) \quad . \quad (68).$$

But, when n is not less than 4,

$$p^2\begin{matrix}\mathrm{I}_n(p)\\ \mathrm{H}_n(p)\end{matrix} = \delta_n\begin{matrix}\mathrm{I}_{n+2}(p)\\ \mathrm{H}_{n+2}(p)\end{matrix} + \epsilon_n\begin{matrix}\mathrm{I}_n(p)\\ \mathrm{H}_n(p)\end{matrix} + \zeta_n\begin{matrix}\mathrm{I}_{n-2}(p)\\ \mathrm{H}_{n-2}(p)\end{matrix}$$

by pp. 460, 469.

Hence

$$(p^2 - q^2)\,A_n I_n(p)\,I_n(q)$$
$$= A_n\delta_n[I_{n+2}(p)\,I_n(q) - I_n(p)\,I_{n+2}(q)] + A_n\zeta_n[I_{n-2}(p)\,I_n(q) - I_n(p)\,I_{n-2}(q)].$$

The corresponding terms in a particular integral of (68) are, by p. 480,

$$\frac{h^2 A_n\delta_n}{2(2n+1)}[I_{n+2}(p)\,I_n(q) + I_n(p)\,I_{n+2}(q)]$$
$$\frac{h^2 A_n\zeta_n}{2(2n-3)}[I_n(p)\,I_{n-2}(q) + I_{n-2}(p)\,I_n(q)].$$

Hence the terms arising in a particular integral of (68) from the terms on the right for which n is not less than 4, may be written

$$\Sigma\{\mathfrak{A}_{n+2}[I_{n+2}(p)\,I_n(q) + I_{n+2}(q)\,I_n(p)] + \mathfrak{B}_{n+2}[H_{n+2}(p)\,I_n(q) + I_{n+2}(q)\,H_n(p)]\},$$

where

$$\mathfrak{A}_{n+2} = \frac{h^2}{2(2n+1)}[A_n\delta_n - A_{n+2}\zeta_{n+2}]$$
$$\mathfrak{B}_{n+2} = \frac{h^2}{2(2n+1)}[B_n\delta_n - B_{n+2}\zeta_{n+2}].$$

The terms on the right of (68), when n is less than 4, are

$$(p^2 - q^2)\cdot\{A_0 + B_0p + [A_1p + B_1]q + [A_2I_2(p) + B_2H_2(p)]\,I_2(q)$$
$$+ [A_3I_3(p) + B_3H_3(p)]\,I_3(q)\}$$
$$= 2A_0I_2(p) + 2B_0I_3(p) - 2[A_0 + B_0p]I_2(q) + 2[A_1I_3(p) + B_1I_2(p)]q - 2[A_1p + B_1]I_3(q)$$
$$+ A_2\delta_2[I_4(p)\,I_2(q) - I_2(p)\,I_4(q)] + B_2\{\delta_2[H_4(p)I_2(q) - H_2(p)\,I_4(q)] - \frac{p}{3}I_2(q)\}$$
$$+ A_3\delta_3[I_5(p)\,I_3(q) - I_3(p)\,I_5(q)] + B_3\{\delta_3[H_5(p)\,I_3(q) - H_3(p)\,I_5(q)] - \tfrac{1}{15}\cdot I_3(q)\}.$$

Thus, the series of terms we have already found to occur in the particular integral, of (68), continues regularly down to $n = 2$, while there are also found the irregular terms

$$-A_0I_2(p) - \frac{B_0}{3}I_3(p)$$
$$+\left[-B_1I_2(p) - \frac{A_1}{3}I_3(p)\right]q$$
$$+\left[A_0 + \left(B_0 + \frac{B_2}{6}\right)p\right]I_2(q)$$
$$+\left[\left(\frac{B_1}{3} + \frac{B_3}{45}\right) + \frac{A_1}{3}p\right]I_3(q).$$

Thus ψ consists of these terms, together with the series

$$\sum_{n=2}^{n=\infty} \{\mathfrak{A}_{n+2}[I_{n+2}(p)\,I_n(q) + I_n(p)\,I_{n+2}(q)] + \mathfrak{B}_{n+2}[H_{n+2}(p)\,I_n(q) + H_n(p)\,I_{n+2}(q)]\},$$

and the series

$$\sum_{n=0}^{n=\infty} [C_n I_n(p) + D_n H_n(p)]\,I_n(q).$$

All the constants above are independent. But referring to p. 490, we see that the conditions for finite and continuous velocity everywhere require that A_0, B_0, A_1, B_1; $\mathfrak{A}_4$, $\mathfrak{A}_5$, &c.; C_0, D_0, C_3, C_4, &c., should all vanish. Hence all legitimate expressions of ψ are included in

$$\begin{aligned}\psi = {} & C_0 + D_1 q \\ & + I_2(q)\,[B_2 p + C_2 I_2(p) + D_2 H_2(p) + \mathfrak{B}_4 H_4(p)] \\ & + I_3(q)\,[B_3 + D_3 H_3(p) + \mathfrak{B}_5 H_5(p)] \\ & + I_4(q)\,[\mathfrak{B}_4 H_2(p) + D_4 H_4(p) + \mathfrak{B}_6 H_6(p)] \\ & + \text{\&c., regularly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),\end{aligned}$$

writing B_2, B_3 in place of $\frac{B_2}{6}$, $\frac{B_3}{45}$ respectively.

The first case to which I shall apply this formula is that in which an obstacle in the shape of a spheroid is opposed to the flow of liquid, and, to begin with, I shall suppose there is no slipping at the surface; so that the boundary conditions are

$$\frac{d\psi}{dp} = 0 \quad \frac{d\psi}{dq} = 0,$$

when $p = p_0$, where p_0 is the parameter of the fixed surface.

These require, omitting the argument, p_0,

$$\begin{gathered} D_1 = 0 \\ B_2 p_0 + C_2 I_2 + D_2 H_2 + \mathfrak{B}_4 H_4 = 0 = B_2 + C_2 P_1 + D_2 Q_1 + \mathfrak{B}_4 Q_3 \\ B_3 + D_3 H_3 + \mathfrak{B}_5 H_5 = 0 = D_3 Q_2 + \mathfrak{B}_5 Q_4 \\ \mathfrak{B}_4 H_2 + D_4 H_4 + \mathfrak{B}_6 H_6 = 0 = \mathfrak{B}_4 Q_1 + D_4 Q_3 + \mathfrak{B}_6 Q_5 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \end{gathered}$$

Hence, three of the above constants are arbitrary, namely, we may give what values we please to C_2, $\mathfrak{B}_4$, B_3, and all the others will then be determinate. If we give a finite value to $\mathfrak{B}_4$ or B_3 we get an infinite series which in each case is divergent (see p. 508). Let us then first consider the solution given by $\mathfrak{B}_4 = 0$, $B_3 = 0$.

We have

$$\psi = \mathrm{I}_2(q)\,[\mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathrm{B}_2 p],$$

where

$$\mathrm{C}_2 p_0 + \mathrm{D}_2\,\mathrm{Q}_1(p_0) + \mathrm{B}_2 = 0$$
$$\mathrm{C}_2\mathrm{I}_2(p_0) + \mathrm{D}_2\mathrm{H}_2(p_0) + \mathrm{B}_2 p_0 = 0.$$

Hence, since

$$\mathrm{H}_2(p_0) = \frac{p_0^2 - 1}{4}\log\frac{p_0 + 1}{p_0 - 1} - \frac{p_0}{2},$$

$$\mathrm{Q}_1(p_0) = \frac{p_0}{2}\log\frac{p_0 + 1}{p_0 - 1} - 1,$$

we get

$$\mathrm{D}_2 = -\frac{p_0^2 + 1}{2\mathrm{L}(p_0)}\mathrm{C}_2,$$

$$\mathrm{B}_2 = -\frac{1}{2\mathrm{L}(p_0)}\mathrm{C}_2,$$

where

$$\mathrm{L}(p_0) = \frac{p_0^2 + 1}{4}\log\frac{p_0 + 1}{p_0 - 1} - \frac{p_0}{2}.$$

Now the term which gives a finite velocity at infinity is $\mathrm{C}_2\mathrm{I}_2(p)\,\mathrm{I}_2(q) = -\mathrm{C}_2\varpi^2/4h^2$. Hence, if this velocity be V from right to left, $\mathrm{C}_2 = 2h^2\mathrm{V}$, and

$$\psi = 2h^2\mathrm{V}\,\mathrm{I}_2(q)\left[\mathrm{I}_2(p) - \frac{p + (p_0^2 + 1)\,\mathrm{H}_2(p)}{2\mathrm{L}(p_0)}\right]. \quad . \quad . \quad . \quad . \quad . \quad (70).$$

From this result we find, by simple reductions, that the velocities U, W (p. 478), perpendicular to the axis and parallel to it, are

$$\mathrm{U} = \frac{\mathrm{V}}{2\mathrm{L}(p_0)}\,\frac{q(p^2 - p_0^2)\sqrt{(1 - q^2)}}{(p^2 - q^2)\sqrt{(p^2 - 1)}},$$

$$\mathrm{W} = \frac{\mathrm{V}}{2\mathrm{L}(p_0)}\left\{\frac{1 + p_0^2}{2}\log\frac{p + 1}{p - 1} + \frac{p(q^2 - p_0^2)}{p^2 - q^2}\,2\mathrm{L}(p_0)\right\} \quad . \quad . \quad . \quad (71).$$

From (70), by the equation $\psi =$ const., we get the forms of the stream surfaces, and from this point of view, an integral which includes (70) has been given by MR. HERMAN, 'Quarterly Journal of Mathematics,' 1889, No. 92. Taking OBERBECK'S values for the velocities due to the steady motion of an ellipsoid through a viscous liquid ('Crelle,' vol. 81), he finds as one surface on which the stream-lines relative to the ellipsoid lie,

$$\left[\int^{\epsilon}\left(1 + \frac{c^2}{c^2 + \lambda}\right)\frac{d\lambda}{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}^{\frac{1}{2}}} - (\mathrm{A}_\epsilon + \mathrm{B}_\epsilon + \mathrm{C}_\epsilon)\,\epsilon\right]xy = \text{constant},$$

where

$$A_\epsilon = \int_\epsilon^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}c^2+\lambda)^{\frac{1}{2}}},$$

and $(a^2+\epsilon)^{\frac{1}{2}}$ is the semi-major axis of the confocal to the moving ellipsoid, which passes through the point x, y, z. I have slightly changed his notation, making the axis of z that along which the ellipsoid is moving.

But if the ellipsoid is one of revolution, so that $a = b$, then

$$h^3 A_\epsilon = h^3 B_\epsilon = \frac{p}{p^2-1} - \tfrac{1}{2}\log\frac{p+1}{p-1}$$

$$h^3 C_\epsilon = \log\frac{p+1}{p-1} - \frac{2}{p}$$

$$h\int_\epsilon^\infty \frac{d\lambda}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}} = \log\frac{p+1}{p-1} = hH_\epsilon$$

in the notation of p. 477, where

$$c^2 + \epsilon = h^2 p^2.$$

Hence, writing Mr. HERMAN'S integral

$$[H_0 + c^2 C_0 - H_\epsilon - c^2 C_\epsilon - \epsilon(A_\epsilon + B_\epsilon + C_\epsilon)]\, xy = \text{const.},$$

and substituting, we find the expression in brackets equal to

$$\frac{1}{h}\left\{4L(p_0) - 2(p_0^2+1)\left[\tfrac{1}{2}\log\frac{p+1}{p-1} - \frac{p}{^2-1}\right] - \frac{4p}{p^2-1}\right\},$$

and

$$xy = \varpi^2 \sin\phi\cos\phi,$$
$$= h^2(p^2-1)(1-q^2)\sin\phi\cos\phi,$$

ϕ being the azimuthal angle about the axis.

That is to say, the surfaces

$$[2L(p_0)\,I_2(p) - (p_0^2+1)\,H_2(p) - p]\,I_2(q)\sin\phi\cos\phi = \text{const.},$$

are composed of stream-lines; and the motion being symmetrical about the axis, $\phi = \text{const.}$ is the other family of surfaces composed of stream-lines. This gives the expression (70) equated to a constant, as representing stream surfaces.

From the expression (70), we can obtain the expression (66), when β is infinite, giving ψ when the obstacle is a sphere. We must suppose h indefinitely small; then we

find $p = \frac{r}{h}$, $q = \cos\theta$, where θ is the angle between the asymptotes of the hyperbola q, and is ultimately the vectorial angle of any point. But by (28), when h is small,

$$\mathrm{H}_2(p) = -\tfrac{1}{3}\frac{h}{r},$$

$$\mathrm{L}(p_0) = \mathrm{H}_2(p_0) + \tfrac{1}{2}\log\frac{p_0+1}{p_0-1} = \tfrac{2}{3}\frac{h}{a}$$

$$p + (p_0^2+1)\,\mathrm{H}_2(p) = \frac{1}{h}\left[r - \frac{a^2}{3r}\right]$$

and

$$\psi = h^2\mathrm{V}\sin^2\theta\left[\frac{r-\frac{a^2}{3r}}{h}\,\frac{3a}{2h} - \frac{r^2}{2h^2}\right]$$

$$= \frac{a^2\mathrm{V}\sin^2\theta}{4}\left[\frac{3r}{a} - \frac{a}{r} - \frac{2r^2}{a^2}\right]$$

which is the expression of p. 498, in which β is supposed infinite.

Three particular cases, besides the sphere, are worthy of notice.

(1.) *The obstacle a paraboloid*, of parameter $2l_0$. Taking $2l$, $2m$ as the parameters of the confocal paraboloids through any point, the solution is found by making h very great in (70).

We find, without difficulty,

$$\psi = \frac{\mathrm{V}}{2\log 4h/l_0}\,m\left[l\log\frac{l_0}{l} + l - l_0\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

(2) *A planetary spheroid of eccentricity* $1/\sqrt{2}$. Hence, by p. 478,

$$p_0 = i,$$

and (70) becomes

$$\psi = a^2\mathrm{V}\,\mathrm{I}_2(q)\left[\mathrm{I}_2(p) + \frac{p}{i}\right],$$

$$= \frac{a^2\mathrm{V}}{4}(q^2-1)(p-i)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where $2a$ is the greater axis.

(3) *A circular disc.* Here $p_0 = 0$,

$$\psi = -2a^2\mathrm{V}\,\mathrm{I}_2(q)\left[\mathrm{I}_2(p) + \frac{2i}{\pi}\{p + \mathrm{H}_2(p)\}\right] \quad . \quad . \quad . \quad . \quad (74).$$

where a is the radius.

Turning to (51) we find the velocities given by this formula are

$$\mathrm{U} = \frac{2\mathrm{V}}{i\pi} \frac{qp^2\sqrt{(1-q^2)}}{(p^2-q^2)\sqrt{(p^2-1)}},$$

$$\mathrm{W} = \frac{2\mathrm{V}}{i\pi}\left[\tfrac{1}{2}\log\frac{p+1}{p-1} + \frac{pq^2}{p^2-q^2} - \frac{i\pi}{2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

vanishing at the edge.

As an example of the analysis to which the use of the expression leads, I will now find the value of ψ for a surface approximately spheroidal, say $p = p_0 + \alpha_n \mathrm{I}_n(q)$, where α_n is a small quantity whose square will be neglected.

First suppose n an even number; let us then assume

$$\begin{aligned}\psi = {} & \mathrm{I}_2(q)\,[\mathrm{B}_2 p + \mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathfrak{B}_4\mathrm{H}_4(p)] \\ & + \mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{H}_2(p) + \mathrm{D}_4\mathrm{H}_4(p) + \mathfrak{B}_6\mathrm{H}_6(p)] \\ & + \ldots \\ & + \mathrm{I}_n(q)\,[\mathfrak{B}_n\mathrm{H}_{n-2}(p) + \mathrm{D}_n\mathrm{H}_n(p) + \mathfrak{B}_{n+2}\mathrm{H}_{n+2}(p)] \\ & + \mathrm{I}_{n+2}(q)\,[\mathfrak{B}_{n+2}\mathrm{H}_n(p) + \mathrm{D}_{n+2}\mathrm{H}_{n+2}(p)],\end{aligned}$$

at which point we shall find that we may stop; in fact, there are $n+3$ constants in the above expression, and $n/2+1$ terms, each of which yields two equations corresponding to $d\psi/dp = 0$, $d\psi/dq = 0$; we are thus left with one constant arbitrary, which is determined when we know the velocity at infinity. It is clear that all the constants except B_2, C_2, D_2 are of the order of α_n, and hence, at the surface, we may put $p = p_0$ in the terms where they arise as coefficients.

The condition $\frac{d\psi}{dq} = 0$ when $p = p_0 + \alpha_n \mathrm{I}_n(q)$ gives

$$\begin{aligned} & q\,[\mathrm{B}_2 p_0 + \mathrm{C}_2\mathrm{I}_2 + \mathrm{D}_2\mathrm{H}_2 + \mathfrak{B}_4\mathrm{H}_4] \\ & \quad + \alpha_n q \mathrm{I}_n(q)\,[\mathrm{B}_2 + \mathrm{D}_2\mathrm{P}_1 + \mathrm{D}_2\mathrm{Q}_1] \\ & \quad + \mathrm{P}_3(q)\,[\mathfrak{B}_4\mathrm{H}_2 + \mathrm{D}_4\mathrm{H}_4 + \mathfrak{B}_6\mathrm{H}_6] \\ & \quad + \ldots \\ & \quad + \mathrm{P}_{n-1}(q)\,[\mathfrak{B}_n\mathrm{H}_{n-2} + \mathrm{D}_n\mathrm{H}_n + \mathfrak{B}_{n+2}\mathrm{H}_{n+2}] \\ & \quad + \mathrm{P}_{n+1}(q)\,[\mathfrak{B}_{n+2}\mathrm{H}_n + \mathrm{D}_{n+2}\mathrm{H}_{n+2}] = 0; \end{aligned}$$

$\frac{d\psi}{dp} = 0$ gives

$$\begin{aligned} & \mathrm{I}_2(q)\,[\mathrm{B}_2 + \mathrm{C}_2\mathrm{P}_1 + \mathrm{D}_2\mathrm{Q}_1 + \mathfrak{B}_4\mathrm{Q}_3] \\ & \quad + \alpha_n \mathrm{I}_2(q)\,\mathrm{I}_n(q)\left[\mathrm{C}_2 + \mathrm{D}_2\frac{d\mathrm{Q}_1(p_0)}{dp_0}\right] \\ & + \mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{Q}_1 + \mathrm{D}_4\mathrm{Q}_3 + \mathfrak{B}_6\mathrm{Q}_5] \\ & + \ldots \\ & + \mathrm{I}_n(q)\,[\mathfrak{B}_n\mathrm{Q}_{n-3} + \mathrm{D}_n\mathrm{Q}_{n-1} + \mathfrak{B}_{n+2}\mathrm{Q}_{n+1}] \\ & + \mathrm{I}_{n+2}(q)\,[\mathfrak{B}_{n+2}\mathrm{Q}_{n-1} + \mathrm{D}_{n+2}\mathrm{Q}_{n+1}] = 0. \end{aligned}$$

Now,

$$I_2(q)\,I_n(q) = \frac{\delta_n}{2} I_{n+2}(q) + \frac{\epsilon_n - 1}{2} I_n(q) + \frac{\zeta_n}{2} I_{n-2}(q) \quad . \quad . \quad . \quad . \quad (17),$$

by p. 460.

Hence the second series of equations requires

$$\begin{aligned}
B_2 + C_2P_1 + D_2Q_1 + \mathfrak{B}_4Q_3 &= 0,\\
\mathfrak{B}_4Q_1 + D_4Q_3 + \mathfrak{B}_6Q_5 &= 0,\\
. \quad . \quad . \quad . \quad .

Last but one,

$$\mathfrak{B}_{n+2}U_{n+2} = \mathfrak{B}_n U_n + \alpha(\epsilon_n - 1)H_n,$$

$$D_n V_n = \mathfrak{B}_n U_n - \alpha(\epsilon_n - 1)\frac{H_{n+2}V_n}{U_{n+2}}.$$

Last but two,

$$\mathfrak{B}_n U_n = \mathfrak{B}_{n-2}U_{n-2} + \alpha\zeta_n H_{n-2},$$

$$= \beta + \alpha\zeta_n H_{n-2},$$

$$D_{n-2}V_{n-2} = \beta + \alpha\zeta_n \frac{H_n V_{n-2}}{U_n}.$$

We have here two values for $\mathfrak{B}_{n+2}U_{n+2}$; reconciling them, we find

$$0 = \beta + \alpha\zeta_n H_{n-2} + \alpha(\epsilon_n - 1)H_n + \alpha\delta_n H_{n+2},$$

or,

$$\beta = -\alpha(p_0^2 - 1)H_n(p_0).$$

With regard to the first pair of terms, we suppose C_2 arbitrary, and B_2, D_2 separated into two parts, of which the former are the same as when the spheroid is perfect, while the latter are of the order α, and fall in regularly with the above.

Hence we get

$$\begin{aligned}\psi = {} & I_2(q)[Bp + CI_2(p) + DH_2(p)] \\ & - \alpha(p_0^2 - 1)H_n(p_0)\left\{I_2(q)\left[\frac{p}{U_2} + \frac{H_2(p)}{V_2} + \frac{H_4(p)}{U_4}\right]\right. \\ & \qquad + I_4(q)\left[\frac{H_2(p)}{U_4} + \frac{H_4(p)}{V_4} + \frac{H_6(p)}{U_6}\right] + \cdots \\ & \qquad \left. + I_n(q)\left[\frac{H_{n-2}(p)}{U_n} + \frac{H_n(p)}{V_n}\right]\right\} \\ & - \alpha\left\{I_{n-2}(q)\left[\zeta_n \frac{H_n(p_0)}{U_n}H_{n-2}(p) - \zeta_n \frac{H_{n-2}(p_0)}{U_n}H_n(p)\right]\right. \\ & \quad + I_n(q)\left[-\frac{\zeta_n H_{n-2}(p_0)}{U_n}H_{n-2}(p)\right. \\ & \qquad \left. + \left(\overline{\epsilon_n - 1}\,\frac{H_{n+2}(p_0)}{U_{n+2}} - \frac{\zeta_n H_{n-2}(p_0)}{V_n}\right)H_n(p) - \frac{\delta_n H_{n+2}(p_0)}{U_{n+2}}H_{n+2}(p)\right] \\ & \quad \left. + I_{n+2}(q)\left[-\frac{\delta_n H_{n+2}(p_0)}{U_{n+2}}H_n(p) + \left(\delta_n \frac{H_{n+4}(p_0)}{U_{n+4}} - \delta_n \frac{H_{n+2}(p_0)}{V_{n+2}}\right)H_{n+2}(p)\right]\right\}.\end{aligned}$$

In this expression, B, C, D have the same values as the corresponding coefficients in (70), and

$$\alpha = \frac{\alpha_n}{2}\left(C + D\frac{dQ_1(p_0)}{dp_0}\right).$$

If n is odd we get a series exactly similar, except that the first terms are

$$-\alpha(p_0^2-1)\,H_n(p_0)\left\{I_3(q)\left[\frac{1}{U_3}+\frac{H_3(p)}{V_3}+\frac{H_5(p)}{U_5}\right]+\&c.\right\}.$$

There is one exceptional case, viz., $n=1$

$$p=p_0+\alpha_1 q.$$

We have

$$\psi=I_2(q)\,[Bp+CI_2(p)+DH_2(p)]$$
$$+\frac{\alpha}{Q_2(p_0)}\,[H_3(p)-H_3(p_0)]\,I_3(q),$$

where

$$\alpha=-\alpha_1\left[C+D\,\frac{dQ_1(p_0)}{dp_0}\right]$$
$$C+D\,\frac{dQ_1}{dp_0}=\frac{h^2V p_0\,(p_0^2+3)}{(p_0^2-1)\,L(p_0)}.$$

I shall now consider those two other functions, which, as we saw on p. 501, are solutions of $D^2\psi=0$, and make $\frac{d\psi}{dp}=\frac{d\psi}{dq}=0$, when $p=p_0$.

One of them consists of a series of terms of odd order only, and the other of terms of even order only.

Consider the latter; put $C_2=B_3=0$. Then we have the unlimited series of equations

$$B_2p_0+D_2H_2+\mathfrak{B}_4H_4=0=B_2+D_2Q_1+\mathfrak{B}_4Q_3$$
$$\mathfrak{B}_4H_2+D_4H_4+\mathfrak{B}_6H_6=0=\mathfrak{B}_4Q_1+D_4Q_3+\mathfrak{B}_6Q_5;$$
$$\cdots\cdots$$

whence, as on p. 506

$$D_nV_n=\mathfrak{B}_nU_n=\mathfrak{B}_{n+2}U_{n+2}$$
$$=B_2U_2.$$

And we find

$$\psi=B_2U_2\left\{\left[\frac{p}{U_2}+\frac{H_2(p)}{V_2}+\frac{H_4(p)}{U_4}\right]I_2(q)+\cdots\right.$$
$$\left.+\left[\frac{H_{n-2}(p)}{U_n}+\frac{H_n(p)}{V_n}+\frac{H_{n+2}(p)}{U_{n+2}}\right]I_n(q)+\&c.,\ ad\ inf.\right\}\quad\ldots\quad(76).$$

Similarly, the series consisting of terms of odd order is

$$\psi=B_3U_3\left\{\left[\frac{1}{U_3}+\frac{H_3(p)}{V_3}+\frac{H_5(p)}{U_5}\right]I_3(q)+\&c.,\ ad\ inf.\right\}\quad\ldots\quad(76\text{A}).$$

Now, the maximum value of the term

$$\frac{H_n(p)}{U_n} = \frac{H_n(p)}{H_n(p_0)\,Q_{n-3}(p_0) - H_{n-2}(p_0)\,Q_{n-1}(p_0)}$$

occurs when $p = p_0$; and this maximum becomes very great when n is very great; for by p. 475, when n is great $H_n(x) = -\frac{1}{2}\sqrt{\frac{\pi}{n^3}}\,\xi^{n-1}(1-\xi^2)^{\frac{1}{4}}$, and (HEINE, § 40) $Q_n(x) = \sqrt{\frac{\pi}{n}}\,\xi^{n+1}(1-\xi^2)^{-\frac{1}{4}}$.

Hence

$$\frac{H_n(p)}{U_n} = -\frac{1}{2\sqrt{\pi}} \cdot \frac{\xi_0^{-n+3}(1-\xi_0^2)^{\frac{1}{2}}}{(n-3)^{-\frac{1}{2}} - n^{\frac{3}{2}}(n-2)^{-\frac{3}{2}}(n-1)^{-\frac{1}{2}}}$$

where $\xi_0 = p_0 - \sqrt{(p_0^2-1)}$, mod. ξ_0 being less than 1. Hence this expression becomes infinite with n; and no infinite series in which such a term is present can represent a motion within the limits of our discussion.

We could have foreseen that these series would be inapplicable, for we have already found one to which there is no objection, and it has been proved by HELMHOLTZ ('Wiss. Abh.,' vol. 1, XII.) that there is only one slow motion which satisfies the equations for a viscous liquid, and obeys definite boundary conditions.

3. *Hyperboloids.*

Let us now consider the motion which may take place within a hyperboloid of one sheet; say, of parameter q_0.

The conditions $d\psi/dp = d\psi/dq = 0$ are evidently satisfied by taking

$$\psi = D_1 q + B_3 I_3(q)$$

where

$$D_1 + B_3 P_2(q_0) = 0\,,$$

or

$$\psi = \frac{B_3}{2}\,q\,(q^2 - 3q_0^2)$$

and

$$B_3 = -2Vh^2/3\,(1-q_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

That is to say, the stream surfaces are hyperboloids of the system confocal with the boundary.

The velocities which (77) gives are

$$\left.\begin{aligned} U &= -\frac{V}{1-q_0^2}\,\frac{(q^2-q_0^2)}{(p^2-q^2)}\,\frac{p}{i}\,\sqrt{\frac{1-q^2}{1-p^2}} \\ W &= \frac{V}{1-q_0^2}\,\frac{(q^2-q_0^2)}{p^2-q^2}\,q \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (77\text{A})$$

where V is the velocity at the centre from right to left.

In the particular case of a flat wall pierced by a circular hole, we have $q_0 = 0$, and

$$\left.\begin{aligned} \mathrm{U} &= -\mathrm{V}\frac{pq^2\sqrt{(1-q^2)}}{i(p^2-q^2)\sqrt{(1-p^2)}} \\ \mathrm{W} &= \mathrm{V}\frac{q^3}{p^2-q^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (77\text{B})$$

both of which vanish at the edge.

4. *Slipping Motions at the Boundary.*

Approaching now as near as may be possible to a discussion of the solutions when the liquid slips at the surface of the obstacle, we find, without difficult or very long calculation, that if on p. 495 we put $\alpha, \beta, \gamma, = p, q, \phi$, and consequently $u, v, =$ P, Q of p. 478, while $w = 0$, we find, I say, $a = b = 0$,

$$2c = \frac{1}{(p^2-q^2)\sqrt{(p^2-1.1-q^2)}}\left\{(1-q^2)\frac{d^2\psi}{dq^2} - (p^2-1)\frac{d^2\psi}{dp^2} + \frac{2q(1-q^2)}{p^2-q^2}\frac{d\psi}{dq} + \frac{2p(p^2-1)}{p^2-q^2}\frac{d\psi}{dp}\right\} \quad \ldots\ldots \quad (78)$$

Now if T is the tangential stress per unit area at any point of the obstacle, whether spheroid or hyperboloid, $\mathrm{T} = 2\mu c$, and a boundary condition being that this tangential stress shall be proportional to the velocity of the liquid, at the surface of the obstacle, we have

for spheroid (p_0) $\quad \mathrm{T} = \beta\mathrm{Q}, \quad \mathrm{P} = 0$, when $p = p_0$

for hyperboloid (q_0) $\quad \mathrm{T} = \beta\mathrm{P}, \quad \mathrm{Q} = 0$, when $q = q_0$

where β is a constant, becoming very large when the slipping is very slight.

Now

$$\begin{aligned} \psi = \mathrm{C}_0 + \mathrm{D}_1 q + \mathrm{I}_2(q)[\mathrm{B}_2 p + \mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathfrak{B}_4\mathrm{H}_4(p)] \\ + \mathrm{I}_3(q)[\mathrm{B}_3 + \mathrm{D}_3\mathrm{H}_3(p) + \mathfrak{B}_5\mathrm{H}_5(p)] \\ + \mathrm{I}_4(q)[\mathfrak{B}_4\mathrm{H}_2(p) + \mathrm{D}_4\mathrm{H}_4(p) + \mathfrak{B}_6\mathrm{H}_6(p)] + \ldots \end{aligned} \quad (69)$$

and remembering

$$\frac{d}{dx}\cdot\mathrm{I}_n(x) = \mathrm{P}_{n-1}(x)$$

$$\frac{d^2}{dx^2}\cdot\frac{\mathrm{I}_n(x)}{\mathrm{H}_n(x)} = -\frac{n(n-1)}{1-x^2}\cdot\frac{\mathrm{I}_n(x)}{\mathrm{H}_n(x)},$$

we see that if P, and, therefore, $d\psi/dq = 0$, for definite values of p, we also have $d^2\psi/dq^2 = 0$.

Hence the above conditions become, *for spheroidal obstacle*, $p = p_0$,

$$\frac{d\psi}{dq} = 0.$$

$$\frac{d^2\psi}{dp^2} - \frac{2p_0}{p_0^2 - q^2}\frac{d\psi}{dp} = \frac{\beta}{\mu}\sqrt{\frac{p_0^2 - q^2}{p_0^2 - 1}}\frac{d\psi}{dp}$$

when

$$p = p_0; \quad \ldots \ldots \ldots \ldots \quad (79)$$

for hyperboloid, $q = q_0$,

$$\frac{d^2\psi}{dq^2} - \frac{p^2 - 1}{1 - q_0^2}\frac{d^2\psi}{dp^2} + \frac{2q_0}{p^2 - q_0^2}\frac{d\psi}{dq} = \frac{\beta}{\mu}\sqrt{\frac{p^2 - q_0^2}{1 - q_0^2}}\frac{d\psi}{dq} \quad \ldots \quad (79\text{A})$$

From these equations we can detect at once a solution in a particular case, viz., the flux through a circular hole in a plane wall, when the result we have already obtained on p. 509, viz., $\psi = -\mathrm{V}a^2/3 \,.\, q^3$ (a being the radius of the hole, and V the velocity at its centre) still applies. In fact, the tangential stress at any point of the wall is zero.

But it is obvious that it is out of the question to apply, in general, the direct method of assuming (69) and substituting, in order to find functions which satisfy (79) and (79A), and it is also clear that the solution for the spheroid is not, as in the case of the sphere, of like simplicity with the case when the slipping is zero. In one case I have followed out the process of solution, viz., when the obstacle is a circular disc, and the slipping slight. We then have $p_0 = 0$, and

$$q\frac{d\psi}{dp} = \frac{\mu}{\beta}\frac{d^2\psi}{dq^2},$$

and assuming that the solution does not differ greatly from the case when there is no slipping, I at first neglected, on the right, all terms except those entering in (70), p. 502.

This led to a set of equations for finding the coefficients

$$\mathfrak{B}_5\mathrm{U}_5 - \mathfrak{B}_3\mathrm{U}_3 = -5 \,.\, \tfrac{2}{3} \,.\, \frac{2\mu a^2\mathrm{V}}{\beta}$$

$$\mathfrak{B}_7\mathrm{U}_7 - \mathfrak{B}_5\mathrm{U}_5 = -\tfrac{9 \cdot 4}{5} \,.\, (\tfrac{2}{3})^2 \,.\, \frac{2\mu a^2\mathrm{V}}{\beta}$$

$$\ldots \ldots \ldots$$

$$\mathfrak{B}_{2n+1}\mathrm{U}_{2n+1} - \mathfrak{B}_{2n-1}\mathrm{U}_{2n-1} = -\frac{(4n-3)(2n-2)}{2n-1}\left(\frac{2n+4\ldots 2}{2n-3\ldots 3}\right)^2 \cdot \frac{2\mu a^2\mathrm{V}}{\beta}.$$

Now, when n is very great, the limit of

$$\frac{2 \,.\, 4 \ldots 2n}{3 \,.\, 5 \ldots 2n+1} \quad \text{is} \quad \tfrac{1}{2}\sqrt{\frac{\pi}{n}}.$$

Hence, when n is great,

$$\mathfrak{B}_{2n+1}\mathrm{U}_{2n+1} - \mathfrak{B}_{2n-1}\mathrm{U}_{2n-1} = -\frac{2\pi\mu a^2\mathrm{V}}{\beta}$$

That is, the coefficients $\mathfrak{B}$ become infinite when n is infinite, and the series which the expression (69) would require us to use for ψ is divergent, showing that the assumption that ψ may be written in a series proceeding like (69) is, in this case, not justified.

It will be seen from the last section, and from p. 508, that an infinite series of the form, satisfying common boundary conditions, frequently becomes divergent. This would not always seem to show that there is no corresponding possible motion, but rather that the mode of expression chosen is not permissible.

Now it is easy to see that there is no solution of $D^2\psi = 0$, consisting of a finite number of terms of (69) which obeys (79) or (79A), the solution $\psi = q^3$ excepted. Hence it seems probable that a series of the form (69) is not appropriate for expressing the motion of a viscous liquid, slipping at the surface of a spheroid or hyperboloid.*

NOTE. (Added, June, 1891.)

At the suggestion of the referees, I add a more detailed consideration of the motion past a disc, and through a hole in a wall.

A circular disc, of radius a, is moved perpendicular to its plane with velocity V; the stream-lines relative to it are given by

$$\psi = -2a^2 \mathrm{V} \mathrm{I}_2(q)\left[\mathrm{I}_2(p) + \frac{2i}{\pi}\{p + \mathrm{H}_2(p)\}\right],$$

or, writing $p = i\mathfrak{p}$,

$$= -\frac{a^2\mathrm{V}}{\pi}(1 - q^2)[(1 + \mathfrak{p}^2)\tan^{-1}\mathfrak{p} - \mathfrak{p}].$$

The stresses at any point of the liquid are

$$\mathrm{P} = -p + 2\mu e; \quad \mathrm{Q} = -p + 2\mu f; \quad \mathrm{R} = -p + 2\mu g; \quad \mathrm{S} = 0 = \mathrm{T}; \quad \mathrm{U} = 2\mu c,$$

and p being the mean of the pressures normal to any three mutually orthogonal surfaces, I shall hereafter allude to it as the "mean pressure" at the point.

Let us first find p.

If $\mathrm{Q} = -\frac{p}{\rho} - \mathrm{V}$, we have†

$$-d\mathrm{Q} = \frac{\nu}{\varpi}\left(\frac{d\chi}{d\varpi}dz - \frac{d\chi}{dz}d\varpi\right)$$

where $\chi = \mathrm{D}\psi$.

Transforming to the elliptic coordinates p, q,

$$-d\mathrm{Q} = -\frac{\nu}{h}\left[\frac{1}{1-q^2}\frac{d\chi}{dp}dq + \frac{1}{1-p^2}\frac{d\chi}{dq}dp\right]$$

* See p. 515.

† BASSET, vol. 2, p. 262.

which is an exact differential by (3*e*) p. 494; and gives, in the case of any spheroid,

$$Q = \frac{\nu V}{hL(p_0)} \cdot \frac{q}{p^2 - q^2}$$

or in the case of the disc, of radius a,

$$Q = \frac{4\nu V}{\pi a} \cdot \frac{q}{p^2 - q^2},$$

and since there are no applied forces, the variable part of the mean pressure is

$$p = \frac{4V\mu}{\pi a} \cdot \frac{q}{\mathfrak{p}^2 + q^2}.$$

Again, by the formulæ, p. 496,

$$e = \frac{1}{h}\sqrt{\frac{p^2-1}{p^2-q^2}} \cdot \frac{du}{dp} - \frac{v}{h} \cdot \frac{q\sqrt{(1-q^2)}}{(p^2-q^2)^{\frac{3}{2}}};$$

$$f = \frac{1}{h}\sqrt{\frac{1-q^2}{p^2-q^2}} \cdot \frac{dv}{dq} + \frac{u}{h} \cdot \frac{p\sqrt{(p^2-1)}}{(p^2-q^2)^{\frac{3}{2}}};$$

$$g = \frac{up}{\sqrt{(p^2-1 \,.\, p^2-q^2)}} - \frac{vq}{\sqrt{(1-q^2 \,.\, p^2-q^2)}};$$

$$a = b = 0;$$

$$2c = \frac{1}{h}\sqrt{\frac{1-q^2}{p^2-q^2}} \cdot \frac{du}{dq} + \frac{1}{h}\sqrt{\frac{p^2-1}{p^2-q^2}} \cdot \frac{dv}{dp} - \frac{vp\sqrt{(p^2-1)}}{h(p^2-q^2)^{\frac{3}{2}}} + \frac{uq\sqrt{(1-q^2)}}{h(p^2-q^2)^{\frac{3}{2}}};$$

whence, in the case of the disc,

$$e = \frac{2V}{a\pi} \cdot \frac{p^2 q(1+q^2-2p^2)}{(p^2-q^2)^2(p^2-1)} = \frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q(1+q^2+2\mathfrak{p}^2)}{(\mathfrak{p}^2+q^2)^2(\mathfrak{p}^2+1)};$$

$$f = \frac{2V}{a\pi} \cdot \frac{p^2 q}{(p^2-q^2)^2} = -\frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q}{(\mathfrak{p}^2+q^2)^2};$$

$$g = \frac{2V}{a\pi} \cdot \frac{p^2 q}{(p^2-q^2)(p^2-1)} = -\frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q}{(\mathfrak{p}^2+q^2)(\mathfrak{p}^2+1)};$$

$$c = \frac{2V}{a\pi} \cdot \frac{pq^2}{(p^2-q^2)^2}\sqrt{\frac{1-q^2}{p^2-1}} = \frac{2V}{a\pi} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2+q^2)^2}\sqrt{\frac{1-q^2}{\mathfrak{p}^2+1}}.$$

These all vanish on the surface of the disc ($p = 0$). Hence the total pressure on the disc is got by integrating the mean pressure over both sides of its surface. This gives $16V\mu a$, agreeing with the known value for any ellipsoid.*

I append tracings of the curves, const. $= \psi/a^3$, $p/2\mu$, e, f, g, c, with numerical values calculated on the supposition that $V/\pi a = 1$. The first shows the stream-lines;

* Basset, § 497.

the second, the mean pressure at any point; the third, fourth, and fifth, the departure from this mean in directions respectively normal to the ellipsoids, to the hyperboloids, and to planes through the axis; the last shows the tangential stress for these particular directions of normal pressure.

With the exception of g, which has a maximum at the point on the axis distant one radius from the disc, all increase if we approach the edge of the disc in particular directions, and we can make this increase very large; but at the edge itself they appear to be indeterminate, not infinite, all the surfaces of equal stress passing through one and the same line; and there seems no ground to suppose that it is impossible to impress a sufficiently great external pressure upon the whole liquid to prevent the pressures becoming negative. But it suggests that the test of zero or finite velocity at a sharp edge is not sufficient to ensure the motion being admissible.

Next, consider motion through a circular hole in a wall.

We have, for any hyperboloid,

$$\psi = \frac{2\mathrm{V}h^2}{3(1-q_0^2)}(q^3 - 3qq_0^2)$$

where V is the velocity at the centre.

Hence

$$\mathrm{Q} = -\frac{4\nu\mathrm{V}}{h(1-q_0^2)}\left\{\frac{p}{p^2-q^2} - \tfrac{1}{2}\log\frac{1+p}{1-p}\right\}$$

and the mean pressure is

$$p = \frac{4\mu\mathrm{V}}{ih(1-q_0^2)}\left[\tan^{-1}\mathfrak{p} + \frac{\mathfrak{p}}{\mathfrak{p}^2+q^2}\right],$$

and for a flat wall ($q_0 = 0$)

$$p = \frac{4\mu\mathrm{V}}{a}\left[\tan^{-1}\mathfrak{p} + \frac{\mathfrak{p}}{\mathfrak{p}^2+q^2}\right],$$

where a is the radius of the hole.

The difference between the pressures at infinity is

$$\Pi = \frac{4\mu\mathrm{V}\pi}{ih(1-q_0^2)}.$$

Again, if F is the total flux per unit time through the aperture,

$$\mathrm{F} = \frac{\mathrm{V}\pi h^2}{3(1-q_0^2)}(q_0-1)^2(2q_0+1).$$

Hence, a being the radius of the aperture, so that $a = ih\sqrt{(1-q_0^2)}$

$$\mathrm{F} = \frac{a^3}{12\mu}\cdot\frac{(2q_0+1)(1-q_0)^{\frac{1}{2}}}{(1+q_0)^{\frac{3}{2}}}\cdot\Pi,$$

or, in the case of the plane wall,

$$\mathrm{F} = \frac{a^3}{12\mu}\Pi.$$

By applying the formulæ given at the beginning of this note

$$e = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2(1+q^2-2p^2)}{(p^2-q^2)^2(p^2-1)} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2(1+q^2+2\mathfrak{p}^2)}{(\mathfrak{p}^2+q^2)^2(\mathfrak{p}^2+1)};$$

$$f = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2}{(p^2-q^2)^2} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2+q^2)^2};$$

$$g = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2}{(p^2-q^2)(p^2-1)} = -\frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2+q^2)(\mathfrak{p}^2+1)};$$

$$c = \frac{2\mathrm{V}i}{a} \cdot \frac{p^2q}{(p^2-q^2)^2}\sqrt{\frac{1-q^2}{p^2-1}} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}^2q}{(\mathfrak{p}^2+q^2)^2}\sqrt{\frac{1-q^2}{\mathfrak{p}^2+1}}.$$

Tracings of sections of these surfaces are given with numerical values on the supposition that $\mathrm{V}/a = 1$, as also for $p/2\mu$. With the exception of the mean pressure, the curves are in general character the same as for motion past a disc.

The surfaces of mean pressure are the same, except for the numerical values attached to them, for all planetary spheroids and for all hyperboloidal boundaries.

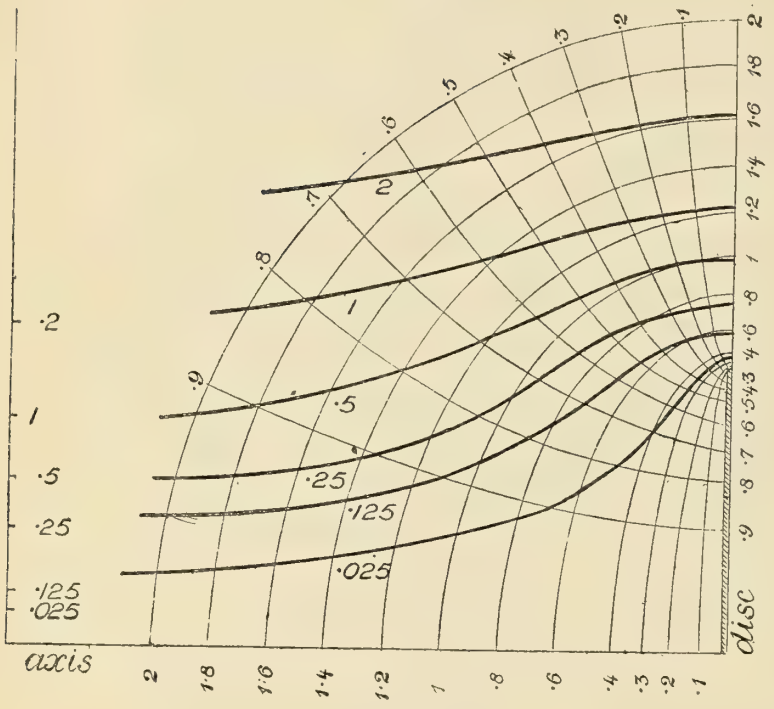
Since the reading of this paper, I have found a proof* that the expansion of any function of q in a linear series of $\mathrm{I}(q)$'s is convergent where q is less than unity. This throws more light on the problem of slipping motion past a disc, attacked in Chapter VI., for it shows that if the expression for any function ψ is a divergent series, then there is no corresponding finite motion.

The work there was conducted on the assumption that the motion, if existing, did not differ greatly when the slipping was small from the case when there was none; and, therefore, we have not a rigorous proof that there is no such motion, though this seems to make it highly probable. As it is not easy to see how there should be a motion with slipping for one particular case of spheroid, viz., a sphere, and none for another, viz., a disc, unless the former is a wholly exceptional case, it is interesting to notice that the probability of it being exceptional is borne out by the analysis (p. 511), for if we approach the case of the sphere, by putting $p = r/h$ where h is indefinitely small in the equation of condition (79), q ultimately disappears from the equation.

It must be remembered that the search for steady motions in viscous liquids proceeds on an assumption, viz., that if the motion is started from rest, the limit of the rate of change of velocity is zero at all points when the time is infinite; therefore, such motions must always be exceptional, and, again, the analysis bears us out in expecting this, for the same thing will be noticed in the two solutions which I give that Stokes pointed out in the case of the sphere—that the possibility of satisfying the boundary conditions $\frac{d\psi}{dq} = 0 = \frac{d\psi}{dp}$ depends on the presence of an irregular term in the series (p. 501), namely, $\mathrm{B}_2 p \mathrm{I}_2(q)$ for spheroids, and $\mathrm{B}_3\mathrm{I}_3(q)$ for hyperboloids.

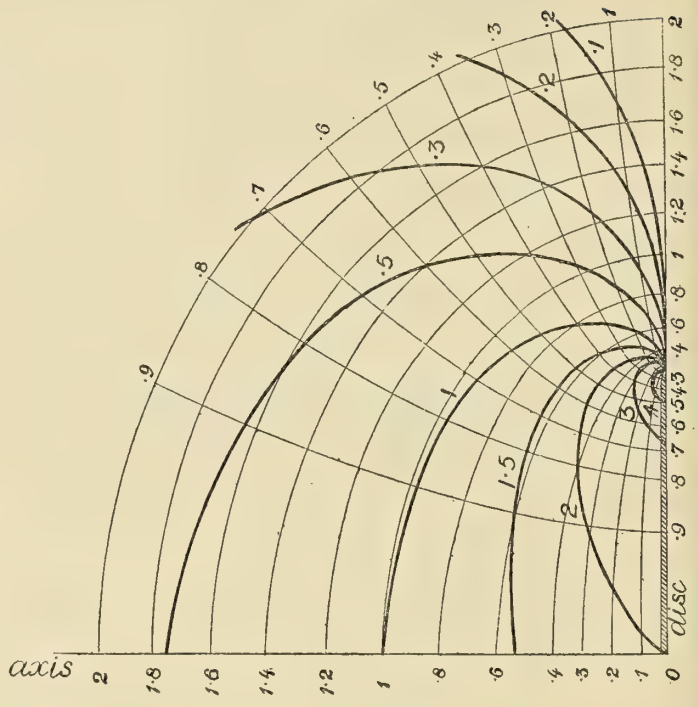
* Inserted in Chapter II.

Fig. 1.



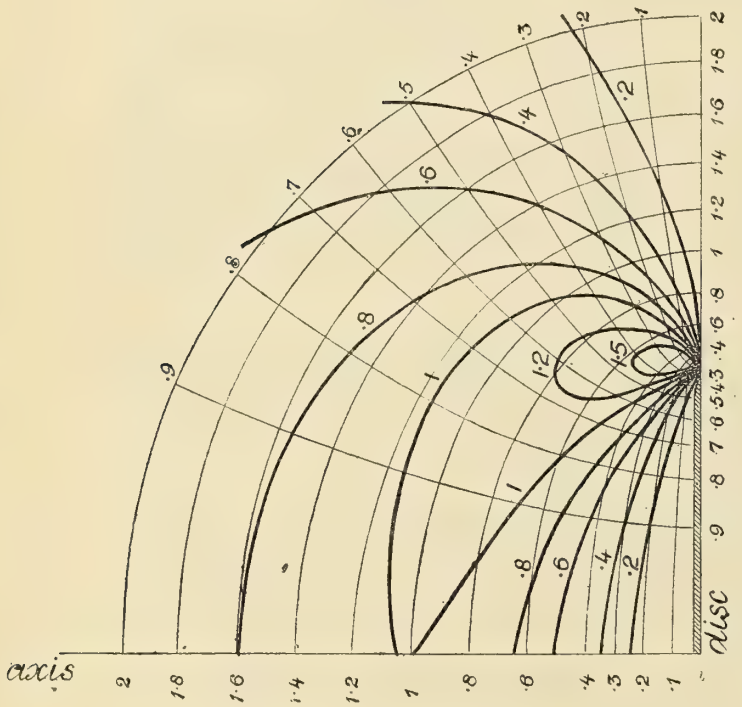
$\psi \cdot \frac{\pi}{\mathrm{V}a^2}$ for disc.

Fig. 2.



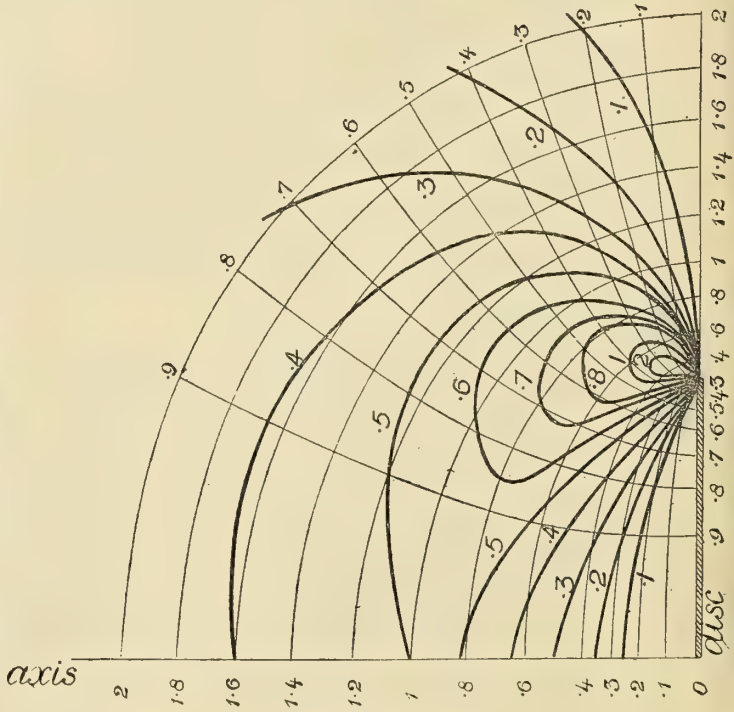
$\frac{p}{2\mu} \cdot \frac{a\pi}{\mathrm{V}}$ for disc.

Fig. 3.



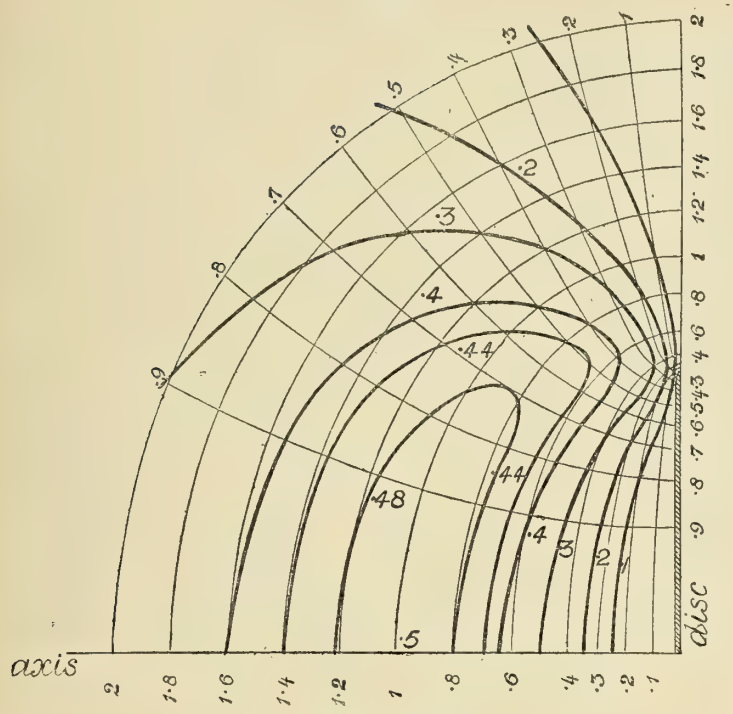
$e \cdot \frac{a\pi}{\mathrm{V}}$ for disc.

Fig. 4.



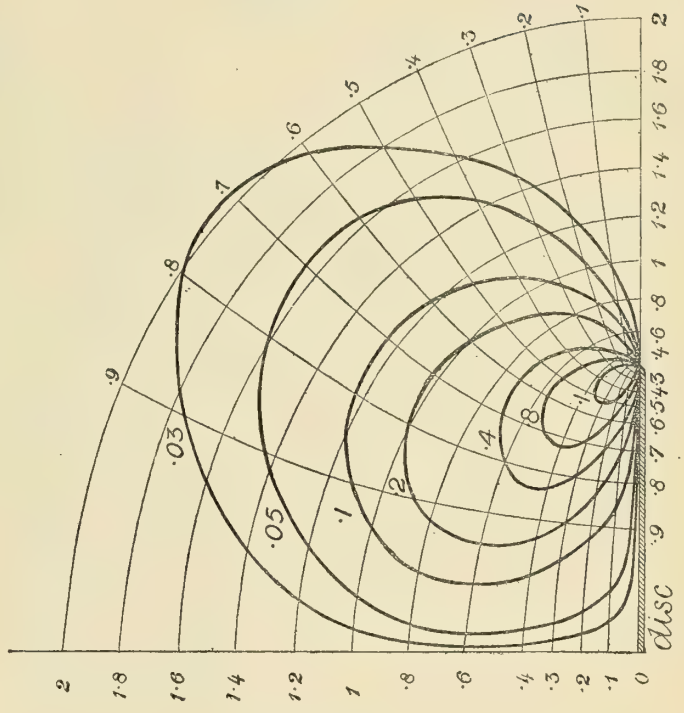
$f \cdot \frac{a\pi}{\mathrm{V}}$ for disc.

Fig. 5.



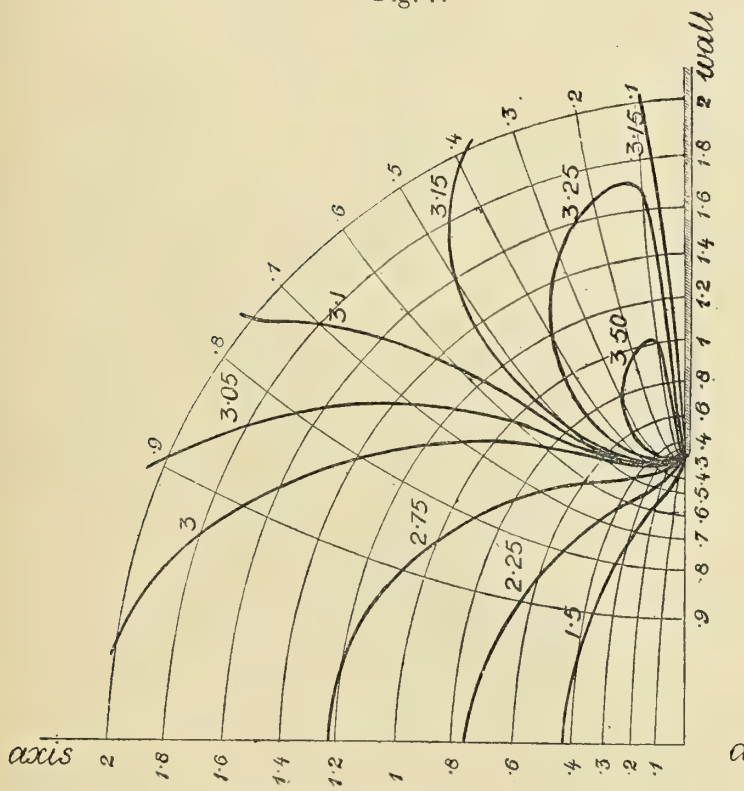
$g \cdot \frac{a\pi}{V}$ for disc.

Fig. 6.



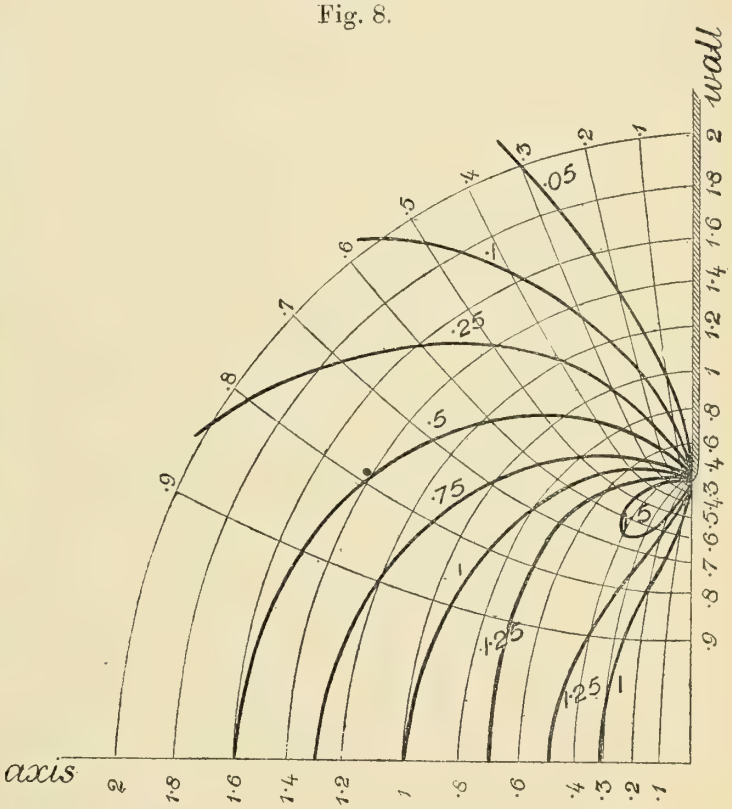
$c \cdot \frac{a\pi}{V}$ for disc.

Fig. 7.



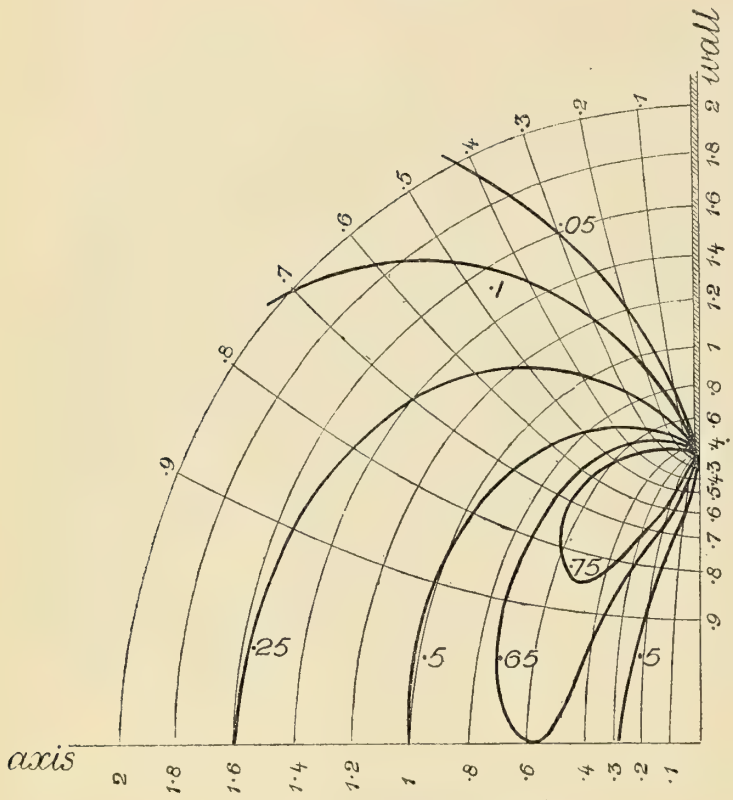
$\frac{p}{2\mu} \cdot \frac{a}{V}$ for hole in wall.

Fig. 8.



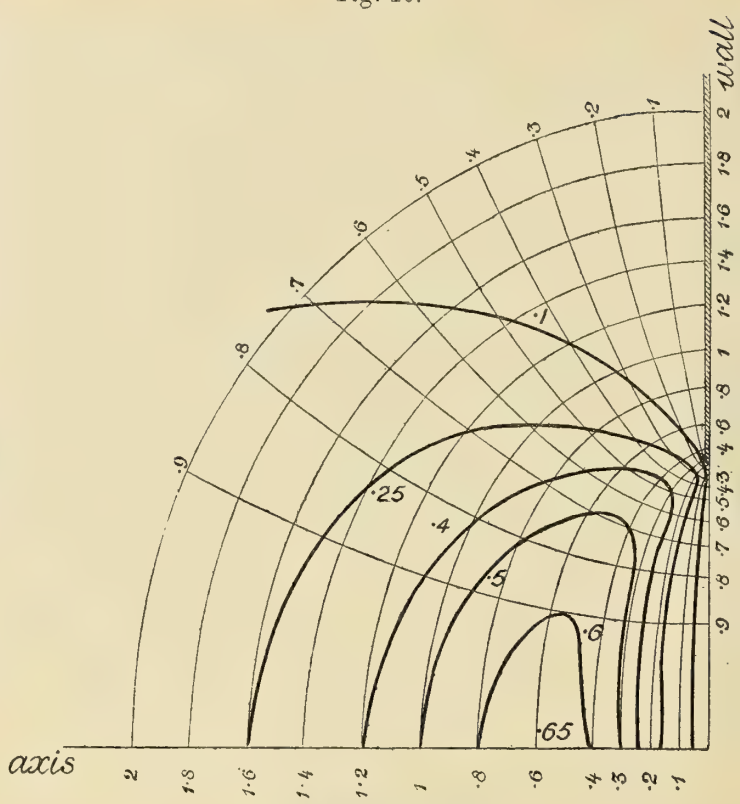
$e \cdot \frac{a}{V}$ for hole in wall.

Fig. 9.



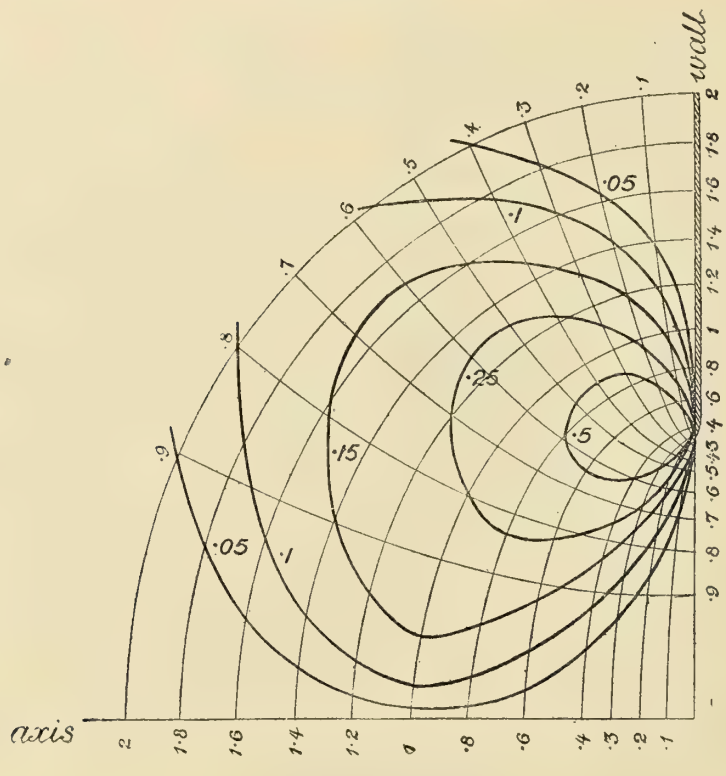
$f \cdot \frac{a}{V}$ for hole in wall.

Fig. 10.



$g \cdot \frac{a}{V}$ for hole in wall.

Fig. 11.



$c \cdot \frac{a}{V}$ for hole in wall.

XIII. *Quadrant Electrometers.*

By W. E. Ayrton, J. Perry, *and* W. E. Sumpner.

Received May 19,—Read June 4, 1891.

[Plates 9-12.]

Table of Contents.

Section.		Page.
I.	General account of the experiments	519
II.	Effect of varying the distance between the fibres	526
III.	Effect of varying the distance between the quadrants	533
IV.	Cause of the peculiar action of the White electrometer	539
V.	Motion of the electrical zero	544
VI.	Electrometer used to measure power	545
VII.	Improved quadrant electrometer, construction	548
VIII.	Results obtained with the improved quadrant electrometer	554
IX.	Sketch of the mathematical theory of the White quadrant electrometer.	561

I.—General Account of the Experiments.

In 1886, on continuously charging up the needle of Sir William Thomson's bifilar quadrant electrometer, No. 5, made by Messrs. White, of Glasgow, and in use at the laboratories at the Central Institution, it was noticed that the deflection of the needle, when the same P.D. (potential difference) was maintained between the quadrants, instead of steadily increasing, first increased and then diminished, so that both for a large charge on the needle as well as for a small, the sensibility of the instrument was small. A similar effect had been described by Dr. J. Hopkinson in the 'Proceedings of the Physical Society,' vol. 7, Part I., for the previous year, and the explanation he gives of this curious result is that if the aluminium needle be below the centre of the quadrants the downward attraction of the needle, which increases with the square of the needle's charge, increases the pull on the bifilar suspension, and so for high charges more than compensates for the increased deflecting couple due to electrical action. On raising, however, the needle of our electrometer much above the centre of the quadrants, the anomalous variation of sensibility of the instrument, with increase of charge in the needle, did not disappear, and even when the needle was raised so that it was very close to the top of the quadrants, and when,

if Dr. HOPKINSON's explanation were correct, the sensibility (or deflection corresponding with a given P.D. between the quadrants) ought to have been very great for a large charge on the needle, it was, on the contrary, found to be small.

As the needle had been somewhat damaged while the instrument was in the possession of the late Mr. CROMWELL VARLEY, before it was so kindly made over to the Central Institution by his son, Mr. CROMWELL VARLEY, Junior, together with other apparatus belonging to his father, a new needle, platinum wire, and weight, were obtained from Messrs. WHITE. On suspending the new needle with the fine platinum wire and weight, as received from Messrs. WHITE, the needle was found to have two modes of vibration, one the ordinary slow one, when the platinum weight turned in the acid at the bottom of the Leyden jar, and the other a very quick one, due to the twisting of the wire itself without the weight moving.* This quick vibration was removed by replacing the fine platinum wire supplied by Messrs. WHITE with a somewhat thicker one. The quadrants were adjusted for symmetry, the silk fibres tightened so as to have equal tension, as shown by the sensibility for a given potential of the needle, being a minimum, and the electrometer again tried, but it was still found that when the quadrants were close to one another, and when, therefore, as the needle was best shielded from external action, the standard formula for the electrometer might be expected to be most nearly fulfilled, the sensibility as before, first increased as the charge in the needle was increased, and then steadily diminished for further increase of charge on the needle. It was now, however, observed that, if the distance separating the quadrants was increased to $\frac{1}{4}$ inch, the sensibility tended to a limit for a large charge instead of first increasing and then diminishing.

Before replacing the new needle, it was carefully weighed with the platinum wire attached and the weight dipping into the acid, and a calculation was made as to the magnitude of the effect that should arise from the change of the pull of the fibres due to any upward or downward attraction of the needle by the quadrants. This calculation showed that for a P.D. of 3000 volts between the needle and the quadrants, the amount of such attraction was quite unable to account for the observed diminution of sensibility with large charges in the needle. Dr. HOPKINSON says in his paper, "Increased tension of the fibres from electrical attraction does not therefore account for the whole of the facts, although it does play the principal part." The experiments that we made at the end of 1886 and beginning of 1887, confirmed by the calculation above referred to, proved that, at any rate in our specimen of the quadrant electrometer, the principal part of the anomalous action was not caused by an increased tension of the fibres, and that, therefore, some other cause must be looked for to explain the observed results. The following investigation was, therefore, undertaken to ascertain the cause, and the means of removing it, and we are happy to say that it

* We do not know whether the weight, with the fine wire attached, supplied us by Messrs. WHITE is exactly the same as is supplied generally by them with the quadrant electrometer, if so the users must experience a very unnecessary difficulty from the Blackburn-pendulum-like vibration of the needle.

has resulted in our finding out that a quadrant electrometer, as made by Messrs. WHITE can, by a special adjustment being given to its quadrants, be made to fulfil the law given in text books for the quadrant electrometer; secondly, that this special adjustment may be dispensed with if the construction of the instrument be altered; and thirdly, that when this alteration in the construction is carried out, it is possible to obtain an instrument having a sensibility many times as great as that obtainable with the White form of construction.

As long as an electrometer is only employed to measure a steady P.D. between the quadrants, it may not be of much interest to know the way in which the sensibility varies with the charge in the needle, unless great sensibility be required; in which case it is important to know for what charge on the needle maximum sensibility is obtained; in fact, for the measurement of steady P.D's. the main thing to be sure of, is that the deflection is proportional to the P.D. between the quadrants when the charge in the needle is constant. But when the electrometer is employed in conjunction with a non-inductive resistance to measure the power given by an alternate or an intermittent current, which may be any function of the time, to a circuit that may possess capacity, or an E.M.F., or self or mutual induction, or all four (in accordance with the method devised by Professor FITZGERALD and ourselves during the meeting of the Electrical Congress in Paris in 1881, and published the same autumn by M. POTIER), it is absolutely necessary that the electrometer should exactly fulfil the law—

$$t = k(Q_1 - Q_2)\left(N - \frac{Q_1 + Q_2}{2}\right),$$

where t is the torque exerted at any moment on the needle by electrical action, and Q_1, Q_2, and N the potentials, at the same time, of the opposite pairs of quadrants and of the needle respectively relatively to the outside case of the instrument, and k is a constant. [See Section VI. "Electrometer used to Measure Power."]

Much has been written during the last three or four years by MM. GOUY, LEDEBOER, and others regarding the departure of the action of the quadrant electrometer from the law given above, but we are not aware that hitherto the cause of the discrepancy has been pointed out, or the means of removing it and obtaining an instrument that fulfils this law.

We thought at first that the curious results might be due to some capillary action between the platinum weight and the sulphuric acid which varied with the potential of the weight, but, experiments having shown that no such action existed, it was thought desirable to measure the P.D. between the needle and the outside case of the instrument by means of one of Sir WILLIAM THOMSON'S absolute electrometers (made especially sensitive by thinning the little coach springs carrying the movable aluminium disc), and so ascertain completely the laws connecting the variation of sensibility of the quadrant electrometer with the potential of the needle, the distance between the silk fibres, the distance apart of the quadrants, &c. The investigation

has taken some years to complete, and in carrying it out this quadrant electrometer has had to be taken entirely to pieces many times. The labour that this has entailed will be appreciated by anyone who has had experience in taking a White electrometer to pieces.

In the ordinary form of the quadrant electrometer as constructed by Messrs. WHITE it is not possible to get at the guard-tube or the lower part, either of the needle, or of the quadrants, without taking the whole interior mechanism bodily out of the Leyden jar. Since in doing this there is danger of breaking the silk fibres from which hangs the aluminium needle with its attached platinum weight, especially as the pull on the fibres is, of course, much increased when the weight is lifted out of the dense sulphuric acid, it may be advisable to clamp the needle and take the stress off the fibres before lifting out the interior of the instrument. And, if, for example, it be desired to adjust the aluminium wire relatively to the guard-tube in

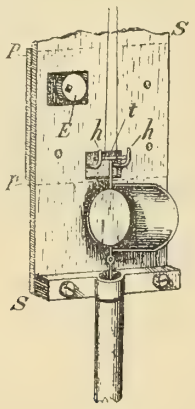


Fig. 1.

which it hangs, or to remove this guard-tube so as to be able to get at the attachment of the aluminium wire, or to straighten the needle if slightly bent, or, generally, to make any adjustment liable to break the silk fibres, it is certainly advisable to clamp the needle before removing the interior of the instrument. Under these circumstances the following operation has to be performed:—1st. Clamp the needle, lift the interior of the apparatus out of the Leyden jar, and place it on some metallic support provided with levelling screws by which the instrument can be levelled. 2nd. Make the necessary adjustment, and unclamp the needle to see whether the adjustment has been correctly made. 3rd. Replace the interior of the apparatus in position inside the Leyden jar.

The method ordinarily adopted for clamping the needle consists in passing a screw through a loop in the aluminium wire connecting the mirror with the needle, and screwing this screw into the insulated framework supporting the apparatus.

There is not much difficulty in putting in this screw by means of the square headed screwdriver provided with the instrument, but there is the greatest difficulty in

removing the screw without breaking the fibres. In fact, the danger of breaking the fibres when making one of the adjustments above referred to without previously clamping the needle is hardly greater than the danger of breaking them when removing the clamping screw.

We, therefore, had fitted to the electrometer the device which is seen in fig. 1* for easily raising and lowering the needle. *E* is a circular piece of metal turning eccentrically about a pin fixed to a small plate *pp*, which can slide up and down at the back of the insulated support *SS*. This plate *pp* carries two little hooks *hh*, which, on being raised by turning the eccentric *E* (by means of the square screwdriver provided with the electrometer), lift up the T-piece *t* at the top of the suspension, and take the weight of the mirror, the needle, and its attachment, off the silk fibres. When then it is desired to remove the interior of the electrometer out of the Leyden jar, all that is necessary to be done is to turn the eccentric *E* one way to raise the needle, and the other way to lower it, either of which operations can be done quickly without the slightest risk of breaking the fibres.

If the instrument is to be packed for travelling it will be well to also clamp the needle by means of the screw inserted as usual in the loop on the aluminium wire, but as the eccentric lifting arrangement which we have added to the electrometer enables the screw to be inserted and removed, if it be desired to use it, when there is no tension whatever on the fibres, the clamping and unclamping by means of the screw can be done with perfect safety. We have to thank Mr. H. B. Bourne for helping us in designing this eccentric attachment for the White electrometer.

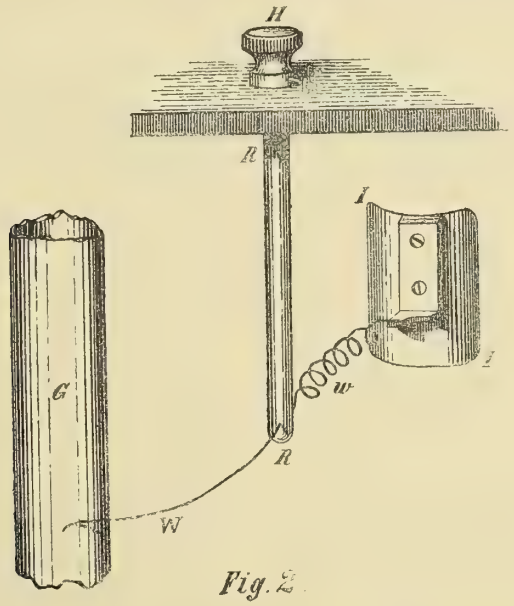


Fig. 2.

We may take this opportunity of mentioning another modification that we have

* In order that the mirror may be more clearly seen, one half of the horizontal guard-tube which surrounds the mirror has been omitted in fig. 1.

introduced, with the assistance of Mr. BOURNE, for the purpose of improving the good working of the quadrant electrometer. In the instrument as ordinarily constructed the framework of the replenisher is always in connection with the sulphuric acid which forms the interior coating of the Leyden jar, and hence electric leakage can always take place over the framework of the replenisher. To avoid this we altered the replenisher in one of our quadrant electrometers, so that by raising a knob on the top outside the instrument the replenisher could be bodily raised up until the platinum wire attached to it was lifted out of the acid. But we think that the following very simple device which we have attached to another electrometer is in every way better. Instead of the replenisher being electrically connected with the interior of the Leyden jar through the acid, as is the case with Messrs. WHITE'S instruments, we arrange that it can be connected with, or disconnected from, the guard-tube G (fig. 2), which is itself always in connection with the acid. This is done thus:—II, the insulated inductor of the replenisher, is attached by means of a flexible wire to a springy wire W carried at the end of an ebonite rod RR. By turning the milled head H this springy wire W can be made to touch, or not to touch, the guard-tube, and thus II can be connected with the interior of the Leyden jar, when the replenisher is to be used, or left wholly disconnected from the Leyden jar, so that there is absolutely no leakage over the replenisher frame, when the replenisher is not in use.

With this device we can dispense altogether with the clamping arrangement that is usually employed for holding the rotatory spindle that supports the carriers of the replenisher in such a position that the carriers are not in contact with the inductors.

In the ordinary use of the quadrant electrometer the charging rod is kept disconnected from the Leyden jar, and therefore the insulation of the charging rod need not be very perfect, since leakage can only occur over it during the time the Leyden jar is being charged; in these experiments, however, as the interior of the Leyden jar of the quadrant electrometer and the needle had to be permanently connected with the lower movable disc of the absolute electrometer, it was necessary to keep the charging rod of the quadrant electrometer always in connection with the Leyden jar of this instrument. This made the experiments very difficult to perform on damp days, indeed on some days hours have been spent without a single trustworthy result being obtained. Hence the results given in the following tables, which are the trustworthy ones, only represent a small portion of the work that has been actually carried out.

As the two electrometers are permanently fixed on two brick pillars, some 14 feet apart, in the laboratory, the absolute electrometer being close to the light, the reflecting quadrant electrometer with its scale well away from the light, the connection between the lower disc of the absolute and the charging rod of the quadrant electrometers was made by a long fine wire supported on a well insulated glass rod, shielded from dust and kept dry by sulphuric acid. The outsides of the two instruments were also connected together by an uninsulated copper wire.

In the earlier of the experiments the Leyden jars of the two electrometers were charged quite irrespectively of one another, sometimes positively, and sometimes negatively, but as the investigation advanced both jars were always charged positively, the P.D. between the inside and outside of the jar of the absolute electrometer being maintained at + 5800 volts, and the force required to bring the movable aluminium disc to the sighted position was always the same, as the reading of the upper micrometer screw was not varied. The attracted aluminium disc was brought to the sighted position by moving the lower plate only, and the potential of the needle of the quadrant electrometer was measured by showing the difference between the readings of the screw moving this lower plate (1) when the lower plate was connected with the outside of the absolute electrometer; (2) when the lower plate was disconnected from the case and connected with the needle of the quadrant electrometer. The absolute electrometer behaved well when the upper micrometer screw was not touched, the earth reading for a fixed P.D. between the inside and outside of the Leyden jar varying but very slightly from day to day.

A series of calibrations of this absolute electrometer carried out in the ordinary way by placing known weights on the movable aluminium disc to reproduce the effect of electric attraction, showed that with the position we gave to the upper micrometer screw, and taking "v" as 30 true ohms, a motion of the lower plate through one division of the vertical scale corresponded with 358·6 volts.

In July, 1888, several large P.D.'s were measured by means of this absolute electrometer (using the constants that we had determined for this instrument), and also by means of one of Sir WILLIAM THOMSON's commercial "electrostatic voltmeters" reading to 20,000 volts, kindly lent us by Messrs. ELLIOTT BROS. The result of these comparisons led first to a correction in the constants that had been previously sent out with the electrostatic voltmeters from Glasgow, and secondly to a new determination of the value of "v." For Sir WILLIAM THOMSON had calibrated these voltmeters by a galvanometer method, based on the value of the electro-chemical equivalent of silver, as determined by Lord RAYLEIGH and Mrs. SIDGWICK,* while we had checked the calibration of the electrostatic voltmeter, by comparing this instrument with the absolute electrometer. The value of "v" thus obtained was 298,000,000 metres per second.

The scale employed with the quadrant electrometer is a circular one at 5 feet distance from the mirror. There are 290 divisions in 1 foot length of scale, and as both scale and electrometer are permanently screwed down in position, one division of the scale always corresponds with $\frac{1}{2900}$ radian deflection of the mirror, or about $\frac{1}{50}$ of a degree. The scale is so placed that the mechanical zero of the electrometer corresponds with 0 on the scale, that is to say, the spot of light is at 0 when the needle is adjusted as symmetrically relatively to the quadrants as it is possible to do with the

* "Electrometric determination of v," 'Brit. Assoc. Report,' 1888. See also the 'Electrician,' p. 448, vol. 22, 1889, for the correction for the effective area of the attracted disc in the absolute electrometer.

eye, the whole instrument being discharged. The *electrical* zero is next made to coincide with the *mechanical* zero, that is to say, on charging the needle when the four quadrants are connected with the outside case of the instrument, the fourth quadrant is slightly moved until the spot of light is brought back to the same place that it occupied on the scale when the needle was discharged. But it was found that if this were done for some particular potential of the needle, a fresh adjustment of the fourth quadrant had to be made for each potential to which the needle was charged, or, in other words, it was practically impossible so to adjust the quadrants that without further touching them the electrical and mechanical zeros should be the same for a considerable variation of the potential of the needle. Hence, in what follows the position of the electrical zero is in each case given. The question of the electrical and mechanical zeros differing from one another will be further dealt with in Section V., entitled "Motion of the Electrical Zero."

II.—Effect of Varying the Distance between the Fibres.

If N be the P.D. in volts between the needle and the outside case of the electrometer, and D the deflection of the needle in microradians produced by a P.D. equal to V between the opposite pairs of quadrants, one pair being connected with the outside case, the ordinary formula given for the electrometer reduces to

$$D = \frac{\sigma}{2} V \left(N - \frac{V}{2} \right),$$

where σ is usually regarded as being constant both for variations of V and of N, the distance between the quadrants and the distance between the fibres being kept constant.

The first thing tested was the constancy of σ for a fixed value of N and for varying values of V. The relative values of V were measured by means of a Thomson's and Varley's potential divider, which enables a P.D. to be accurately subdivided to the $\frac{1}{10000}$th part if required, and it was found that for any large and fixed charge on the needle less than the value which produced instability, and for any fixed position of the quadrants and distance between the fibres, the deflection of the needle D was proportional to V to within 2 per cent., that is to say, when V and D alone were the variables σ was always constant to within 2 per cent. In the great majority of cases the constancy of σ was much closer than this, but, as it was found necessary to vary the position of the fourth quadrant in order that the deflections for all the different values used for V should be on the scale when the charge in the needle was varied, the want of constancy of σ for unsymmetrical positions of the instrument was sometimes nearly 2 per cent.

On Sheet I. (Plate 9) are seen two curves connecting the double deflection of the

electrometer needle with the deflecting P.D. between the quadrants. One set of results was obtained when the P.D. between the needle and the outside case of the instrument was maintained at 1793 volts, and the other when the needle's potential was 2510 volts. In each case the curve is practically a straight line, and it will be observed that in spite of the needle's potential having been increased about 40 per cent., the combination of these two lines is practically the same.

The next point tested was the constancy of σ when N was varied. When N is large the preceding formula is approximately

$$D = \frac{\sigma}{2} VN ,$$

so that

$$\sigma = \frac{2D}{VN} .$$

In the following tables σ is the deflection of the ray of light in *microradians*, divided by the product of the deflecting P.D. in volts maintained between the quadrants into the P.D. in volts between the needle and the case of the electrometer. σ is thus the *sensibility factor* of the instrument for a given position of the quadrants, and for a given distance between the fibres; and the different values of σ not only show how the behaviour of this electrometer departs from the simple mathematical law, but enables its sensibility to be compared with the sensibility of any other quadrant electrometer which may be tested *absolutely* in this manner.

On November 3, 1887, the fibres were adjusted so as to be parallel, and at about 2 millims. apart, and the results of the experiments are given in Table I. The values for the double deflection per one Leclanché cell for the different values of N are also plotted in the curve on Sheet II. (Plate 9), corresponding with "Fibres Parallel." It is interesting to notice that while σ is far from being constant for different values of N it has the same value for the same value of N, although V and D are very different, thus confirming the experiments previously made with fixed values of N and different values of V.

TABLE I.—November 3, 1887. Fibres Parallel.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
1·45	1886	30	130	16·4	Adjustable quadrant at +100.
	1543	20	118	17·9	
	1219	10	104	20·3	
	944	..	86	21·5	
	705	..	66	22·4	
	466	..	43	22·0	
	158	0	14	21	The potential of the needle could not be very accurately measured when less than about 360 volts.
	1513	− 20	118	18·6	
	2733	− 120	154	13·4	
	2378	− 50	138	13·9	Points marked ⊙ on the curve, Sheet II. (Plate 9).
	1914	− 30	128	15·4	
2·9	2726	− 150	152	13·3	Adjustable quadrant at + 92·5.
	2259	− 60	140	14·7	
	1919	− 40	128	16·0	
	1822	− 35	126	16·4	
	1531	− 25	115	18·0	
	1305	− 20	104	18·9	
	1201	− 18	99	19·6	
	1015	− 15	87	20·4	Points marked △ on the curve, Sheet II. (Plate 9).
	914	− 12	84	21·8	
	792	− 11	70	21	
	638	− 9	59	22	
	599	− 9	54	21·4	
	398	− 5	36	21·6	
	262	− 3	25	..	The potential of the needle could not be very accurately measured when less than about 360 volts.
	204	− 4	18	21·0	
	129	− 2	7	..	
1·45	1951	29	127	15·1	Adjustable quadrant at 113.
	746	13·5	70	22·4	
	1161	22	99	20·2	
	1560	29·5	120	18·3	
	2636	8	147	13·3	
	2621	10	148	13·4	Points marked × on curve, Sheet II. (Plate 9).
	2392	24	143	14·3	
	2435	20	145	14·2	
	2199	28	119	..	
	1829	31	127	16·5	
	610	9	57	22·3	

The fibres were now put nearer together at the top, and the experiments repeated. The results obtained are given in Table II. and are plotted on the curve on Sheet II. (Plate 9), corresponding with "Fibres near at top." The departure from the law that the sensibility of the electrometer is proportional to the potential of the needle, *i.e.*, the want of constancy of the "sensibility factor" is here very marked, a maximum sensibility being obtained when the P.D. between the needle and the metallic case of the electrometer is about 1800 volts.

TABLE II.—November 4, 1887. Fibres near together at the Top.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
1·45	2420	0	176	17·3	Adjustable quadrant at 160.
	2223	20	187	20	
	2080	30	192	21·9	
	1919	38	195	24·2	
	1768	40	197	26·5	
	1606	43	198	29·3	
	1506	45	193	30	
	1338	44	191	33·9	
	1112	37	180	38·5	Points marked ⊙ on the curve,
	961	35	170	42	Sheet II. (Plate 9).
	889	31	162	43·3	
	753	21	146	46·1	
	674	18	137	48·3	
	581	15	120	49·1	
	484	9	103	50·6	
	376	5	81	51·2	
	255	2	58	54·2	
December 3, 1887.					
	240	1	49	48·5	The potential of the needle
	359	2·5	70	46·4	could not be very accurately
	538	7	100	44·2	measured when less than
	717	13	133	44·1	about 360 volts.
	896	20	156	41·4	
	1076	26	174	38·5	
	1255	31	185	35·1	Adjustable quadrant at 50.
1·45	1434	36	194	32·2	Points marked × on the curve, Sheet II. (Plate 9).
	1610	39	199	29·4	
	1793	42	201	26·7	
	1972	42	202	24·4	
	2152	41	200	22·1	
	2331	38	198	20·2	
	2510	32	195	18·5	
	2659	20	190	16·8	
	2869	8	186	16·1	
	3048	−2	184	14·4	

TABLE II. (continued).—December 4, 1887. Some Experiments repeated to Test the Accuracy of those previously made.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double Deflection per Leclanché cell.	Sensibility factor.	Remarks.
	717	9	136	45·1	Adjustable quadrant at +50.
	1434	32	194	32·2	
	2152	32	200	22·1	Points marked × on the curve, Sheet II. (Plate 9).
	2869	−5	187	15·5	

From the dates given for the tests contained in Table II., it will be seen that this group was made at intervals lasting over one month. The sensibility factor is very far from being constant, varying from about 14 when the potential of the needle was about 3000 volts to about 50 when it was about 300 volts; but nevertheless for each potential of the needle with the constant distance between the quadrants, and between the fibres, the sensibility has practically a perfectly constant value. The following examples selected from Table II. illustrate this:—

Date.	Potential of needle in volts.	Sensibility factor.
Nov. 4	753	46·1
Dec. 3	717	44·1
Dec. 4	717	45·1
Nov. 4	1506 } Mean 1444	30 } Mean 32
	1338 }	33·9 }
Dec. 3	1434	32·2
Dec. 4	1434	32·2
Nov. 4	2080	21·9
Dec. 3	2152	22·1
Dec. 4	2152	22·1
Nov. 4	2420	17·3
Dec. 3	2869	16·1
Dec. 4	2869	15·5

The small differences between the values of the sensibility factor on November 4, and on the later dates for the same potential of the needle, arises probably from the fact that in the interval the fibres were separated far apart at the top to enable the next set of experiments to be made, and approached again on December 3, until

they were again, as far as the eye could judge, in the same position as on November 4. In changing the distance between the fibres, as well as in altering the tension of the fibres to adjust in each case for minimum sensibility (corresponding with equal tension) only one of the rollers on which the fibres are wound was touched, in order that a position could be reproduced as nearly as possible. But it was, of course, impossible to exactly reproduce a previous position of the fibres, and the sensibility factor for a fixed potential of the needle, and for a so-called fixed position of the fibres was, consequently, not absolutely constant.

In view, however, of the extreme delicacy of the tests and the great variation of the sensibility factor with a variation of the potential of the needle, the wonder is that for the same potential of the needle the sensibility factor was as constant as it was.

On November 8, 1887, the fibres were separated until they were far apart at the top, the tension adjusted for minimum sensibility, and experiments were made on November, 8, 10, 11, and December 2, to determine the values of the sensibility factor. The results are given in Table III., and the values of the double deflection per Leclanché cell for the different P.D's. between the needle and the outside case are plotted in the curve on Sheet II. (Plate 9), corresponding with "Fibres far apart at the Top."

TABLE III.—November 8, 1887. Fibres far apart at the top.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
4·35	2256	— 13	89·6	9·4	Adjustable quadrant at 68.
	1936	— 6	81	9·9	
	1596	— 2	71	10·6	
	1338	— 1	63·6	11·3	Points marked ⊙ on the curve, Sheet II. (Plate 9).
	1047	— 0·5	52·3	11·9	
	778	0	41·3	12·6	
	513	0	29	13·4	
	323	0	19	14	
November 10, 1887.					
2·9	359	— 1	19	12·6	Adjustable quadrant at — 50. Points marked △ on the curve, Sheet II. (Plate 9).
	717	— 4	38	12·6	
	1076	— 8	53	11·7	
	1434	— 15	67	11·1	
	359	2	18·5	12·3	Adjustable quadrant at 0.
	717	8	39	12·9	
	1076	16	53·5	11·8	
	1434	28	68·5	11·3	
November 11, 1887.					
2·9	2869	— 42	110	9·1	Adjustable quadrant at — 100.
	2510	— 50	93	8·8	
	1793	— 35	75	9·9	
	1076	— 18	54	12	Points marked △ on the curve, Sheet II. (Plate 9).
	2152	— 43	84	9·3	
	2869	— 42	106	8·8	
	3048	— 35	118	9·2	
December 2, 1887.					
1·45	2252	— 29	87	9·2	Adjustable quadrant at — 100. Weather unfavourable, leakage rather rapid. When the potential of the needle was high the needle was drawn aside, causing the electrical zero to differ much from the mechanical zero, and rendering the sensibility factor unusually small. When this happened, a further increase in the potential of the needle caused it to stick. The experiment with the needle at a potential of 3184 volts is an example of this. Points marked × on the curve, Sheet II. (Plate 9). *Needle stuck.
	2887	— 27	111	9·1	
	2986	— 20	118	9·4	
	3077	— 15	133	10·3	
	3023	— 13	124	9·9	
	3184	— 100	64	4·8	
	3137	— 33	108	8·2	
	3087	+ 2	136	7·3	
	3055	— 8	127	9·9	
	2900	— 26	108	8·9	
	2840	— 29	105	8·8	
	3045	— 10	124	9·7	
	3077	— 8	127	9·8	
	1832	— 18	76	9·9	
	2180	— 26	84	9·2	
	2714	— 33	97	8·5	
	2883	— 30	104	8·6	
	3163	+ 8	138	10·4	
	3586*	..	..	..	

From the Tables of Results and from the curves on Sheet II. (Plate 9), we see that when the fibres are parallel the sensibility factor falls off somewhat as the potential of the needle is increased, whereas with the fibres far apart at the top the sensibility factor first falls off, and then increases again as the potential of the needle steadily increases. With the fibres near together at the top the curve is perhaps the most interesting, as here the sensibility factor falls off so much for high potentials of the needle that the deflection of a fixed P.D. betweeen the quadrants is actually smaller for a high potential of the needle than for a lower potential.

III.—Effect of Varying the Distance Between the Quadrants.

The reasoning which leads to the ordinary formula for the quadrant electrometer

$$D = \frac{\sigma}{2} V \left(N - \frac{V}{2} \right)$$

(where D is the deflection produced by a P.D. equal to V between the opposite pairs of quadrants, one pair being connected with the outside case of the instrument, and N the P.D. between the needle and the outside case), shows that σ is proportional to the variation of capacity of the needle per radian deflection. Consequently it might be expected that since the needle is less covered by the quadrants when they are drawn out, and, since the variation of capacity of the needle per radian deflection is less when the quadrants are pulled out than when they are in, that the instrument would be less sensitive when the quadrants are drawn out than when they are in. But those who are accustomed to use the quadrant electrometer know that the opposite is the case. Here again then there is a marked difference between the actual behaviour of the instrument, and the behaviour that the simple formula would lead one to expect.

In all the preceding experiments three of the quadrants had been left untouched with their edges at from $2\frac{1}{2}$ to 3 millims. apart, and the fourth quadrant, the one to which the micrometer screw is attached, had alone been moved. But at the beginning of 1888 experiments were made to determine the way in which the sensibility factor varied with the potential of the needle for different distances between the quadrants and for different distances between the fibres. When the quadrant under the induction plate was moved far out, it seemed quite possible that the latter would exert some influence on the curves. The deflections, however, were found to be practically unaffected by disconnecting the induction plate from one pair of quadrants, and connecting it with the other pair. As a rule the induction plate was kept electrically connected with the quadrant immediately underneath it. The distances between the

quadrants given in the following tables were estimated by eye only, and are, therefore, only approximately correct.

Table IV. gives the results for four positions of the quadrants when the fibres were near together, and the curves on Sheet III. (Plate 10) show the results recorded graphically.

The quadrants are called respectively Q_1, Q_2, Q_3, and Q_4; Q_4 being the one adjustable by means of the adjustable screw.

TABLE IV.—January 23, 1888. Fibres near together at the top.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
Quadrants about $2\frac{1}{2}$ millims. apart.					
1·45	359	1	80	53·2	
	717	3	134	44·6	
	1075	7	167	37·5	
	1434	8	179	29·8	
	1793	+ 7	182	24·2	Points marked × on the curve, Sheet III. (Plate 10).
	2152	− 1	175	19·3	
	2510	− 17	166	17·6	
	2869	− 40	162	13·5	
	3227	− 86	157	11·6	
	3407	− 120	160	11·2	
	3586	− 165	175	11·6	
Quadrants about $3\frac{1}{2}$ millims. apart.					
1·45	1076	..	182	40·3	
	1793	33	220	29·3	Points marked ⊙ on the curve, Sheet III. (Plate 10).
	2510	− 8	209	19·9	

January 24, 1888.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
		Quadrants about $3\frac{1}{2}$ millims. apart.			
1·45	359	6	80	53·2	
	717	12	142	47·0	
	1076	21	181	40·2	
	1434	26	207	34·3	
	1793	25	218	29·0	Points marked ⊙ on the curve,
	2152	16	217	24·0	Sheet III. (Plate 10) ⊙.
	2510	— 3	212	20·2	
	2869	— 40	205	17·0	
	3227	— 102	215	15·9	
	3586	— 277	255	17·0	
		Quadrants about 4 millims. apart.			
	1793	— 11	..		
	1434	2	332	54·2	
	1255	3	279	52·9	Points marked △ on the curve,
	1076	3	235	52·1	Sheet III. (Plate 10).
	896	3	196	52·2	
	538	1	121	53·5	
	1793	5	331	44	Quadrant Q_1 pushed inwards
	1076	19	216	47·8	very slightly, the three other
	2152	— 23	370	41·0	quadrants remaining un-
	2331	— 43	369	37·7	touched. Points marked ▽
	2331	..	376	38·3	on the curve, Sheet III.
	2510	— 75	383	36·3	(Plate 10).
					Q_3 pushed inwards very slightly,
					and Q_4 slightly altered to
	1793	13	290	38·5	bring electrical zero to near
	2152	— 8	313	34·7	the middle of the scale, the
	2689	— 82	331	29·4	other quadrants remaining
					untouched. Points marked
	2869	Off scale			⊡ on the curve, Sheet III. (Plate 10).
		Quadrants about 1 millim. apart.			
	717	11	128	42·5	
	1076	11	145	32·2	
	1434	14	153	25·4	Points marked ⊙ on curve,
					Sheet III. (Plate 10).
	2152	11	127	14·0	
	2869	2	107	8·9	
	1434	3	142	23·6	
	717	3	128	42·5	Q_3 slightly shifted.
	2152	— 3	126	13·9	
	2869	— 27	110	9·1	
	2869	— 195	106	8·8	
	2152	— 147	126	13·9	Q_4 slightly shifted.
	1434	— 107	142	23·6	
	717	— 47	130	43·2	

An examination of the preceding results, and of the curves on Sheet III. (Plate 10), corresponding with them, shows that when the quadrants are near together the sensibility factor diminishes at first slowly and afterwards rapidly as the potential of the needle is steadily increased. As the distance between the fibres increases, the rate of diminution of the sensibility factor, with increase of the potential of the needle, grows less until, when the quadrants are about 4 millims. apart, the sensibility factor actually increases with increase of potential of the needle, so that the curve connecting deflection for a constant P.D. between the quadrants with a varying P.D. between the needle and the outside case, becomes convex instead of concave to the axis along which the potential of the needle is reckoned. Hence, with our specimen of the White electrometer, *when the distance between the quadrants is about* 3·9 *millims. the sensibility factor is a constant, and the conventional formula for the quadrant electrometer becomes the true formula.*

The next set of experiments was made with the fibres far apart at the top, the fibres being carefully adjusted for equality of tension after having been separated. The results are given on Table V., and they are also plotted on Sheet IV. (Plate 10); it will be seen that the general shape of the curves for the various distances between the quadrants is the same as was obtained when the fibres were near together.

TABLE V.—January 30, 1888. Fibres far apart at the top.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
		Quadrants about 1 millim. apart.			
1·45	1793	− 20	88	11·7	
	2152	− 21	90	10·0	
	2510	− 23	91	8·6	
	2869	− 25	90	7·5	
	3227	− 30	92	6·7	
	3586	− 25	111	7·4	
	359	0	28	18·6	
	717	0	52	17·3	
	1076	0	71	15·7	
	1434	0	81	13·4	Points marked × on the curve, Sheet IV. (Plate 10).
	1793	− 1	88	11·7	
	2152	− 4	90	10·0	
	2510	− 5	90	8·6	
	2869	− 8	89	7·4	
	3227	− 12	91	6·7	
	3407	− 16	94	6·6	
	3586	− 11	103	6·7	
	2940	..	90	7·2	
	1434	− 1	82	13·6	
	359	0	29	19·2	
		Quadrants about $2\frac{3}{4}$ millims. apart.			
1·45	1793	0	101	13·4	
	2152	− 4	110	12·2	
	2510	− 13	118	11·2	
	2869	− 38	123	10·2	
	3227	− 63	133	9·1	
	3586	− 115	147	9·8	Points marked ⊙ on the curve, Sheet IV. (Plate 10).
	359	0	27	18	
	717	1	54	18	
	1076	2	74	16·4	
	1434	2	89	14·7	
	1793	1	102	13·6	
	2510	− 12	118	11·2	
	3227	− 64	135	10	
		Quadrants about 4 millims. apart.			
1·45	1793	7	144	19	
	2510	− 63	237	22·5	
	3227	off the scale			
	2152	− 5	188	20·8	Points marked △ on the curve, Sheet IV. (Plate 10).
	1793	2	149	19·9	
	1076	5	85	18·8	
	359	1	30	20	

The curves contained on Sheets III. and IV. (Plate 10) show that this quadrant electrometer may be adjusted so that the variation of sensibility with the potential of the needle may be made to follow one or other of *three distinct laws*. If the quadrants be near together there are certain limits between which the potential of the needle may vary without producing more than a small change in the deflection corresponding with a fixed P.D. between the quadrants; for example, when the quadrants were about $2\frac{1}{2}$ millims. apart, and the fibres near together at the top, the deflection produced by a P.D. of 1·45 volts between the quadrants only varied about 11 per cent. when the potential of the needle was increased from 896 to 3586 volts. When the fibres were far apart at the top it was when the quadrants were about 1 millim. apart as seen in Sheet III. (Plate 10) that a similar flatness was obtained in the curve connecting deflection with potential of the needle. In this case the deflection of the needle was practically quite constant when its potential varied between 2152 and 3227 volts; and even when the potential of the needle was increased from 1434 to 3407 volts, that is, by nearly 2000 volts, the deflection only increased by less than 9 per cent. This arrangement of the quadrants gives but a comparatively small sensibility, but, where great sensibility is not required, it would be a convenient one to employ, as leakage of the Leyden jar, or loss of potential of the needle due to the rapid absorption that occurs when the jar is first charged, would only slightly affect the deflection for a fixed P.D. between the quadrants.

When the quadrants were at about 3·9 millims. apart the deflection for a given P.D. between the quadrants was almost directly proportional to the potential of the needle. This then would be the arrangement to employ when the electrometer is used with alternating P.D's. And, lastly, when the quadrants were 4 or more millims. apart the deflection increased much more rapidly than the potential of the needle, so that maximum sensibility, bordering on instability, is obtained with this arrangement of the quadrants.

For each distance between the quadrants less than the distance at which instability begins to occur, the shape of the sensibility curve is perfectly definite, but its shape alters very much for an exceedingly small alteration in the position of the quadrants when the distance separating them is some three or more millims.

It might have been anticipated that the conventional mathematical law for the quadrant electrometer, viz. :—

$$D = \frac{\sigma}{2}(Q_1 - Q_2)\left\{N - \frac{Q_1 + Q_2}{2}\right\}$$

(where D is the deflection, and Q_1, Q_2, and N the potentials of the two pairs of quadrants and of the needle relatively to the outside case) would have been most nearly fulfilled when the quadrants were pushed far in, so that the needle was well under cover of them. But the preceding results show that this is very far from being the case, and that constancy in the sensibility factor σ, for different values of N, is

obtained *only* when the quadrants are so far separated that the sensibility of the instrument for the particular adjustment of the distances between the fibres is fairly large.

IV.—Cause of the Peculiar Action of the White Electrometer.

Towards the end of 1887 the cause of the irregularity in the action of the Thomson quadrant electrometer, as constructed by Messrs. White, began to dawn on us. The wire supporting the aluminium needle, as well as the wire which connects the needle with the sulphuric acid in the Leyden jar, is enclosed in a metallic guard-tube to screen the wire from external action. But in order that the needle may project outside the guard-tube, GG′, openings are made in its two sides, as seen in fig. 3. Hence

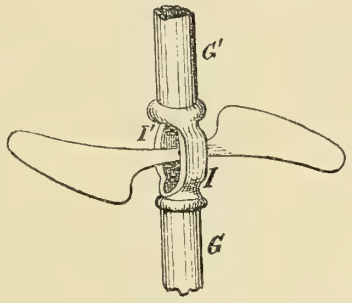


Fig. 3.

the moment the needle is deflected from its zero position each half of the needle becomes unsymmetrically placed relatively to the two bits I and I′ of the guard-tube. Therefore, in spite of the needle and the guard tube being always maintained at the same potential, there is a repulsion between the charges on the two bits I and I′ of the guard-tube, and the charges on the two halves of the needle; and this repulsion might not only have the defect of seriously diminishing the sensibility of the quadrant electrometer, as made by Messrs. White, but might cause the variation of sensibility of the electrometer with variation of the P.D. between the needle and the outer coating of the Leyden jar to follow a far more complicated law than that expressed by the conventional formula just given.

To test this theory, that the peculiarities in the law of the quadrant electrometer are due to the electric action of the guard-tube on the needle, in consequence of the special shape of the former, the want of symmetry of the guard-tube was intensified and varied by attaching a piece of thin aluminum foil to it above and below the needle, and experiments on the variation of sensibility of the electrometer with variation of the potential of the needle made for the positions of this piece of aluminium foil, F, shown in figs. 4 and 5. These figures show a section of the guard-tube through the needle, I and I′, figs. 4 and 5 giving the sections of the connecting pieces I and I′ seen in fig. 3. The difficulty connected with these experiments was

very great, arising from the almost impossibility of attaching the aluminium foil to the guard-tube without slightly altering the position of the tube. And as the slightest alteration of the position of the guard-tube considerably altered the electrical zero, it was necessary in each case to readjust the movable quadrant in order that the electrical zero might coincide with the mechanical zero. This, in itself, altered the sensibility curves slightly, and the effect due to this cause must be kept in mind when comparing the sensibility curves made before attaching the foil, for the various positions of the foil, and after removing it. These experiments on the guard-tube were more troublesome than any of the others, and very great care was expended on them. Many sets of experiments were made and curves drawn to record the results; but, since in the majority of cases it was considered that too much disturbance of the position of the guard-tube had been produced in the act of attaching the foil or of altering its position, the majority of the results were rejected.

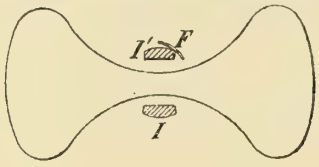


Fig. 4.

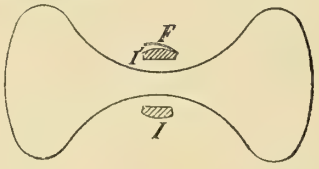


Fig. 5.

The following sets, given in Tables VI. and VII., were obtained on December 6, 7, and 8, 1887. They correspond with positions of the adjustable quadrants varying between -50 and $+50$, and, as subsequent experiments made on December 9 (the results of which are given in Table IX.) confirm the idea that this amount of variation of the adjustable quadrant alone would not account for the variations in the results that were obtained on attaching the thin piece of aluminium foil to the side of the guard-tube, and on moving this foil from the side to the back of the guard-tube, the results given in Tables VI. and VII. may be regarded as correct. They may, therefore, be compared with those contained in the latter part of Table II., and obtained on December 3 and 4 before the foil was attached, as well as with those contained in Table VIII., and obtained on December 8, after the foil had been removed. For in all these experiments the variation in the position of the adjustable quadrant was kept between the same limits -50 and $+50$, the position of the other quadrants remained the same, and there was the same distance between the fibres, which were adjusted for minimum sensibility when the potential of the needle was about 1800 volts.

The results are all plotted on Sheet V. (Plate 11), and it is important to notice that:—

1. The curves obtained with "*no foil*" were practically the same before attaching and after removing the foil, and had ordinates much higher than when the foil was attached.

2. The curves obtained with "*foil at the side of the guard-tube*" had the next higher ordinates.

3. The curves obtained with "*foil at the back of the guard-tube*" had the lowest ordinates.

We may, therefore, conclude that the attachment of a small piece of foil to the guard-tube not only altered the shape of the curve connecting the sensibility of the electrometer with the potential of the needle, but, whether the foil was put at the side or at the back of the guard-tube, it diminished the sensibility for high potentials of the needle, in fact, exaggerated the very peculiarity in the behaviour of the instrument which is under investigation. This makes it almost certain that the reason why, under ordinary circumstances, a high potential of the needle does not produce a high sensibility, lies in the fact that the guard-tube is not continuous all round in the neighbourhood of the needle.

The bending up which may be seen at the ends of the curves on Sheet V. (Plate 11), as well as at the ends of many of the curves on the preceding sheets, is probably due to a slight tilting of the needle at high potentials. Such a tilting brings the ends of the needle near the quadrants, and much increases the sensibility. (See Section IX., "Sketch of the Mathematical Theory of the White Electrometer.")

TABLE VI.—December 6, 1887. Thin Piece of Aluminium Foil at the side of the Guard-Tube.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
1·45	359	— 1	72	47·8	Adjustable quadrant at — 50.
	717	— 5	130	43·2	
	1076	— 7	174	38·6	
	1434	— 13	186	30·8	
	1615	— 17	189	27·9	
	1793	— 23	188	25·0	
	1972	— 29	186	22·5	Points marked × on the curve, Sheet V. (Plate 11).
	2152	— 38	183	20·3	
	2331	— 43	181	18·5	
	2510	— 51	177	16·8	
	2689	— 60	173	15·4	
	2869	— 77	173	14·4	
	3043	— 93	170	13·3	
	3227	— 122	176	12·9	
	3407	— 155	184	12·9	
	3586	— 218	223*	14·8	* This observation confirmed many times.
	717	— 5	140	46·5	
	1434	— 23	186	30·8	
	2152	— 53	183	20·3	
	2869	— 98	173	14·3	
	3048	— 119	172	13·4	
	3227	— 124	172	12·7	
	3406	— 157	173	12·1	
	3586	— 228	221	14·7	

TABLE VII.—December 7, 1887. Thin Piece of Aluminium Foil at the back of the Guard-Tube.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double Deflection per Leclanché cell.	Sensibility factor.	Remarks.
1·45	359	4	82	54·5	Adjustable quadrant at + 50.
	717	11	130	43·2	
	1076	21	160	35·5	
	1255	26	167	31·7	
	1434	29	172	28·6	
	1615	30	171	25·2	
	1793	31	170	22·6	
	1972	29	166	20·0	
	2152	27	162	17·9	
	2331	23	158	16·1	
	2510	17	153	14·5	
	2689	8	150	13·5	Points marked △ on the curve,
	2869	— 1	145	12·0	Sheet V. (Plate 11).
	3048	— 13	141	11·0	
	3227	— 31	137	10·5	
	3407	— 49	138	9·6	
	3586	— 95	142	9·4	
	3765	— 127	152	9·6	
	2987	— 14	142	11·3	
	1972	27	165	19·9	
	1651	33	170	24·5	
	1004	29	157	37·3	
	538	24	114	50·5	
	179	21	53	70·5	Reading on the absolute electrometer not very trustworthy at such low volts.
December 8, 1887.					
					Adjustable quadrant at
1·45	359	— 41	79	52·5	— 50
	717	— 92	134	44·5	— 50
	717	— 51	136	45·2	0
	896	— 61	152	40·4	— 50
	896	— 148	151	40·1	0
	1076	..	154	34·2	— 50
	1076	..	162	35·9	0
	1434	..	170	28·2	— 50
	1434	— 96	168	27·9	0
	717	..	134	44·5	— 50
	1793	— 240	170	22·6	— 50
	1793	..	166	22·1	0
	2152	..	167	18·5	— 50
	2152	— 137	158	17·5	0
	2510	— 165	151	14·3	0
	2869	— 194	145	12·0	0
					Points marked ⊡ on the curve, Sheet V. (Plate 11).

TABLE VIII.—December 8, 1887. Aluminium Foil Removed from the Guard-Tube.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
					Adjustable quadrant at
1·45	359	..	82	54·5	0
	717	— 30	138	45·9	0
	1076	— 63	175	39·0	0
	1434	..	190	31·6	0
	1793	— 128	194	25·7	0
	2152	— 158	189	20·9	0
	2152	..	203	22·5	+ 50
	2510	— 52	197	18·6	+ 50
	2510	..	187	17·7	0
	2869	..	190	15·7	0
	2869	— 228	187	15·5	+ 50
	1793	..	207	27·5	+ 50
					Points marked ⊙ on the curve, Sheet V. (Plate 11).
					The points represented by the ▽ sign on the curve, Sheet V., were taken on Dec. 3 and 4. See Table II. and Sheet II.

The two last sets of experiments made with the aluminium foil at the back of the guard-tube, and after the foil had been removed, show that no considerable change is made in the double deflection by turning the micrometer screw attached to the adjustable quadrant through 50 divisions, that is, turning it through 180°.

Some experiments were therefore made to obtain further information on this point, especially to see how a motion of the adjustable quadrant affected the sensibility for different potentials of the needle, and the results are given in Table IX. It will be observed that, for a given potential of the needle, the effect produced on the sensibility, by attaching or altering the position of the foil, is much greater than the effect produced by the corresponding motion of the adjustable quadrant, so that we are justified in regarding the greater part of the difference between the curves on Sheet V. (Plate 11), as really due to the presence and position of the foil on the guard-tube.

TABLE IX.—December 9, 1887. Effect of Varying the Position of the Adjustable Quadrant. Double Deflection for a Deflecting P.D. of 1·45 Volts.

Potential of needle in volts.	Position of micrometer screw attached to movable quadrant.							
	− 50	− 25	0	+ 25	+ 50	+ 75	+ 100	+ 150
359	84	..	83	..	82	..	83	83
717	..	..	143	..	..	..	149	153
1076	..	..	175	..	182	..	194	
1434	190	..	189	..	200	..	260	
1793	204	193	191	194	207			
2152	..	198	190	189	206			
2510	..	203	185	185	197			
2869	..	..	187	..	189			

V.—MOTION OF THE ELECTRICAL ZERO.

It was found to be practically impossible to adjust the needle, quadrants, and other parts of the electrometer, so that the electrical and mechanical zeros should be the same for all P.D.'s between the needle and the outside of the instrument, although, when the controlling force was made somewhat large by separating the suspending fibres at the top, it was possible with a careful adjustment to render the difference between the electrical and mechanical nearly nought for all potentials of the needle up to some 1400 volts. For example, on January 30, 1888, when the fibres were separated as far as is possible with the ordinary construction of the WHITE form of quadrant electrometer, and the quadrants and needle very carefully adjusted for symmetry, the electrical zero only differed by three scale divisions from the mechanical, when the needle was gradually charged up from 0 to 1434 volts. On another occasion this motion was reduced to two divisions for the same limits of the potential of the needle, and in a third case there was no perceptible motion of the spot of light on charging up the needle to this potential. But even in these cases there was a considerable motion of the electrical zero for still higher P.D.'s between the needle and the outside case of the electrometer.

On one occasion, when the fibres were very near together, and the controlling force therefore weak, the electrical zero only moved through ten divisions from the mechanical zero when the needle was charged up to as high as 2869 volts. This was, however, a totally exceptional result, and was obtained, as if by accident, after very many trial adjustments. The adjustments of the quadrants, however good they may seem to the eye, are not accurate enough to cause the electrical and mechanical zero to remain constant for a weak control of the needle as the needle is steadily charged up. It does not seem to be sufficient to have only one quadrant adjustable by means

of a micrometer screw, and we would suggest that, if two quadrants were provided with a micrometer screw motion, much time would be saved. It is, of course, possible, by moving the adjustable quadrant, to make the electrical zero coincide with the mechanical zero for some one potential of the needle, but although the coincidence is made to occur when the potential of the needle is, say, 2000 volts, it by no means follows that the two zeros will agree when the needle's potential is 1000 volts. And the experiments, of which some of the results are given in the following Table X., and plotted on Sheet VI. (Plate 12), show that when the four quadrants are adjusted symmetrically, as far as the eye can judge, the following result is obtained. First, the electrical zero moves away from the mechanical zero as the potential of the needle is increased; then, having reached a certain distance from the mechanical zero the electrical zero returns towards the mechanical zero as the needle's potential is still further increased; next it passes through the position of the mechanical zero, and finally it moves away farther and farther down the other side of the scale.

TABLE X.—November 8, 1887. Motion of the Electrical Zero. Position of the Electrical Zero Measured from the Position of the Mechanical Zero.

Potential of needle in volts.	Reading on micrometer head attached to movable quadrant.								
	− 100	− 50	0	+ 25	+ 50	+ 75	+ 100	+ 150	+ 200
359	− 52	− 28	− 17	..	− 5	..	9	21	35
538	− 85	− 62	− 31	..	− 4	..	24	54	89
717	− 140	− 100	− 56	..	− 10	+ 15	40	92	150
896	− 200	− 140	− 78	..	− 10	..	63	145	235
1076	− 260	− 183	− 100	..	− 12	..	83	195	
1255	..	− 220	− 122	..	− 10	..	108	250	
1434	..	− 258	− 143	..	− 15	..	120	316	
1793	..	..	− 189	− 100	− 22	+ 63	154		
2152	..	..	− 228	− 135	− 37	+ 60	168		
2510	..	..	..	− 177	− 66	+ 38	158		
2869	..	..	..	− 238	− 110	+ 3	129		

Fibres near at the top. Quadrants about $2\frac{1}{2}$ millims. apart.

VI.—ELECTROMETER USED TO MEASURE POWER.

When the power given by any current to any circuit ab (fig. 6) has to be measured by means of a quadrant electrometer the method is as follows:—In series with ab is connected a non-inductive circuit bc, of resistance r; the opposite pairs of quadrants are joined respectively to a and b, and two observations are taken, one when the needle is joined to b, and one when it is joined to c. Then the difference between these readings divided by r is proportional to the power given to ab. A second method consists in connecting the opposite pairs of quadrants respectively with b and c,

and to take two observations, one when the needle is joined to b, and one when it is joined to a. Then, as before, the difference between these two readings divided by r is proportional to the power given to ab.

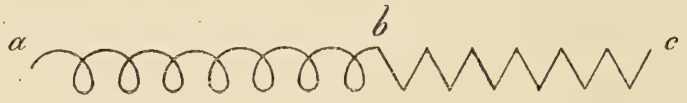


Fig. 6.

If ab be the primary coil of a transformer, the P.D. between the terminals of which will be 1000 or 2000, or more, volts, the first of the two methods just described of joining up the electrometer will be inapplicable, for, with the ordinary adjustment of this instrument, the spot of light would fly off the scale if one pair of quadrants were connected with a and the other pair and the needle with b, and the P.D. between a and b were some thousands of volts. Hence the second method of joining up the apparatus is alone applicable in such a case.

When the quadrants are joined to b and c, the needle to b, and the outside case of the electrometer to c, the instrument is being used idiostatically. The text-book law

$$D \propto (Q_1 - Q_2)\left\{N - \frac{Q_1 + Q_2}{2}\right\},$$

where D is the deflection, Q_1, Q_2, and N the potentials of the opposite pairs of quadrants, and of the needle relatively to the outside case of the instrument, reduces to

$$D \propto N^2,$$

when the needle is connected with one pair of quadrants, and the outside case with the other pair. Now a P.D. of only some 100 volts between the needle and the outside case will deflect the spot of light to the end of the scale, when the electrometer is used idiostatically. And we can see that the first part of each of the curves on Sheets II., III., and IV. is straight, that is, that the text-book law holds for the quadrant electrometer made by Messrs. WHITE when the P.D. between the needle and the case does not exceed a hundred volts or so. Hence we should anticipate that when this instrument is used idiostatically the deflection would be directly proportioned to the square of the P.D. between the quadrants for any fixed position of the quadrants.

And experiment proves this to be the case, for a number of tests made to check the accuracy of the square law showed that the numbers obtained by dividing the deflection by the square of the potential of the needle did not differ from one another by more than one per cent. for deflections in all parts of the scale. The various P.D.s were obtained and measured by connecting up 200 silver chloride cells with the terminals of a Thomson-Varley sliding P.D. divider, the "zero" point being joined to

one pair of quadrants and to the outside of the electrometer, while the "index" was joined to the other pair of quadrants and to the needle.

The square law obtained on using the quadrant electrometer idiostatically is never quite true for small P.D.s, since, as pointed out by Dr. HOPKINSON and the authors some years ago, the contact P.D. between the needle and the quadrants causes the deflection to be different for the same P.D. between the quadrants, depending on whether the needle is charged positively or negatively.

When making the second observation with the second method of joining up the apparatus for measuring the power given to ab, we have the pairs of quadrants joined respectively to b and c, between which there may be a P.D. of 1 or 2 volts, and the needle joined to a, the potential of which, when ab is the primary of a transformer, may differ by some thousands of volts from that of the quadrants or of the outside of the Leyden jar. In other words, we are now using the electrometer heterostatically in exactly the way in which it is ordinarily employed, excepting that, if an alternating or intermittent current be passing through the circuit abc, all the potentials will be varying.

Now we have shown that when the P.D. between the needle and the outside case is some hundreds of volts or more, the formula

$$D \propto (Q_1 - Q_2)\left\{N - \frac{Q_1 + Q_2}{2}\right\}$$

does not hold unless the quadrants be in one definite position. Hence the deflection will not measure the mean value of the expression on the right hand side of the last equation unless the quadrants be in this definite position. But the electrometer method of measuring power is based on the assumption that the deflection measures the mean value of

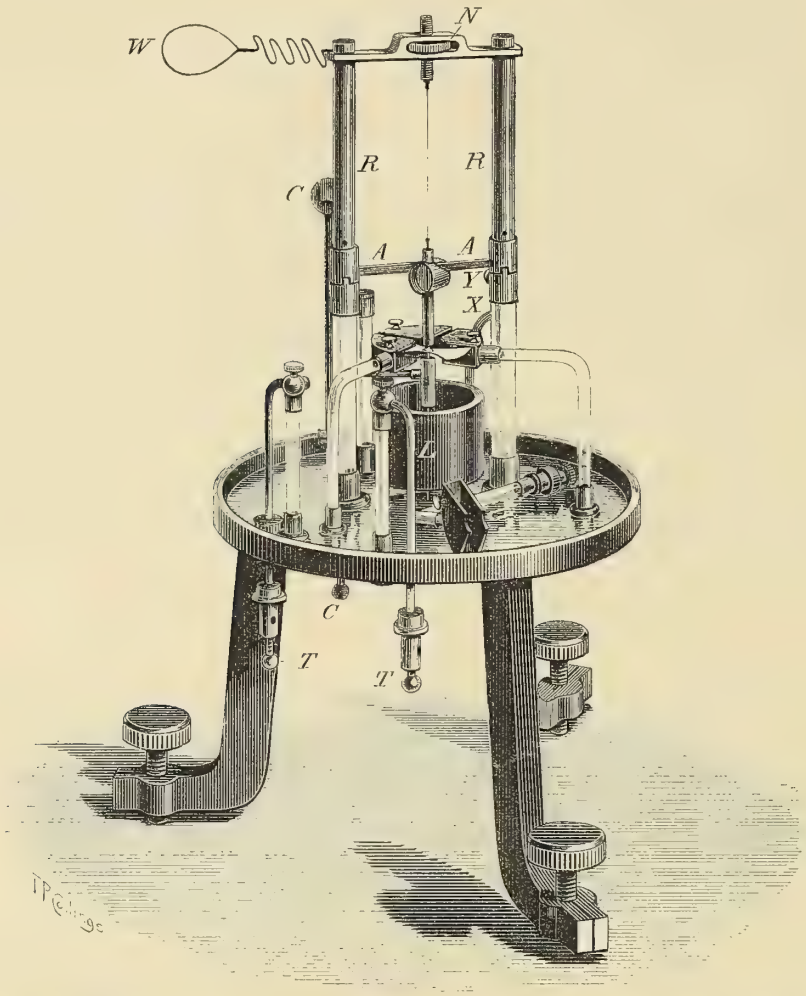
$$(Q_1 - Q_2)\left\{N - \frac{Q_1 + Q_2}{2}\right\}.$$

Hence it follows that *the quadrant electrometer, if measuring the power given to the primary coil of an ordinary transformer, will not give correct results unless the quadrants be in the one special position that we have pointed out.*

It might be thought that, although the electrometer did not obey the usual law given for it, it would be possible to calibrate it empirically as a watt meter. As regards one of the observations, viz., that made when the needle and one pair of quadrants are joined to b (fig. 6) and the outside case and the other pair of quadrants to c, it is possible to calibrate the electrometer empirically, whatever its law may be. For any special deflection can, for a given adjustment of the instrument, be only produced by one special P.D. between the quadrants. But on making the second observation when the quadrants are joined respectively to b and c and the needle to a the same deflection may be obtained with many different potentials of the points a, b, and c. Hence it follows, that unless the deflection is connected with the P.D.

between the quadrants and the P.D. between the needle and case by a law of a certain special form, it will be impossible to calibrate the quadrant electrometer empirically as a watt meter.

Fig. 7.



Improved Quadrant Electrometer. Glass shade (which is also the Leyden jar) and the Magnetic Control removed.

VII.—Improved Quadrant Electrometer, Construction.

The first condition to be fulfilled in designing an electrometer, indeed, we venture to think in designing any electrical measuring instrument, consists in supporting *all* the working parts from the base, so that on removing the cover, all the parts can be got at and adjusted *in position*. We, therefore, commenced by turning Sir William Thomson's electrometer upside down, so to say, supporting the quadrants, needle, charging rod, &c., from the base, and using the glass shade which covers up our instrument as the Leyden jar.

Fig. 7 shows an instrument with this glass shade removed, the springy wire *W* being

for the purpose of making connection with the interior of the Leyden jar, and it will be seen that every part of the interior can be easily got at without interfering with the adjustment and without having to first clamp the needle, a precaution necessary to be adopted in the quadrant electrometer as made by Messrs. WHITE. Further, not merely are the insulating stems that support the needle and the quadrants protected from dust and moisture, when the cover is on, but also the insulating stems that

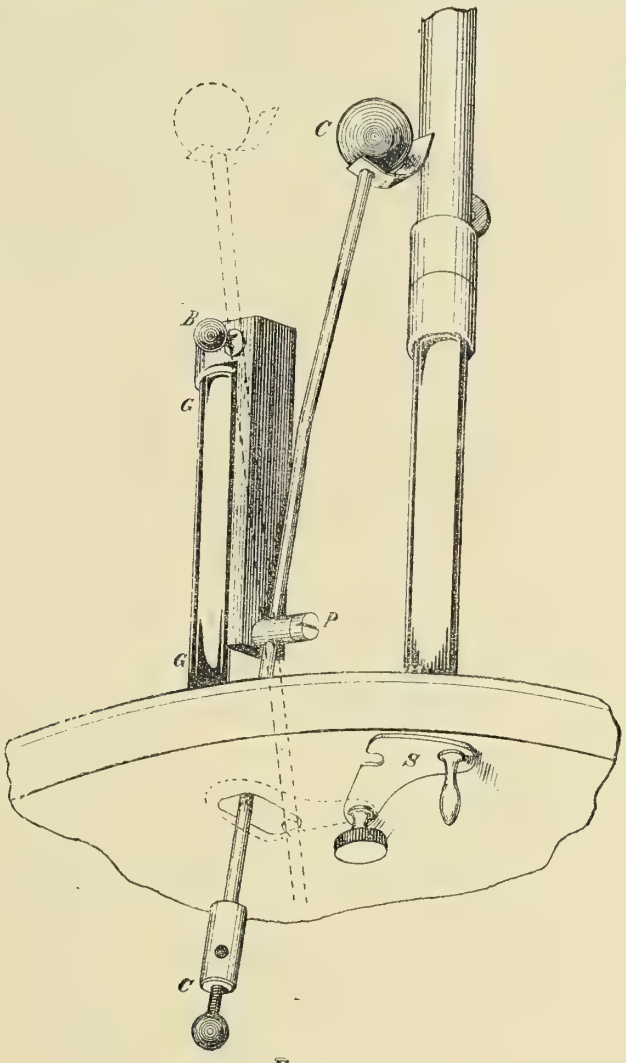


Fig. 8.

support the brass terminals, *T*, *T* and the charging rod *CC*. Hence there are no exposed ebonite stems requiring daily rubbing to remove dust and moisture, an operation that not unfrequently electrifies them, resulting in a slow charging of the quadrants when they are insulated. When the instrument is not in use the holes in the base plate through which pass the terminal rods *T*, *T* are closed by means of brass plugs sliding on the terminal rods and pushed up from below.

In the WHITE form of electrometer the charging rod can be electrically disconnected from the interior by the short ebonite tube, which holds it, being turned by

the fingers. To do this, however, the ebonite tube must be tightly grasped in one's fingers, and as they are necessarily chemically dirty, there is the possibility of the Leyden jar being partially discharged in the act of disconnecting the charging rod. This objection is entirely overcome with the arrangement, seen enlarged in Fig. 8, which we have adopted for the charging rod in the improved quadrant electrometer. At the top of a glass rod *GG* is cemented a cap, to which is attached a piece of brass, to the lower end of which is fixed a pivot *P*, round which the charging rod *CC* can turn. The ball *C* at the upper end of this rod is made of lead, and its weight causes the rod to remain in either the position shown by the full line, or by the dotted line. When in the position shown by the full line, the ball *C* is in electric contact with the needle, guard-tube, and the interior of the Leyden jar, the area of contact being increased by the bending of the little piece of flat spring fastened to the ball. The Leyden jar can now be charged, and at the moment that the right amount of charge has been given, the charging rod *CC* is tilted over into the position shown by the dotted line, and the electric connection is broken. This tilting over is easily done by pushing the lower end of the rod with the electrophorus, if an electrophorus be used to charge with, or by means of a glass rod, or by slightly pulling the wire attached to the binding screw at the lower end of *CC*, if such a wire be used to charge through. The slide *S* is now closed to exclude dust and moisture.

When this electrometer is employed for alternating current work it is desirable to connect the binding screw *B* by means of a fine wire with a binding screw attached to the interior of the instrument, so that the electric contact between the charging rod and the interior of the instrument (and which should be very good for very rapid reversals of the charge) does not depend solely on the pressure of the leaden ball at the top of *CC*. To make such a wire connection would be practically impossible with the WHITE form of the quadrant electrometer, since the charging rod must be bodily removed from the instrument before the part of it inside the electrometer can be got at. It is also desirable, when an electrometer is employed for measuring a rapidly alternating P.D., that the electric connection between the electrodes and the quadrants should be far better than can be obtained with the comparatively slight pressure that exists between the ends of the electrodes and the quadrants in the WHITE form of instrument; hence in the improved quadrant electrometer the connection with the quadrants is made by means of screwed contacts. Further, for alternate current work it is not only important to have as good contacts as possible, but also as little capacity as possible in the electrometer. This result we attain by removing the springy wire *W*, and electrically connecting the interior metallic coating of the glass shade with the exterior.

The next point to consider in designing this instrument was the size to give to the needle and quadrants. Theoretical reasoning, confirmed by some roughly made experiments, showed that the sensibility of the quadrant electrometer, as made by Messrs. WHITE, could be much increased if the needle and the quadrants were made

smaller than in that instrument. We therefore made the diameter of the quadrants, their vertical depth, and the length of the needle about six-tenths of the size adopted in the White pattern. Feeling, however, that the needle in the White pattern was not broad enough at the widest part to prevent its edges approaching too near the intersection of the quadrants when the needle was deflected, we thought it better not to reduce the width of our needle in the same proportion as its length. Therefore while we reduced its length to six-tenths we left its breadth eight-tenths of the sizes adopted in the White electrometer.

Then as regards the controlling force. In our laboratories are electrometers with a bifilar suspension control, a torsional control, and a magnetic control respectively, and of these three forms we have found that the magnetic control is the best to be used in an electrometer, whose sensibility we require to be able to easily vary. The bifilar suspension control is unnecessarily troublesome, arising partly from the frequent necessity for adjusting for equal tension in the two fibres. For, in the White form of quadrant electrometer, no alteration in the centering of the needle, or in the height of the needle, or in the distance between the fibres, can be effected without a fresh adjustment being rendered necessary for equality of tension in the fibres.

The magnetic control has no doubt the defect that the position of the needle is liable to be affected by outside magnetic disturbance. But this source of error can be practically annihilated if the magnet attached to the suspension wire *be very short and be very weakly magnetised*, and if the requisite magnetic couple be produced by the use of a *very powerful controlling magnet* placed near to the suspended system, and we venture to think that it was a neglect of this principle that caused the abandonment of the magnetic control originally employed by Sir William Thomson in his quadrant electrometers. Another plan that we have found very successful is to fix on the suspension wire an *astatic* combination of magnets, and to place the controlling magnet nearer one of the little magnets than the other.

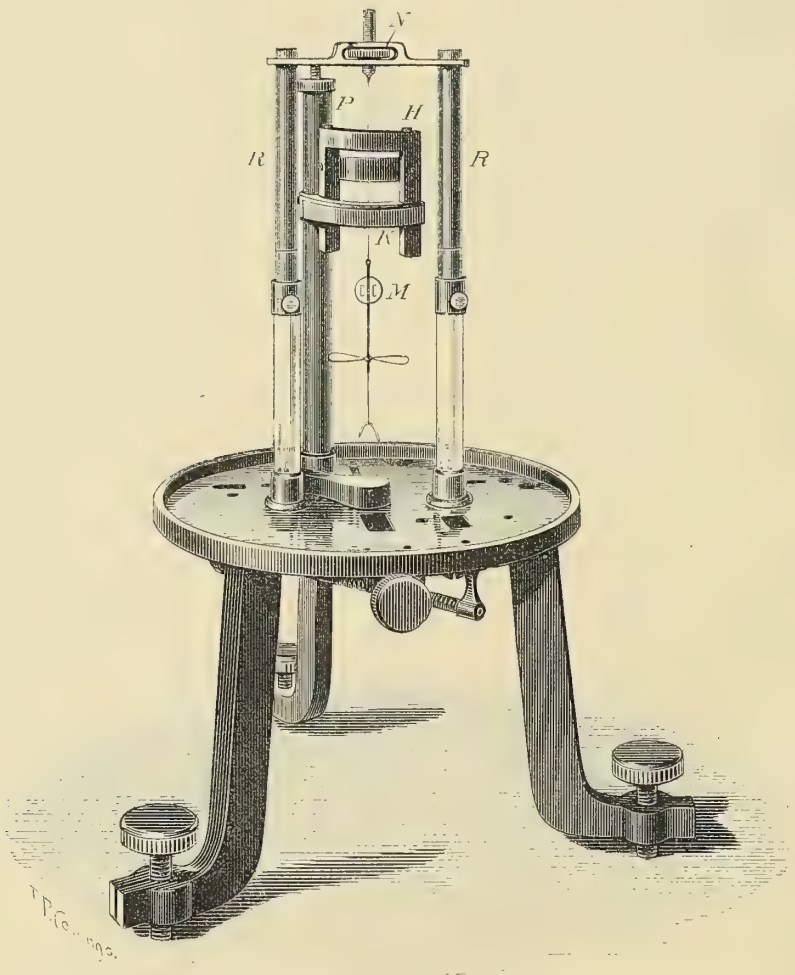
But probably the most perfect plan is to use, as the astatic combination, two little *vertical* magnets, which may be conveniently attached to the back of the mirror, *M*, as seen in fig. 9, since *such a combination remains necessarily astatic as far as rotation round a vertical axis is concerned, no matter how much one or both of the magnets may weaken with time.*

Where completeness in the electrometer is of more importance than the expense, the controlling magnet may be a vertical horseshoe carried inside the instrument from a support attached to the base, as seen in fig. 9, which shows the system of magnetic control adopted in the improved electrometer. Two adjustments are provided, one worked by a milled head at the base of the supporting pillar, *P*, which projects about an inch through the base plate to enable this milled head to be easily turned ; the other by a tangent screw fixed under the base plate. Turning the milled head at the base of the supporting pillar slides the soft-iron keeper, *K*, over the poles of the horseshoe magnet, *H*, and enables the strength of the controlling field to be varied

within wide limits; while, on the other hand, a rotation of the tangent screw causes the supporting pillar with the horseshoe magnet and the keeper which it carries to rotate about the central axis of the instrument, and so enables any variation of the zero to be effected.

Lastly, we supported the lower part of the guard-tube quite independently of the upper part, and hence the objectionable connecting pieces I, I' (fig. 3) are entirely dispensed with, and the result is that, whether the quadrants be pushed far in, or

Fig. 9.



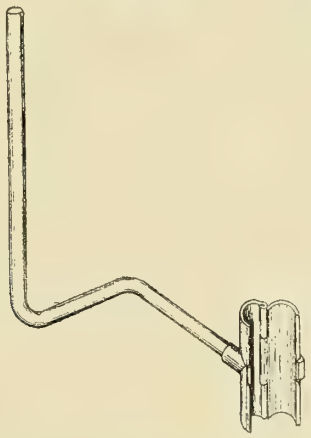
Improved Quadrant Electrometer showing the Magnetic Control and the arrangements for varying the strength of the field, also the direction of the field. Quadrants &c., removed.

pulled far out, this instrument obeys the recognised formula for a quadrant electrometer.

The brass rods, R, R (fig. 7), which are each supported on a glass rod, carry all that portion of the apparatus that is at the potential of the needle, except, of course, the interior of the Leyden jar glass-shade. The needle hangs by means of a silk fibre

from a screw which can be raised or lowered by turning the nut N, a vertical slot in the screw and a horizontal pin projecting from the framework preventing the screw itself turning round. The platinum wire at the bottom of the needle dips into some sulphuric acid contained in a leaden cup L. This cup is supported and electrically connected to RR by a stout wire X, the upper end of which slides through a clamping screw Y at the back of the lower end of the right-hand brass rod R. The leaden cup can, therefore, be easily raised or lowered and also centred. Similarly the lower portion of the guard tube is carried by a stout wire passing through a clamping screw at the back of the lower end of the left-hand brass rod R. This portion of the guard-tube is hinged and is shown open in fig. 10. After being put in position the tube is

Fig. 10.



closed round the platinum wire, which projects down from the needle but so as not, of course, to touch this wire.

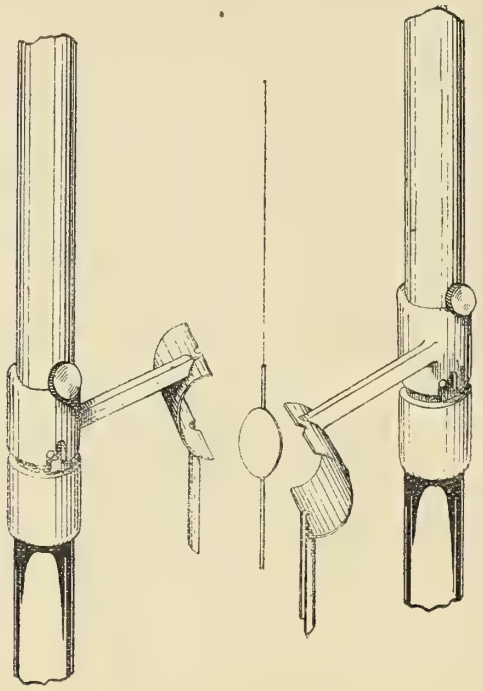
The upper portion of the guard-tube is made in two halves which are carried by the arms A, A (fig. 7) attached to collars sliding with a bayonet-joint motion on the rods R, R. By raising up these collars and turning them round the two halves of the upper portion of the guard-tube can be opened as seen in fig. 11, and the mirror and the wire attached to it got at. In fact, by opening and turning aside both the lower and the upper portions of the guard-tube, the mirror, needle, and its attached wire, can be left hanging freely from the silk fibre suspension, and it can be seen whether they are hanging properly, or whether they require adjustment.

The two upper portions of the guard-tube are made skew, as shown in fig. 11, in order that when the mirror is parallel to the needle, and the latter symmetrical with the quadrants, the glass supports of the latter may clear the glass supports of R, R.

To avoid confusion in the illustrations the fine wire spirals connecting the opposite pairs of quadrants together, and with the terminal screws at the tops of the glass rods supporting the terminal rods T, T, have been omitted.

We have to thank Mr. Mather for taking a leading part in designing and constructing this improved quadrant electrometer.

Fig. 11.



VIII.—Results obtained with the Improved Quadrant Electrometer.

A large number of experiments have been made with the improved quadrant electrometer, and marked differences are observed between its behaviour and the behaviour of the White instrument.

First, altering the distance between the quadrants of our instrument does not materially affect the sensibility. For example, increasing the distance between adjacent quadrants from $2\frac{1}{2}$ to 4 millims. hardly varies the sensibility at all; whereas with the White electrometer, drawing out the quadrants increases the sensibility (a result which, as we have pointed out, is quite opposed to the text-book formula for a quadrant electrometer), and this increase in sensibility is brought about by a complete change in the shape of the curve connecting the potential of the needle with the deflection produced by a given P.D. between the quadrants.

Secondly, whereas with the White instrument it is only with one position of the quadrants that the potential of the needle is even approximately proportioned to the deflection produced by a given P.D. between the quadrants, this result is obtained with considerable accuracy with *any* symmetrical position of the quadrants in our instrument.

The following table is a sample of the results obtained, no special adjustment of the quadrants having been made to obtain constancy in the sensibility factor. This sensibility factor, as already explained, is the ratio of the deflection, in microradians, of the reflected ray of light to the product obtained by multiplying together the

deflecting P.D. in volts, and the P.D. between the needle and the outside case also in volts.

TABLE XI.—July 13, 1888. Improved Electrometer.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.	Remarks.
4·35	359	1	13·5	9·0	Strong magnetic controlling field
	717	1·5	26·2	8·7	
	1,076	4	39·0	8·6	
	1,434	14	54·2	9·0	
	1,793	34	69·0	9·2	
	2,152	92	87·8	9·7	
	1,434	36	53·7	8·9	
	717	8	26·0	8·6	
	359	4	13·8	9·2	
	179	3	7·0	9·3	

Further, it is possible to practically obtain far greater sensibility with this improved electrometer than with the specimen we possess of the quadrant electrometer, as made by Messrs. WHITE. The greatest double deflection for a Leclanché cell that we have been able to obtain, with our specimen of the White pattern, is 386 scale divisions, the P.D. between the needle and the outside case being 2500 volts, the scale being at 5 feet distance from the mirror, and 1 foot length of scale containing 290 divisions. It would, however, be impossible to work with this adjustment, as the instability was too great. Practically, therefore, about 250 scale divisions, as a double deflection for a Leclanché cell, or, what is the same thing, about 180 divisions per volt P.D. between the quadrants, is all that we can obtain with our specimen of the White electrometer.

With the improved form, on the contrary, very great sensibility indeed can be obtained by weakening the controlling magnetic field, the only limit being caused by the increased importance of want of symmetry, the greater disturbing effects produced by capillary attraction, by the motion of exterior magnetic bodies, &c., as the sensibility is made greater and greater. When the controlling field was made very weak it was not found advisable to make the P.D. between the needle and the outside case more than 1800 volts. Under these circumstances *it was possible to obtain, as a double deflection,* 100 *scale divisions for a P.D. between the quadrants of only one hundredth part of the E.M.F. of a Leclanché cell* corresponding with a sensibility factor of 580. The needle, it is true, was very unstable, and the deflection was not definite enough for ordinary practical use. Possibly, however, the instrument might be used very well in this condition for null methods when great sensibility is required.

However, with a somewhat less sensibility, a double deflection of 44 divisions being

obtained for one hundredth of the E.M.F. of a Leclanché cell, the improved electrometer worked more satisfactorily than did the White instrument, when adjusted to have its maximum sensibility, so that there is no doubt that the improved electrometer is at least ten times as sensitive as our specimen of the White pattern, when the two instruments are adjusted to be in an equally trustworthy condition as regards definiteness of the zero, definiteness of a given deflection, &c.

Tests were also made of the two instruments when used idiostatically; and, as regards the White pattern, it was found, as stated by Professor GRAY on p. 299, vol. 1, 'Absolute Measurements in Electricity and Magnetism,' that "The quadrant electrometer may be used idiostatically for the measurement of differences of potential of not less than 30 volts." But, on the contrary, we found that three volts would give a double deflection of about 50 scale divisions with the improved electrometer when used idiostatically. *Hence the improved instrument is also ten times as sensitive as the White pattern when used idiostatically.*

Spring, 1891.—In what precedes we have shown that the improved electrometer is free from the defect in the White pattern, which forms the subject of this paper. We have attributed this defect to the electrical action of certain pieces I, I' (fig. 3), which connect the upper and lower portions of the guard-tube in the White pattern. But as our instrument differs in many points from the White pattern, not possessing, for example, a bifilar suspension which, according to Dr. HOPKINSON, is a necessary feature in the electrometer in order to explain the major part of the anomalous action of the White instrument, we determined not to publish this paper until we had experimentally proved, beyond all doubt, that our explanation was correct. This we were unable to do until the year 1891, various reasons having prevented our taking up the electrometer investigation again from the point we left it some years ago.

First, as regards a bifilar suspension being necessary, in order that the deflection produced by a given P.D. between the quadrants should be very far from proportional to the P.D. between the needle and the outside case. This question is settled by the experiments that we have made this year on a quadrant electrometer made by Messrs. WHITE, in the possession of the Royal College of Science, and the needle of which has been suspended with a single quartz fibre by Professor BOYS. The following were the results that we obtained, the quadrants being about 2 millims. apart, and the instrument in excellent condition in so far that the spot of light returned absolutely to zero, and a deflection could be read with confidence to $\frac{1}{10}$th millim. The slight irregularities in the curve on Sheet VII. (Plate 11), which graphically represents the results contained in the following Table XII., arise from the potential of the needle having been read by means of a portable pivotted electrostatic voltmeter, the suspension of which was, of course, far more frictional than the quartz fibre of the quadrant electrometer. This voltmeter was previously calibrated by comparison with the absolute electrometer.

TABLE XII.—February 3rd, 1891. White Electrometer with Single Quartz Fibre Suspension.

Deflective P.D. in volts.	Potential of needle in volts.	Electrical zero.	Deflection per one volt.	Sensibility factor.
0·5	0	23·9	..	..
	260	30·6	134	510
	300	31·5	152	510
	335	32·8	178	530
	370	34·1	204	550
	390	36·2	246	630
	410	37·7	276	670
	427	39·4	310	720
	437	40·3	328	750
	440	40·4	330	750

From the preceding we see that even when the needle is suspended by a single fibre, an increase in the potential of the needle from 260 to 440 volts alters the sensibility factor by 50 per cent. It is, therefore, quite certain that the peculiar action in the White electrometer can occur without a bifilar suspension.

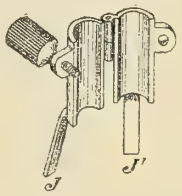
Next we had made the collar, or saddle, shown in fig. 12, which could be fastened on the lower part of the upper guard-tube of the improved electrometer, the pieces J, J', projecting down into the quadrants on each side of the needle. As these projections J, J', are at the same potential as the needle, there will be the same sort of electrical action between them and the needle as there is between the pieces I, I' (fig. 3) of the White instrument. Hence, if our explanation of the defect in the White instrument be correct, the application of the collar with its projections J, J' (fig. 12), to the guard-tube of the improved electrometer ought to destroy the straight-line-law which the numbers given in Table XI. show that this instrument possesses.

In Table XIII. are given the results obtained with the improved electrometer just before attaching the saddle piece (fig. 12), while those given in Table XIV. were obtained after merely attaching the saddle (fig. 12), no other change whatever being made in the instrument.

The adjustment of the quadrants, and the position of the controlling magnet were quite different when obtaining the results given in Table XIII. from what they were when the numbers given in Table XI. were obtained nearly three years ago with the same instrument. The sensibility factor had, therefore, quite a different value in March, 1891, from what it had in July, 1888, but, just as a large variation in the potential of the needle caused at that date practically no change in the sensibility factor with the improved electrometer, so now the sensibility factor is practically independent of the potential of the needle before attaching the saddle. After, however, the saddle has been attached, we see first that the

sensibility factor varies rapidly with the potential of the needle, and, secondly, that it is much less than before the saddle was attached, its nearly constant value (22), obtained before the saddle was used, being only reached with the saddle on when the potential of the needle has been lowered to some 200 volts.

Fig. 12.



The results given in the Tables XIII. and XIV. are plotted in the curves shown on Sheet VIII. (Plate 12), and prove without doubt that the pieces *I*, *I'* (fig. 3), in the White electrometer are the cause of all the trouble.

TABLE XIII.—March 20, 1891. Improved Electrometer before Attachment of Saddle (fig. 12).

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.
1·45	1250	+ 29	104	21·5
	1170	+ 29	99	21·8
	905	+ 2	75·5	21·5
	860	+ 1	72	21·5
	812	0	68·5	21·8
	740	− 3	61·5	21·5
	700	− 2	59	21·8
	660	− 1	55·5	21·7
	579	− 1	48	21·3
	505	− 2	43	22
	415	− 3	34·5	21·5
	340	− 2	29	22
	1305	− 40	111	21·8
	1090	− 30	93·5	22
	1140	− 42	97	22
	915	− 29	75·5	21·3
	860	− 23	72·5	21·8
	965	− 34	79·5	21·3
	1015	− 28	85·5	21·7
	1275	− 63	108·5	21·8
	1152	− 54	97	21·7
	1277	− 68	110·5	22·2
	1235	− 65	107	22·4
	1140	− 53	99	22·4
	1087	− 49	93·5	22·2

TABLE XIV.—March 20, 1891. Improved Electrometer, Saddle Attached.

Deflecting P.D. in volts.	Potential of needle in volts.	Electrical zero.	Double deflection per Leclanché cell.	Sensibility factor.
1·45	1430	240	28·5	5·1
	1330	232	30	5·9
	1260	222	31·5	6·5
	1160	207	32·5	7·2
	1005	185	35	9·0
	955	173	36	9·7
	910	164	35	9·9
	850	153	35	10·6
	795	143	35·5	11·6
	705	128	34·5	12·7
	670	117	33·5	13·3
	610	106	33	14
	570	99	32·5	14·8
	530	89	31·5	15·4
	490	80	31	16·4
	450	71	29·5	17·1
	400	59	27·5	17·8
	355	50	25·5	19·2
	305	41	23·5	19·9
	260	32	21	20·8
	210	23	18	22·3
	165	17	14·5	22·7

As the electrical zero altered very much in the above set of measurements made with the saddle attached, experiments were subsequently tried to see whether this alteration of zero affected the sensibility. In each of the above cases, the controlling magnet was turned so as to alter the direction of the controlling field without altering its strength, and, as it was found that the change of zero thus produced did not materially alter the sensibility for any one of the potentials of the needle given in the above table, it was inferred that the mere alterations of the zero mentioned in the table did not materially affect the sensibilities.

[Lastly, an experiment was made to try how inaccurate the improved quadrant electrometer might be, if no care were taken to adjust the quadrants even for symmetry, and if the needle were suspended with a very coarse silk fibre, which had neither been boiled nor subsequently cleaned with absolute alcohol, in accordance with the plan that we usually adopt with silk fibres for suspension. The keeper, K, of the horseshoe magnet H (fig. 9) was placed so that the magnetic control was a maximum, and the instrument was used idiostatically. In order that, although the electrometer was used idiostatically, the deflection might be reversed and zero errors eliminated, the needle and the outside case were attached respectively to two of the terminals of a reversing key, and the pair of quadrants with the other two terminals respectively.

The results obtained are given in Table XV., and, considering that the instrument was set up in the rough way above described, the constancy of the sensibility factor is sufficiently striking.

TABLE XV.—July 1891. Improved Quadrant Electrometer used Idiostatically.

Reading of potential divider.	P.D. between quadrants, in volts.	Double deflection of reflected ray, in scale divisions.	Sensibility factor, or double deflection of needle, in micro-radians divided by square of P.D., in volts.
10	17	9	11·0
15	25	20	10·8
20	33·6	34·5	10·58
25	41·9	53·4	10·43
30	50·3	77	10·43
35	58·8	103·6	10·38
40	67	134	10·00
45	75·5	169	9·98
50	84	208·5	9·98
55	92·3	252	9·95
60	100·6	299	9·95
65	109	350	9·93
70	117·5	405·5	9·91
75	126	465	9·91
80	134	529	9·91
85	142	597	9·91
90	151	669·5	9·91
70	117·5	405·5	9·91
50	84	207·5	9·93
30	50·3	75·5	10·00
20	33·6	34	10·40

N.B.—The fibre suspending the needle was a very coarse one, and the instrument was tested without any previous adjustment of the quadrants even for symmetry.

One hundred and seventy old silver chloride cells were connected with a Thomson and Varley potential divider. The potential of one hundred of these cells was tested by an electrostatic voltmeter, and found to be 99 volts. One scale division corresponded with an angular movement of the needle equal to $\frac{1}{2900}$ radian.—*July* 20, 1891.]

We cannot conclude this account of the numerous defects of the Thomson quadrant electrometer, as hitherto made by Messrs. WHITE, and of the way to overcome them, without expressing the deep debt of gratitude that we, in common with electricians of all countries, owe to the inventor of the electrometer, for having provided us twenty years ago with an instrument that has been of inestimable value, both for scientific research and for the commercial testing of submarine cables.

IX.—Sketch of the Mathematical Theory of the White Electrometer.

Before we were certain of the cause of the peculiarities in the action of the White electrometer, we devoted a large amount of time to a mathematical investigation of its behaviour. This gave us many weeks' hard work, as the investigation is a complicated one. For, in order to obtain an equation (for the deflection of the needle in terms of its potential and the P.D. of the quadrants) that would fit all the various curves obtained experimentally, we found it necessary to take into account:—

1st. The repulsion between the pieces I, I' (fig. 3), of the guard-tube and the needle;

2nd. The diminution of this repulsion as the quadrants were separated symmetrically;

3rd. The alteration that the tilting of the needle at high potentials produces on this repulsion;

4th. The alteration that the tilting of the needle produces on the rate of variation, with motion of the needle, of the coefficient of induction between the insulated pair of quadrants and the needle.

As, however, subsequently to this mathematical work, we completed the experimental proof already described of the cause of the curious action in the White electrometer, we think that it will be unnecessary to give more than a brief sketch of the mathematical investigation.

Let N be the P.D. between the outside of the Leyden jar of the White electrometer and the system of conductors, consisting of the needle, guard-tube, sulphuric acid, &c. Let $2n$ be the capacity of this system of conductors. Let the pair of quadrants that is connected with the outside of the Leyden jar be regarded as being at potential nought, and let V be the potential of the other pair. Let $2q$ be the capacity of the insulated pair of quadrants, and s the coefficient of induction between this pair and the needle. Then the energy stored up in the system is

$$\mathrm{N}^2 n + \mathrm{V}^2 q + \mathrm{NV}s,$$

therefore, if $m\theta$ be the moment exerted by the bifilar suspension, when the needle is deflected through an angle θ from the equilibrium position,

$$m\theta = \mathrm{N}^2 \frac{dn}{d\theta} + \mathrm{V}^2 \frac{dq}{d\theta} + \mathrm{NV} \frac{ds}{d\theta}.$$

Now, it is usually assumed that, in consequence of symmetry,

$$\frac{ds}{d\theta} = 2 \frac{dq}{d\theta} = k, \text{ a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$\frac{dn}{d\theta} = 0,$$

which leads to the ordinary equation,

$$m\theta = kV\left(N - \frac{V}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The repulsion between the projections I, I' (fig. 3), of the guard-tube would not necessarily render assumption (1) unjustifiable, since the rate of variation of the capacity of the insulated pair of quadrants, and the rate of variation of the coefficient of induction between this pair and the needle, would not necessarily be affected by the action of the guard-tube. But in all probability this action between the projections I, I', and the needle would affect the distribution of electricity on I, I', and on the needle near them as the needle moved, so that the capacity of the system of conductors at potential N would vary somewhat as the needle moved. In other words, $dn/d\theta$ would not equal nought, and the equation for the White electrometer, taking into account this action of the guard-tube, would be

$$m\theta = kV\left(N - \frac{V}{2}\right) + N^2\frac{dn}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

instead of the text-book equation (2).

When the fibres are near together so that the control they exercise is small, experiments show that θ does not increase as fast as it ought to do when N is increased, and that for moderately high values of N, θ actually diminishes instead of increasing. From this it follows that $dn/d\theta$ in equation (3) is negative, a result that might be expected, seeing that, as the needle is deflected, so that portions of it approach the projections I, I' of the guard-tube, it is probable that the action between these projections and the needle at the same potential tends to diminish the capacity of the needle and guard-tube.

Further, it is clear that k diminishes as the quadrants are pulled out symmetrically so as to increase the distance between them; consequently, if (2) were the true formula for the White electrometer, the sensibility ought to diminish as the quadrants are drawn out. But, as seen from our curves on Sheets III. and IV. (Plate 10), and as is probably generally known, the sensibility actually increases as the quadrants are withdrawn symmetrically. Hence $dn/d\theta$ for a given value of θ must diminish numerically as the quadrants are drawn out, and this diminution in the negative value of $dn/d\theta$ must more than compensate for the diminution in k. This is consistent with what we know, since $-N\,dn/d\theta$ represents the couple exerted between the charge on the needle and the charges on the projections I, I', of the guard-tube, and it might be expected that the repulsion between two conductors at the same fixed potential would be increased by enclosing them in a conductor like the quadrants at a low potential, and diminished as this cover is more and more withdrawn.

Although the electrical zero is made to correspond with the mechanical zero by an adjustment of the fourth quadrant, the position is not necessarily a position in which

the electrical attraction of the quadrants on the needle is zero, but is a position in which the action of the guard-tube on the needle is balanced by the action of the quadrants when all are at the same potential. Let α be the angle between the zero position of the needle, and the position in which the force exerted by the guard-tube on the needle is nought, then $\theta - \alpha$ is the angle through which the needle has been moved against the action of the guard-tube on the needle. This force will vary in sign with the sign of $\theta - \alpha$, therefore, if $-dn/d\theta$ be expanded in powers of $\theta - \alpha$, it must be of the form

$$a(\theta - \alpha) + b(\theta - \alpha)^3 + c(\theta - \alpha)^5 + \&c.,$$

so that if $\theta - \alpha$ be small we may say that approximately

$$-\frac{dn}{d\theta} = a(\theta - \alpha),$$

that is

$$m\theta = kV\left(N - \frac{V}{2}\right) - aN^2(\theta - \alpha),$$

or

$$\theta = \frac{kV\left(N - \frac{V}{2}\right) + a\alpha N^2}{m + aN^2} \quad \ldots\ldots\ldots (4).$$

If then ϕ represents the double deflection of the needle obtained by reversing V (the P.D. between the quadrants), and ψ represents the position of the electrical zero, that is, the position of the needle when all four quadrants are at the same potential,

$$\phi = \frac{2kVN + a\alpha N^2}{m + aN^2} \quad \ldots\ldots\ldots (5),$$

$$\psi = \frac{a\alpha N^2}{m + aN^2} \quad \ldots\ldots\ldots (6),$$

and equations of this form would probably give the value of the double deflection and the electrical zero of the White electrometer, if it were possible to hang the needle so symmetrically that there was no tilting even when it was raised to a high potential.

The tilting of the needle will increase the value of k, and this probably accounts for the rise which occurs at the ends of many of the sensibility curves obtained with the White electrometer and shown on Sheets II., III., IV., and V. Indeed, if there were no action due to the guard-tube it is probable that for all symmetrical distances between the quadrants and for all distances between the fibres the sensibility would increase more rapidly than the potential of the needle, the bending in the sensibility

curve being the greater the more easily the needle could be tilted. With the improved quadrant electrometer there is but little tilting of the needle, but in the highest part of the curve, obtained with this instrument when the saddle was not on Sheet VIII. (Plate 12), there is an indication of the sensibility curve bending up towards the vertical axis along which deflection is measured. With the White electrometer we can, in spite of the opposing action of the guard-tube projections I, I' (fig. 3), obtain a sensibility curve concave to the axis along which deflection is reckoned, that is, we can make the sensibility increase more rapidly than the potential of the needle in one or other of two ways. If a bifilar suspension be employed, which tends to resist tilting of the needle, then the quadrants must be drawn out so far that, even for high potentials of the needle, the repulsive action of the guard-tube shall be relatively not large. Examples of this are seen in the curves marked "Quadrants about 4 millims. apart" on Sheets III. and IV. (Plate 10). If a unifilar suspension be employed, which enables tilting to take place much more easily, then a sensibility curve concave to the axis along which deflection is reckoned can be obtained, without drawing the quadrants out far, if the control be so weak that a deflection to the end of the scale is obtained for a potential of the needle too small for the control due to the guard-tube to be important. An example of this is seen in the curve on Sheet VII. (Plate 11), obtained with a White electrometer having its needle suspended by a fine quartz fibre.

And generally it would appear that for a White electrometer to have a sensibility directly proportional to the potential of the needle, it is necessary to separate the quadrants by such an amount that the increase in the repulsion between the guard-tube and the needle exactly balances the effect due to increased tilting as the potential of the needle is raised, or, in other words, it is necessary that the quadrants should be at such a distance apart that the increase of $N^2\, dn/d\theta$ (equation 3) exactly balances the effect of increase of k in the expression $kV\,(N - V/2)$ as N increases.

We had, therefore, to express k, α and $\alpha\alpha$ in equations (5) and (6) as functions of N, and after much working we succeeded in obtaining expressions for ϕ and ψ, each involving the first, second, third, and fourth power of N, and three constants, which agreed very fairly well with the various sensibility curves obtained with the White electrometer and with the curves on Sheet VI. (Plate 12) for the "Motion of the Electrical Zero" as the potential of the needle and the position of the adjustable quadrant were changed.

XIV. *On a Determination of the Mean Density of the Earth and the Gravitation Constant by means of the Common Balance.*

By J. H. Poynting, *D.Sc., F.R.S., Professor of Physics, Mason College, Birmingham.*

Received May 13,—Read June 4, 1891.

[Plates 13–19.]

I. Account of Apparatus and Method.

In a paper printed in the 'Proceedings of the Royal Society,' No. 190, 1878 (vol. 28, pp. 2–35), I gave an account of some experiments undertaken in order to test the possibility of using the Common Balance in place of the Torsion Balance in the Cavendish Experiment. The success obtained seemed to justify the intention expressed in that paper to continue the work, using a large bullion balance, instead of the chemical balance with which the preliminary experiments were made.

As I have had the honour to obtain grants from the Royal Society for the construction of the necessary apparatus, I have been able to carry out the experiment on the larger scale which appeared likely to render the method more satisfactory, and this paper contains an account of the results obtained.

At the time I was making the preliminary experiments the late Professor v. Jolly was already employing the balance for gravitation investigations ('Wiedemann's Annalen,' vol. 5, p. 112), though I was not aware of the fact. Later he published an account ('Wied. Ann.,' vol. 14, p. 331) of a determination of the Mean Density of the Earth by the use of the Balance. Still more recently Drs. Kœnig and Richarz have devised a method of using the balance for the same purpose ('Nature,' vol. 31, pp. 260 and 475), and I believe that their work is still in progress. It might appear useless to add another to the list of determinations, especially when, as Mr. Boys has recently shown, the torsion balance may be used for the experiment with an accuracy quite unattainable by the common balance. But I think that in the case of such a constant as that of gravitation, where the results have hardly as yet begun to close in on any definite value, and where, indeed, we are hardly assured of the constancy itself, it is important to have as many determinations as possible made by different methods and different instruments, until all the sources of discrepancy are traced and the results agree.

The apparatus for the experiments described in this paper was first set up in the Cavendish Laboratory at Cambridge through the kindness of Professor Clerk Maxwell. After spending some months in working at the experiment, but without much success beyond the detection of some sources of error, I left Cambridge, and ultimately the apparatus was again set up at the Mason College, Birmingham. The difficulties in carrying out the work with any approach to exactness have been far greater than were anticipated, and many times work has been begun and results have been obtained, but examination has shown them to be affected by large errors which could be traced and eliminated by further improvements in the apparatus.

At the beginning of 1890, however, the apparatus was brought into fair working order, and during the course of the year I made a number of experiments with the results recorded in this paper.

The Principle of the Experiment.

The object of the experiment, in common with all of its class, may be regarded, primarily, as the determination of the attraction of one known mass M on another known mass M′ a known distance d away from it. The law of universal gravitation states that when the masses are spheres with centres d apart this attraction is GMM'/d^2, G being a constant—the gravitation constant—the same for all masses. Astronomical observations fully justify the law as far as M'/d^2 is concerned. They do not, however, give the value of G, but only that of the product GM for various members of the solar system.

To determine G we must measure GMM'/d^2 in some case in which both M and M′ are known, whether they be a mountain and a plumb bob, as in Maskelyne's experiment, the surface strata and a pendulum bob, as in Airy's experiment, or two spheres of known mass and dimensions, as in all the various forms of Cavendish's experiment.

Knowing the gravitation constant G, we may at once find the mean density of the earth Δ. For if V be the volume of the earth—regarded as a sphere of radius R—the weight of any mass M′, being the attraction of the earth on it, is

$$GV\Delta M'/R^2.$$

But if g is the acceleration of gravity the weight is also expressible as $M'g$.

Equating these we get

$$\Delta = gR^2/GV.$$

Method of Using the Common Balance.

In using the common balance to find the attraction between two masses, perhaps the most direct mode of proceeding would consist in suspending a mass from one arm

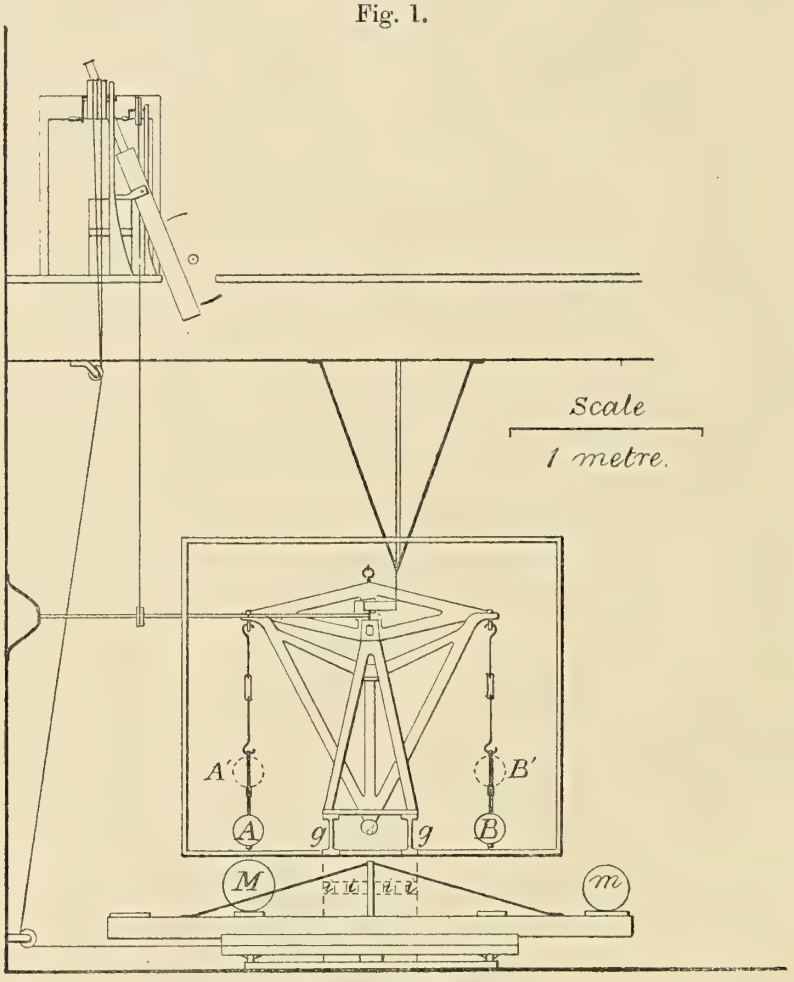
of a balance by a long wire, and counterpoising it in the other pan. Then bringing under it a known mass, its weight would be slightly increased by the attraction of this mass. The increase would be the quantity sought if the attracting mass had no appreciable effect before its introduction beneath the hanging mass, and if, when beneath it, the effect on the balance could be neglected. This is very nearly the principle of the method used by VON JOLLY, and it is that of the method used in the preliminary experiments referred to above, in which a mass of 453 grms. of lead was hung from one arm of a chemical balance (about 40 centims. beam) by a wire 1·8 metres long, and was attracted by a mass of 154 kilogrms. of lead. But the attraction to be measured was exceedingly small, rather less than 0·01 milligrm., and it therefore appeared advisable to use a much larger balance with a larger hanging mass so that the attraction might be made comparable with the weight of exactly determined riders. Other anticipations as to proportionate increase of sensibility and diminution of effect of air currents, have hardly been justified in the way I expected, though, by the ultimate form of the apparatus, they have, I think, been more than realised.

With increase in the length of beam, a differential method became applicable, by means of which the attraction of the mass on the beam was eliminated, and the necessity for prolonging the case to allow of a long suspending wire was removed. This will be seen from a consideration of fig. 1. Let AB represent equal masses suspended from the two arms of the balance, and let M be the attracting mass put first under A, the position of the beam being noted. If M is then placed under B its attraction is not only taken away from A but added to B, so that the tilting of the beam is that due to nearly double the attraction to be measured. Of course there are what we may term *cross-attractions*, in the first position, of M on B, and in the second position, of M on A, but these may be allowed for in the calculations. We cannot give any mathematical expression for the attraction of M on the beam and suspending wires, owing to their irregularity of shape. But this attraction is eliminated if a second experiment is made in which A and B are raised equal known distances to A′ and B′. For the *difference* between the two increments of weight on the right, is due solely to the alteration of the positions of A and B relative to M, the attraction on the beam remaining the same in each. From the observed effect of a known alteration of distance the attraction at any distance can be found.

This is, shortly, the method adopted. The arrangement was ultimately complicated by the addition of a second mass *m*. Originally the mass M was alone on a turn-table which revolved about a vertical axis immediately under the central knife-edge of the balance. And some experiments which I made led me to suppose that mere change of position of the mass did not affect the level of the balance. However, after a complete determination in 1888 of the mean density, when I supposed that the work was finished, an examination of the results showed some curious anomalies, which I could only ascribe to a tilting of the whole floor on the displacement of the mass. Making new tests as to the effect of removal of the mass, I found that the

previous tests had been quite wrong in principle, and that there was a very appreciable effect quite visible in the telescope when the masses A and B were removed, and M was removed from one side to the other, the slope of the floor changing by an angle comparable with a third of a second. If this had been absolutely constant in amount, the differential method would have eliminated it; but, probably, it varied slightly in successive motions of the turn-table, and the results showed that

Fig. 1.



Elevation of balance room and observing room. The front of the case is removed, and the front pillar is not shown. The pointer and mirrors are at the back.

there was also a secular change, the amount of tilt gradually increasing. This secular change was probably due to increasing rigidity of the floor, so that it tilted over bodily, moving the supports of the balance with it, an increase partly due, perhaps, to the pressure of the building, which had only been erected ten or twelve years, but chiefly, I think, to a gas engine recently erected next door. When this was doing heavy work, the vibrations were very plainly felt, and no doubt they greatly aided the floor in "settling down." A second balancing mass m was therefore added, half as

great as M, and on the opposite side of the turn-table, but twice as far from the axis. The resultant pressure was now always through the axis, and I could detect no tilting of the floor when the turn-table was moved. Of course the balancing mass acted somewhat to reduce the effect of the larger attracting mass, but in a calculable ratio.

Finally, in order to eliminate or reduce the effect of any want of symmetry in the moving parts or in the masses, a second set of experiments was made with all the masses turned over and moved from left to right, and the mean of the first and second set was taken.

I now proceed to a detailed description of the various parts of the apparatus and the mode of experiment.

The Balance Room.—The balance room is in the basement of the Mason College, immediately under my room, and about 20 metres from the street. On one side were three windows looking on to a small courtyard, entirely surrounded by high buildings, but the windows have been bricked up. On the two adjacent sides are two other rooms, and on the opposite side a closely fitting door opening on a short corridor with doors at each end. There is no chimney in the room, and only an opening in the ceiling through which the balance was observed from the room above. The floor is of brick, resting on earth, and is very firmly laid.

The temperature of the room was taken by means of a thermometer with a protected bulb at the end of a long wooden rod hanging down from the room above. The thermometer was about 6 feet from the floor, near one end of the case, and it could be rapidly pulled up into the room above and read by the observer before its temperature sensibly varied. The temperature never appeared to vary so much as 0·1° C. in the course of two or three hours.

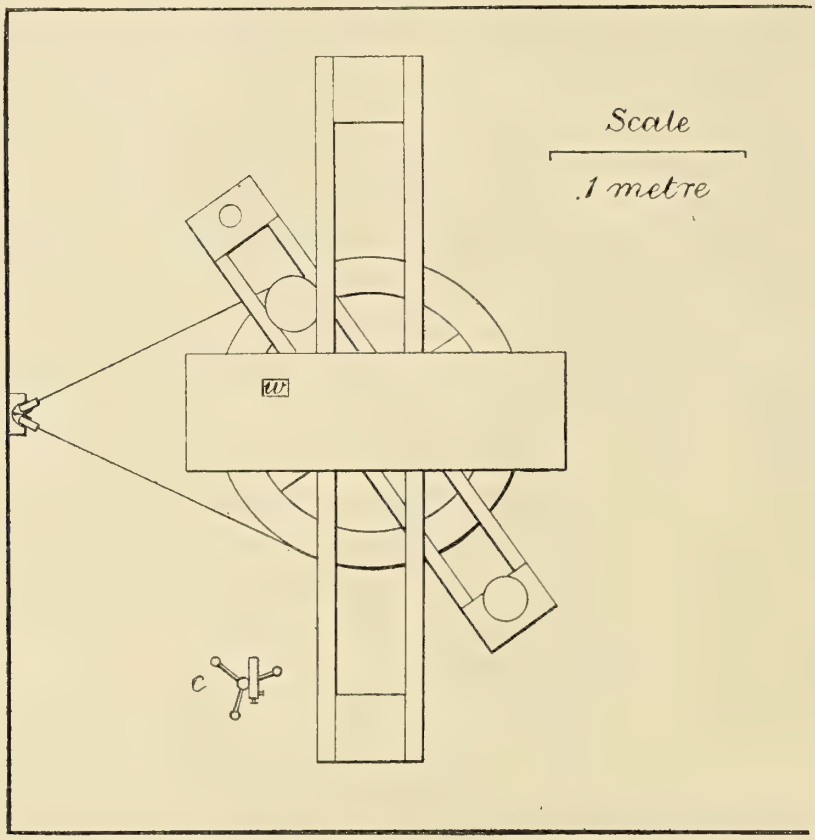
The Balance Case and its Supports.—The case (fig. 1) is a large cabinet of $1\frac{1}{4}$ inch wood, 1·94 metre high, 1·63 metre wide, ·61 metre deep, with three large doors in front giving access to the hanging masses and riders, and a small door at the back near the mirror hereafter described. It is lined inside and out with tinfoil, and under each of the suspended masses is a double bottom with a layer of wool between, making a total thickness of about $1\frac{1}{2}$ inch or 4 centims. At the top is a small window about 10 centims. square, through which the oscillations of the beam were observed. On each side within the case are placed three horizontal partitions, like shelves, to hinder circulation of the air.

The larger attracting mass and the attracted masses are gilded, and it is possible that some advantage may arise from having the surface of the case of different metal. For if it, too, were gilded, it would readily absorb radiation from the large mass, and when the inside temperature changed, the suspended masses would readily absorb radiation from the inner surface of the case. But gold probably absorbs considerably less of tin radiation than it absorbs of gold radiation, and so temperature changes are probably lengthened out more than if the case were gilded.

It was necessary to support the case so that the attracting masses could be moved

about underneath it, and also to make it independent of the floor. Two brick pillars, 58 centims. × 36 centims. and 56 centims. high, were therefore built on thick beds of concrete under the floor, and about $3\frac{1}{2}$ metres apart. They rise up free from the bricked floor. Stretching between them are two parallel iron girders (g, g), about 30 centims. apart, and with their under side 56 centims. above the floor. The balance case is placed across the middle of these girders (see plan, fig. 2), with its

Fig. 2.



Plan of turn-table, girders, pillars, and balance case. w. Window in case. c. Usual position of cathetometer.

under surface level with that of the girders. The square base plate of the balance is placed on the girders on three levelling screws. Two horizontal screws attached to the girders bear against each edge of the base plate, so that it can be adjusted and fixed in any position.

To lessen vibration one tier of bricks is removed from each pillar, and in its place are inserted eight cylindrical blocks of indiarubber (i, i, fig. 1), originally 7·5 centims. diameter and 7·5 centims. high. These crushed down almost 1 centim. at once, but have not shown any further measurable contraction in the course of several years. Their effect in deadening vibration has been surprisingly great.

The Turn-table.—On a bed of concrete, and quite free from the brickwork of the floor, is a circular rail of cast iron, 1·3 metre in diameter. On this, on conical brass

wheels and pivoted at the centre, runs the turn-table, about 1·5 metre in diameter. This is made of wood and covered with tinfoil. It is like a wheel with a flat circular rim, and with four flat spokes arranged as a cross. It is as nearly symmetrical as possible, and at opposite ends of a diameter are placed two shallow cups, in either of which the large attracting mass may rest. The centres of these cups are a distance apart, equal to the length of the balance beam. There are cut slots through the bottom of each cup, so that the bottom of the mass can be seen for the purpose of measuring the vertical diameter.

Two beams, 2·74 metres long, run across the turn-table 26 centims. apart, with the cups between them, and across the ends are two boards, each with a circular hole 12 centims. in diameter, and in either of these the smaller, or balancing mass, may rest. These beams are braced by brass rods to brass uprights at their middle points to diminish bending.

The turn-table is moved by an endless gut rope passing round it, and fixed at one point of the rim. The two sides of the rope pass over pulleys on to a drum in the room above. There are stops on the circular rail, against which come brass pieces on the turn-table when the masses are in position at either end of the motion. The drum can be turned easily by the observer at the telescope. Since the knife-edges and planes of the balance are of steel, all the moving parts of the apparatus are free from iron. As an illustration of the necessity of this, I may mention that for some time I used what I supposed to be a brass wire rope to move the turn-table, but on looking out for the explanation of some irregularities, I found that the brass was wrapped round a core of steel wire, which acquired poles at the highest and lowest points in the position in which it always rested between different sets of weighings. These poles had quite an appreciable action on the balance beam.

The Balance.—This is of the large bullion balance type, with gun-metal beam and steel knife-edges and plates. It was made specially for the experiment by Mr. Oertling, with extra rigidity of beam. Its performance has shown the great excellence of the design. The central knife-edge is supported on a steel plate by a frame-work rising 107 centims. above the base plate, and the usual moveable frame can be raised or lowered from outside the case, fixing the beam or setting it free to oscillate. The beam has often been left free to oscillate for months at a time, with the full load of 20 kilogrms. on each side, but I have no reason to suppose that the knife-edges have suffered at all.

The length of the beam was measured by taking the length of each half separately by a beam compass, and the mean of several measurements gave 123·329 centims. as the total length. The standard scale used throughout was that of a cathetometer made by the Cambridge Scientific Instrument Company. This scale has been verified at the Standards Office, and taking its coefficient of expansion as $\frac{1}{60000}$, it may be regarded for our purpose as perfectly correct at 18°, any errors being at that temperature much less than the errors of experiment. Comparing the beam compass

with this scale, it was found that ·06 centim. must be subtracted, reducing the length to 123·269 centims. Now both beam and scale are of gun-metal and may, therefore, without serious error, be assumed to have the same coefficient of expansion, so that this is the length of the beam at 18°. At 0° it is 123·232 centims.

Mirrors, Telescope, and Scale.—At first a mirror was attached to the centre of the beam and the reflection of a scale in it was observed, either in the ordinary method or in the method described in the former paper ('Roy. Soc. Proc.,' No. 190, 1878), where a second fixed mirror is used to throw the ray of light a second, or even a third time back on to the moving mirror, each return increasing the deflection of the ray. But it was then necessary to make the time of vibration very long, and even when the time was three minutes, the tilt due to the attraction, *i.e.* the change of resting point, did not amount to more than two or three scale divisions. Now certain irregularities observed when the apparatus was first set up at Cambridge, led to experiments on the time taken by heat to get through the case in sufficient quantity to affect the balance, and I found that a coil of copper wire placed close under the case on one side (the bottom of the case being then solid, 1 inch thickness), heated by a current yielding 100 calories per minute, began to produce an appreciable disturbance on the balance in about 10 minutes, doubtless by the creation of air currents from the heated floor of the case. It appeared advisable, therefore, to reduce the time of a complete experiment to less than this if possible, and, consequently, the time of a single swing very much below 3 minutes. This could only be done if at the same time the optical sensibility were very greatly increased.

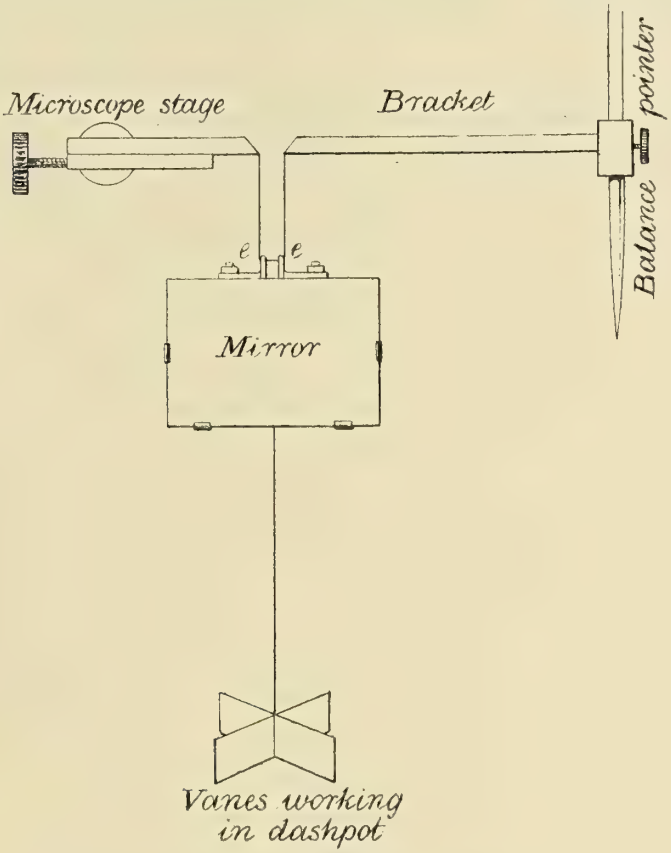
The employment of what may be termed the double-suspension mirror method due I believe to Sir William Thomson, and used by Messrs. G. H. and Horace Darwin in their experiments on the Lunar Disturbance of Gravity ('Brit. Assoc. Rep.,' 1881), has very satisfactorily solved the problem, giving a greatly increased deflection on the scale, even when the time of oscillation is as short as twenty seconds.

This method, which deserves to be more generally known and applied for the detection of small motions, consists in suspending a mirror by two threads, one from a fixed point, the other from the point which moves. The angle through which the mirror turns for a given motion of the latter point is inversely as the distance between it and the fixed point, so that by diminishing this distance the sensibility of the arrangement may be almost indefinitely increased.

To apply it to the balance, a small bracket (fig. 3) is fixed to the ordinary pointer of the balance, about 60 centims. below the central knife-edge. This projects horizontally at right angles to the axis of the beam, and it is bevelled at the edge. Close to it is another bevelled edge attached to a microscope stage movement which is fixed on to the central pillar of the balance. A thread of silk (as supplied for the Kew magnetometer) is fastened to the stage, passes over the bevelled edge, through two eyes, *e e*, on a light frame holding the mirror up over the bevelled edge of the bracket, and is

fastened to the bracket. The microscope stage movement allows the distance between the threads to be adjusted, and also enables the azimuth of the mirror to be altered.

Fig. 3.



Double Suspension Mirror (half size).

Of course, if the mirror were weightless, it would not affect the sensibility of the balance, and the threads might be brought very close together. But the weight of the mirror—it is silver on glass, 56 millims. × 38 millims. × 10 millims.—has a considerable effect on the sensibility, diminishing it with decrease of distance between the points of suspension. In practice it has been found convenient to work with the threads parallel, and from 3 to 4 millims. apart, the time of swing one way being adjusted to about 20 seconds. A less time hardly suffices for a correct determination and record of the scale reading. Taking 4 millims. as the distance, and supposing the bracket to be 600 millims. below the knife-edge of the balance, the mirror evidently turns through an angle 150 times as great as that through which the beam turns.

The drawback to this method of magnification is that the mirror has its own time of swing and is easily disturbed. The swings of the mirror and the disturbances are, however, effectually damped by having four light copper vanes attached to the end of

a thin wire, projecting down from the mirror and working in a dash pot with four radial partitions not quite meeting in the centre, one vane being in each compartment. I found that mineral lubricating oil is very suitable for the dash pot, as the surface keeps quite clean and there is little evaporation. The swings of the balance are also very greatly damped by this arrangement, but the effect of this will be discussed later.

The telescope and scale are in the room over the balance room (see fig. 1), a hole being cut through the floor, and a small glass window being fixed in the top of the case. As the suspended mirror is in a vertical plane it is necessary to have an inclined mirror fixed in front of it to direct the light from the scale horizontally on to it and back again to the telescope. With the magnification used it was necessary for good definition to have an exceedingly good inclined mirror, and several were rejected before a suitable one was obtained. That finally used is a silver on glass oval mirror, 60 millims. × 40 millims., by Browning. The glass window in the case is optically worked and carefully adjusted to be normal to the path of the light.

The telescope has a 3-inch object glass of about 4 feet focal length. It is fixed on a brick pillar built on one of the brick arches, which form the ceiling of the balance room, and it rises free from the floor of the observing room. To destroy vibration one course of bricks is replaced by blocks of india-rubber. The scale has 50 divisions to the inch (say $\frac{1}{2}$ millim.), ruled diagonally, and divided to tenths by cross lines. It is photographed on glass from a scale drawn on paper with very great care, 50 inches long (say 127 centims.), and with 500 divisions. The photograph is $\frac{1}{10}$th of this length, and only the central part of the scale, about 60 divisions in length, has been used. The diagonal ruling enables a tenth of a division to be read with certainty, and the readings recorded in the Tables, pp. 625–655, are in tenths. Though the lines appear somewhat coarse, I have not been able to find another scale equal to it in distinctness and in ease of reading. As all the results depend on the ratio of measurements, taken almost simultaneously, of deflection due to attraction and rider respectively, in the same part of the scale, I have not thought it necessary to calibrate it.

The scale is fixed horizontally on the end of the telescope close to the object glass with a piece of ground glass over it. It was illuminated in general by an incandescent lamp placed above it, once by an Argand burner.

The distance from the scale to the mirror and back is about 5 metres. It follows that 1 division of the scale corresponds to an angular motion of the mirror through ·0001 radian. But this is at least 150 times the angle through which the beam turns for the same deflection. So that 1 scale division implies an angular motion of ·0000006 radian, or $\frac{2}{15}''$ in the beam. As the total length of swing in Table III. is never more than 12 divisions, the angular vibrations of the beam are at the most about $1''{\cdot}6$, and the linear vibrations of the masses, since the half beam is about 60 centims., are at the most about ·005 millim. This shows that it is quite unneces-

sary to consider any change of distance due to vibration. The greatest deviation from the mean in any of the series of weighings recorded is about 1 per cent. of the rider value, corresponding to about $\frac{1}{10}$th of a division, or an angle of $\frac{1}{75}''$ in the beam, and a distance of ·00004 millim., say $\frac{1}{600000}$ inch, in the motion of the masses. This seems to show that the method is accurate as well as sensitive.

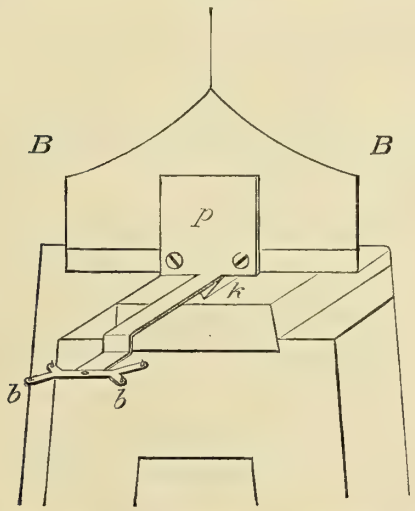
Determination of the Value of the Scale Divisions by means of Riders.—This was done by means of centigramme riders (fig. 4), these being the least weights which

Fig. 4.

Rider, actual size, and end of lifting rod, *r*.

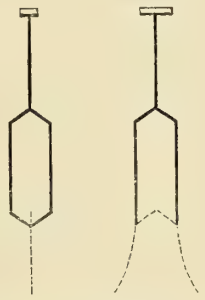
appeared capable of sufficiently accurate determination. Instead of transferring the same rider from point to point, it was much easier to use two equal riders, and to take one up while the other was being let down a given distance from it. The distance selected was about 2·5 centims., since the deflection due to the transfer of one centigramme so far along the beam was nearly equal to that due to the greatest attraction to be measured.

Fig. 5*a*.



Subsidiary rider beam, *bb*, attached to centre of balance beam, *BB*, by plate *p* just above central knife-edge, *k* (half size).

Fig. 5*b*.



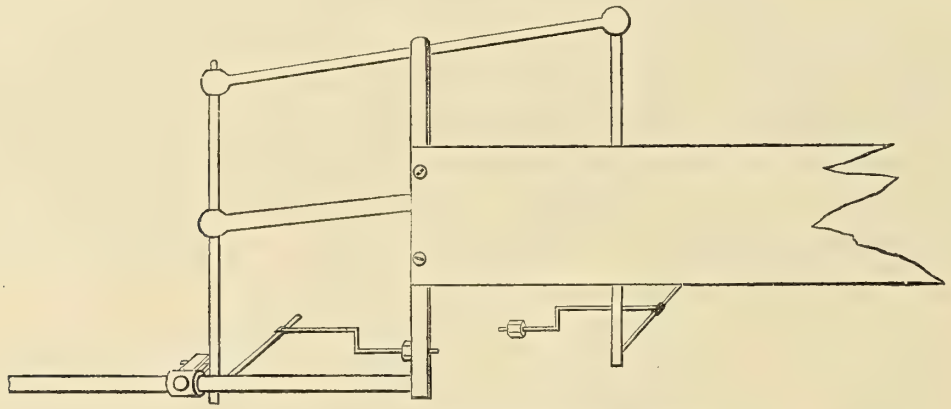
Wire frames depending like scale pans from ends of *bb*, fig. 5*a*, side and end views (half size).

At first the riders when on the beam rested in V notches in a pair of parallel brass strips fixed on and parallel to the beam. But this plan was soon abandoned, as there

was no certainty about the position of the rider in the notches. The riders were then supported in little wire frames, each hung by two cocoon fibres from the edges of a plate fixed to the beam, the edges being parallel to the central knife-edge. The only objection to this method was the very considerable time spent in replacing the fibres after the breakages which occurred on dusting or any readjustment of the balance.

Ultimately a small subsidiary beam, about 2·5 centims. long, was attached to the centre of the balance beam just above the knife-edge (fig. 5*a*), the scale pans being represented by small wire frames in which the riders could rest (fig. 5*b*). These frames depend from agate pieces resting on steel points at the extremities of the subsidiary beam in the way now usually adopted in delicate assay balances. This mode of supporting the riders appears to be perfectly satisfactory.

Fig. 6.



Lifting rods to raise or lower riders (half size).

To raise or lower the riders two short horizontal lifting rods parallel to the beam move up and down within the supporting wire frames with a nearly parallel motion, and on them are two metal pieces with their upper surfaces shaped so that the riders rest on them without swinging (fig. 4, *r*). They are the extremities of L-shaped projections from a jointed parallelogram framework (fig. 6), supported on an upright in front of the subsidiary beam. The framework is moved by a tongue engaging with it, and projecting from a horizontal rod, which rotates about its axis in bearings, one within the case and the other outside. The rod is turned through an angle of about 30° between stops by an endless string passing upwards and round a wheel in the observing room.

The parallelogram framework and the bearing of the rotating rod within the case are both supported independently of the case from the ceiling. At first they were supported respectively on the central pillar of the balance and on the case; but when the increase of optical sensitiveness enabled me to detect small irregularities, I

realised how essential it was for accurate weighing that all parts of the apparatus moved from the outside should be supported quite independently of the balance. Even the string moving the rod transmitted great and continual vibration. The rod and the framework with the lifting levers were, therefore, supported by iron rods coming down from the ceiling through holes in the top of the case, large pieces of cardboard stretching from these rods over the holes to hinder the passage of dust into the case. Once or twice in the course of preliminary experiments irregularities were traced to accidental contact of outside bodies with the case.

It appeared just possible that there might be electrification of the riders by friction with the lifting rods, especially when they were supported by cocoon silk. It was, therefore, advisable that the surface of the lifting rods should be of the same kind as that of the riders. As the latter are silver wire gilded, the lifting rods are also gilded. It may not be uninteresting to note here a curious phenomenon which occurred during some early preliminary experiments. The shaped pieces on the lifting rods were then of wood covered with gold leaf, put on with ordinary paste. After they had been on for some months, I obtained some very various results for the deflection due to the riders, and on examining the lifting rods I found that a number of long needle growths projected from the wood pieces and interfered with the supporting wire frames. At first I thought these were organic, but my colleague, Professor HILLHOUSE, examined them and found that they were crystalline. Doubtless, the hygroscopic paste set up electric action between the gold leaf and the brass to which the wood pieces were attached, and the crystals were probably zinc sulphate. The wood was then replaced by brass gilded, and no further difficulty of the kind was experienced.

The length of the subsidiary beam was kindly determined for me by Mr. GLAZEBROOK at the Cavendish Laboratory. The steel points are hardly sharp enough to determine the distance to 1 in 10,000, but the mean of the results is sufficiently exact. The following are Mr. GLAZEBROOK's determinations; the four points being denoted by *abcd*:—

Date.	Temperature.	Number of readings.	*a* to *b*.	Number of readings.	*c* to *d*.
	°		inches.		inches.
1889 July 4 .	22·5	6	·9985	6	·9979
July 11 .	21·5	3	·9990	3	·9982
July 12 .	23	3	·9988	3	·9979

These are in terms of a gun-metal standard of which the error is only 3 in 100,000 at 0°, and, therefore, for my purpose negligible. The beam is of brass, and we may assume with sufficient exactness that it has the same expansion as the standard. The

temperature may, therefore, be left out of account. The mean value of $\frac{1}{2}(ab + cd)$ is therefore ·9983375 inch, or taking 2·539977 centims. to the inch we obtain

Length of beam at 0°, 2·53575 centims.

There is an advantage in fixing this beam at the centre, which should be noted here. Suppose the riders are not quite equal, but have values w and $w + \delta$. Let the two ends of the subsidiary beam be distant a and $a + l$ from the central knife-edge. Then the effect of picking up the rider w from the nearer, and letting down the rider $w + \delta$ on the further end, is equivalent to putting at unit distance

$$(w + \delta)(a + l) - wa = wl + \delta(a + l) = wl\left(1 + \frac{\delta}{w}\frac{a + l}{l}\right),$$

or the error δ/w is multiplied by $(a + l)/l$, and, if the beam is not central, $(a + l)/l$ may be greater than 1, so that the error is magnified.

If, however, the small beam is central, l is equal to $-2a$, and the error is multiplied by $+\frac{1}{2}$.

If the riders are interchanged and the weighings are then repeated, the mean result is the same as if riders with the mean value were used for

$$w(a + l) - (w + \delta)a = wl - \delta a = wl\left(1 - \frac{\delta}{w}\frac{a}{l}\right),$$

and the mean of this and the above is

$$\left(w + \frac{\delta}{2}\right)l.$$

The Attracting and Attracted Masses.—These are all made of an alloy of lead and antimony, for the sake of hardness, the specific gravity in each case being about 10·4. They were made at various times and places, the large attracting mass M being made more than 12 years ago by Messrs. Storey, of Manchester. The smaller balancing mass m was made in 1889 by Messrs. Heenan, of Manchester and Birmingham. These were both cast with a "head" on, and then turned. The attracted masses A and B were made by Messrs. Whitworth, and subjected to hydraulic pressure before turning. The dimensions have been measured from time to time, and there is no evidence of any sensible change of shape.

The larger mass M and the attracted masses A and B were weighed at the Mint through the kindness of the Deputy Master and Professor Roberts-Austen. For the weight of the balancing mass m, I am indebted to Messrs. Avery, of Birmingham. The large mass M has suffered two accidents since it was weighed, once being slightly cut into by a saw during some alteration of the case, and once being scratched by coming into contact with a piece of metal fixed to the turntable in taking it out of

its place. The saw-cut was carefully filled in with lead, and the scratch removed only a fraction of a gramme, as was determined by taking a mould of the hollow. I should be glad to think that the determination of the attraction was sufficiently exact to make reweighing necessary, but I am afraid that the alteration in weight is very far beyond the important figures, and I therefore take the original weight as sufficiently near the truth. The masses A and B have been gilded since the original weighing, but I carefully determined their increase of weight by the balance used in the gravitation experiment.

The values given below in the second column are the true masses. In the third column are the masses of M and m, less the air displaced by them, this being taken as 18·41 and 9·2 grms. respectively. It will be shown later that the true masses of A and B and the reduced masses of M and m may be used in the calculations of the result.

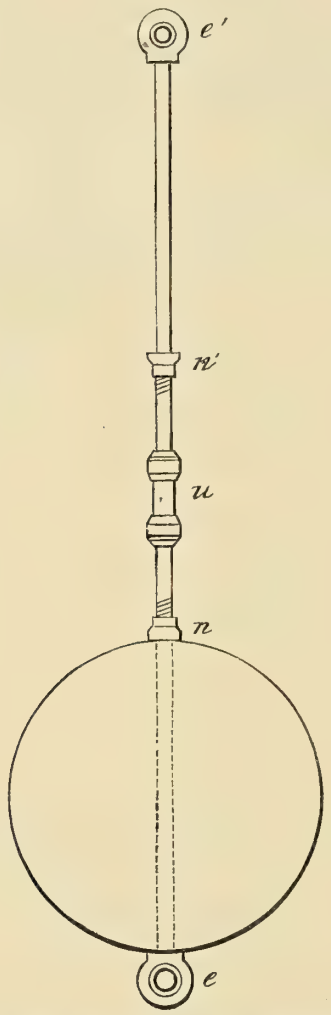
	True mass in grammes.	Mass less that of air displaced.
M	153407·26	153388·85
m	76497·4	76488·2
A	21582·33	
B	21566·21	

Suspension of the Attracted Masses.—Each of the attracted masses is drilled through along a diameter, the hole being ·6215 centims. in diameter, and a brass rod (fig. 7) terminating in an eye e below, is passed through the hole. The mass is secured in position by a nut n working in a screw thread cut for a short distance in the rod. An exactly similar rod terminating in a similar eye e', and with a similar nut n', is fastened end to end to this by a union u. The nuts and the inner sides of the enlargements for the eyes are hollowed out so as to fit exactly on to the spheres.

From the ends of the balance beam hang down stout brass wires terminating in hooks. If these hooks are passed through the eyes e' the attracted masses are close to the floor of the balance case, and their centres are adjusted to be about 32 centims. from the centre of the large attracting mass when under either of them. If the masses are turned over so that the hooks pass through the eyes e, they are about 30 centims. higher or at nearly double the distance, the length ee' being about 48 centims. The rods being perfectly symmetrical about the union u, the attraction on them is the same in either position. The weight of each is about 212 grms., or about $\frac{1}{100}$ of the attracted mass, so that any small variation in their position would produce a negligible variation in the total attraction. By the differential method, the attraction on them entirely disappears from the results.

The mode of support of the attracting masses M *and* m.—This has already been described when describing the turntable.

Fig. 7.



Suspender for Attracted Mass (one-fourth size).

The Riders.—Four centigrm. riders, A, B, C, D, of silver wire gilt were made by Mr. Oertling of the form shown in fig. 4. These were weighed in 1886 at the Bureau International des Poids et Mesures, by M. Thiesen. The following is an extract from the certificate :—

"*Densité et volume.*—Comme densité on a accepté celle de l'argent, et par conséquent comme volume de chacun des cavaliers, 0·0010 millilitre.

"*Détermination des poids des cavaliers.*—L'étude des poids de ces quatre cavaliers a été faite par M. le Dr. Thiesen, adjoint du Bureau International, chargé de la section des pesées. M. Thiesen au moyen de la balance Stüchrath, destinée à des poids au dessous du gramme, a d'abord déterminé les différences entre les quatre

cavaliers pris deux à deux dans les six combinaisons possibles, et ensuite la différence entre l'ensemble des quatre cavaliers et le poids de 40 milligrms. de la série 0 du Bureau, série en platine iridié récemment étalonnée par M. THIESEN. Les comparaisons ont été faites du 19 au 29 Mars, 1886

"*Résultats.*—De l'ensemble de ces comparaisons résultent les poids :—

A = 10·1247 milligrms.
B = 10·0615 „
C = 10·1196 „
D = 10·1262 „

"L'incertitude de ces déterminations ne dépasse pas 0·001 milligrm."

A and D were selected for use as being the nearest to each other in value. B and C were kept untouched in boxes till 1890. In the various experiments made between 1886 and the final weighings, A and D had necessarily been handled to some extent, especially through the frequent breaking of the silk fibre suspension used before the subsidiary beam described above, and it appeared possible that their weights might be altered. It was also necessary to determine whether an appreciable amount of dust was deposited on them in the course of several weeks as it was inconvenient to dust them frequently. The riders B and C might be assumed to have the same weight as in 1886, and could be taken as standards.

To make the weighings a 16-inch chemical balance was arranged with a double suspension mirror on exactly the principle already described for the large balance. The apparatus was put together quickly with materials at hand, and might easily be greatly improved. It is only described here to show how accurate the method is, even with such rough apparatus, and that it is applicable to a small as well as a large balance.

A cork sliding on the pointer with a horizontal needle stuck in it, served to support one thread of the mirror; a stand with a projecting arm—one made to hold platinum wires in a Bunsen flame—served to support the other thread. A wire with a small copper vane depended from the mirror and was immersed in an oil dashpot. The telescope and a millimetre scale were on a level with the mirror about 2 metres distant on one side of the balance. Two brass strips, parallel to each other and the beam, were fixed on the top of one arm of the beam, and in each of these were two V notches in which centigramme riders could rest. Two levers, worked by cams on a rod rotated by the observer, picked one rider up and let down the other, so that the effect was equivalent to the transfer of 1 centigrm. from one notch to the other. Their distance apart was such that this was equivalent to the addition of ·3284 milligrm. to one pan of the balance. This was the arrangement described in my former paper. Attached to one pan was a pair of brass strips parallel to each other, and such that the riders A, B, C, or D, would just rest across them. Two lifting rods worked up and down between these strips, so that of the two riders to be compared, one could be picked up at the

instant the other was let down. The lifting rods were worked by a rod rotated by the observer and supported quite independently of the balance, and of the slab on which it rested. By this plan the value of the scale divisions and the shifting of the centre of swing on changing the weights to be compared, could all be determined without raising the beam of the balance between the successive weighings, an essential condition, I believe, for exact work.

The weighings were made in the large room of the Physical Laboratory, and no precaution was taken to protect the balance case beyond placing a board in front of it. The room is draughty and subject to great variations of temperature, so that the weighings were made under very disadvantageous circumstances. One result of this was a rapid and sometimes very great change of resting point in the course of a few hours, so that the scale passed out of the field of view. In order to bring it back without opening the case, two glass tubes passed through the top of the case, almost down to the scale pans, and small bits of wire could be dropped through these on to either pan as needed. Caps fitted on to the tubes to prevent draughts. This plan appears worthy of mention, as it suggests a mode of determining the value of a scale division when a balance is either too sensitive for riders or has no special arrangement for their accurate use. If a piece of wire weighing, say, 1 milligrm. is cut into say ten nearly equal parts, and if these are dropped on to the two pans alternately the shiftings of the centre of swing will be to and fro, about equal distances, due to about ·1 milligrm., but the sum of the shiftings will be that due to 1 milligrm., and the balance at the end will be nearly in the same position as at the beginning.

The following is an abstract of the comparisons of the riders. They were made soon after the first determinations of attraction on February 4, when A and D had not been dusted for three months.

In each case three extremities of swing were observed, and the centre of swing was determined from these by the graphic construction described later (p. 595).

The centres of swing were combined in consecutive threes in the usual way to give the differences in scale divisions.

Thus, in the first series, the successive centres of swing with D and A alternately in the scale pan were

D	A	D	A	D
231	223	217	211·9	208

whence

$$(\mathrm{D} - \mathrm{A})_1 = \frac{217 + 231}{2} - 223 = + 1{\cdot}0 \text{ division.}$$

$$(\mathrm{D} - \mathrm{A})_2 = 217 - \frac{223 + 211{\cdot}9}{2} = - 0{\cdot}45 \text{ division.}$$

$$(\mathrm{D} - \mathrm{A})_3 = \frac{217 + 208}{2} - 211{\cdot}9 = + 0{\cdot}6 \text{ division.}$$

$$\text{Mean D} - \text{A} = {\cdot}38 \text{ division.}$$

Successive values of the differences alone are given below.
The time of swing one way was about 16 seconds.

February 16, 1890.

(1.) Comparison of A and D, undusted.

Deflection due to ·328 *milligrm.* 83·45, 82·45, 84·45 divisions. Mean 83·45 divisions.

$$D - A = 1{\cdot}0, -{\cdot}45, +{\cdot}06 \text{ division. Mean } {\cdot}38 \text{ division;}$$

therefore

$$D = A + {\cdot}0015 \text{ milligrm.}$$

(2.) Comparison of A undusted, D dusted.

Value of scale division taken as in the last.

$$D - A = -{\cdot}5, +{\cdot}3, -{\cdot}1, -{\cdot}4, +{\cdot}25. \text{ Mean } -{\cdot}09 \text{ division;}$$

therefore

$$D = A - {\cdot}0004 \text{ milligrm.}$$

February 17, 1890.

(3.) Comparison of A and D, both dusted.

Value of scale division taken as below (4).

$$D - A = -{\cdot}1, -{\cdot}2, +{\cdot}3, -{\cdot}3, -{\cdot}8. \text{ Mean } -{\cdot}22 \text{ division;}$$

therefore

$$D = A - {\cdot}0008 \text{ milligrm.}$$

(4.) Comparison of C and D.

Deflection due to ·328 *milligrm.*, 85·35, 85·4, 84·65. Mean 85·13 divisions.

$$D - C = +{\cdot}15, {\cdot}00, +{\cdot}05, -{\cdot}15, +{\cdot}05, +{\cdot}3, +{\cdot}05, -{\cdot}05, -{\cdot}35, {\cdot}05, {\cdot}35, 45, {\cdot}50, {\cdot}2. \text{ Mean } {\cdot}114 \text{ division;}$$

therefore

$$D = C + {\cdot}00044 \text{ milligrm.}$$

February 18, 1890.

(5.) Comparison of C and D repeated.

Deflection due to ·328 *milligrm.*, 92·75, 92·3, 91·65. Mean 92·23 divisions.

D − C = ·35, − ·05, − ·8, − ·95, − ·1, + ·05, 0, − ·15, − ·1, + ·05. Mean − ·17 division;

therefore

$$D = C - \cdot0006 \text{ milligrm.}$$

Combining this with the last, and weighting them in the ratio of the numbers of determinations in each,

$$D = C + (\cdot00044 \times 14 - \cdot0006 \times 10) \div 24 = C - \cdot0000 \text{ milligrm.}$$

(6.) Comparison of A and D.

Value of scale division taken as above, ·328 milligrm. = 92·23 divisions.

D − A = ·45, ·25, ·1, − ·2, − ·1, ·35, ·25, ·45, ·60, ·5, ·5, ·55, ·5, ·75, ·55, ·1, ·05, ·45, ·10, ·30, ·8, ·9, ·35, ·2, ·30, ·55, ·50, ·35, ·45, ·45. Mean ·378 division;

therefore

$$D = A + \cdot00134 \text{ milligrm.}$$

Examining the values obtained in (1), (2), and (3), it will be seen that no trustworthy evidence is given of a difference due to dusting. Any existing difference was probably under ·002 milligrm, and since the weighings on February 4, before dusting, were made with the attracted masses in the upper position, when the attraction was only one-fourth of that on which the final results depend, we may safely neglect the effect. After this the riders were dusted more frequently, so that we may probably assume their values more constant.

The comparisons of C and D, and of A and D, in (4), (5), and (6), were made more carefully. That of A and D in (6) is much the best of the series, the air in the laboratory happening to be steadier while it was made. The range between the greatest and least values of the difference is one scale division, or ·0036 milligrm., and the different results are grouped fairly closely about the mean.

The centres of swing and the differences are plotted in Diagram VIII. I do not claim that these results show any remarkable accuracy when compared with those obtained at the Bureau International des Poids et Mesures, but remembering how rough the apparatus was, and how little precaution was taken to ward off air currents,

I have not the slightest doubt that, with special design of apparatus and more suitable locality, the results could be very greatly improved, and the accuracy carried far beyond anything hitherto reached. As they stand, they seem to show the value of the combination of a short time of swing with optical magnification.

The result of comparisons (4), (5), and (6), is, that if C has its Paris value, viz., C = 10·1196 milligrms., then, A = 10·1183 milligrms., and D = 10·1196 milligrms. ; whence $\frac{1}{2}(A + D) = 10{\cdot}119$ milligrms. This value may be used in calculating the result, since the riders were interchanged before Set II. was taken.

The losses experienced since 1886 by A and D are respectively, by A ·0064 milligrm., and by D ·0066 milligrm., *i.e.*, they have diminished by practically equal amounts. This was to be expected as they have probably received equal amounts of rough usage.

The substitution of the subsidiary beam for the cocoon fibre suspension of the riders having greatly diminished the handling to which they were subjected, I have not thought it necessary to weigh them again during the work.

Linear Measurements.

In the mathematical theory it will be shown that the lengths required are those marked in fig. 15, viz., the horizontal distances, L and l, and the vertical distances, D_1 D_2, d_1 d_2, H_1 H_2, h_1 h_2.

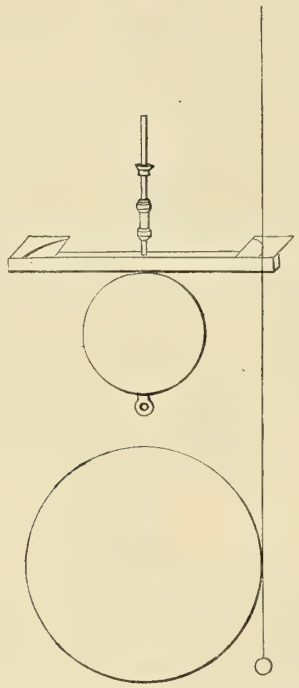
The Horizontal Distances.—Except when estimating the moment of the rider, the distance L is really that between the verticals through the centre of M and the centre of the more distant attracted mass. But the verticals through the centre of M in each position, so nearly passed through the centre of the mass above it, and, therefore, through the knife-edge from which it hung, that L was taken as equal to the length of the beam (p. 571).

The accuracy of this adjustment was secured as follows. A horizontal cross-piece was fixed on the top of each attracted mass, with two horizontal cards at its two ends, each with a portion of a circular arc on it, with radius equal to that of the large mass M, and with centre over that of the attracted mass (fig. 8). A plumb line was then hung just in front of the case, and the balance was moved by the horizontal screws bearing against the base plate until the plumb line always appeared to touch the circular arc above, when it appeared to touch the large mass below. The adjustment was not quite perfect, but the error in the worst case was probably not more than 1 millim., and certainly less than 2 millims. Such an error in the horizontal distance is negligible.

The distance l had different values for the two positions occupied by m on the turntable. Calling these values l_1 and l_2 respectively, $l_1 + l_2$ was found by measuring a, the inside distance between M and m, arranged as in Set II., and b, the inside distance between them, when m was put on the same side of the turntable as M, and

adding to $a + b$ the sum of the diameters of M and m in the radial direction of the turntable as taken by square calipers.

Fig. 8.



Plumb line Adjustment of Masses.

The following are the values in terms of the cathetometer scale already referred to, the temperature being 15° C.:—

a =	157·01
b =	33·95
Diameter of M =	30·52
,, ,, m =	24·23
therefore $l_1 + l_2$ =	245·71

The value of $l_1 - l_2$ was found by measuring the shortest distance of m from the wall when respectively in the first and the second positions on the turntable. It was found that

$$l_1 - l_2 = \quad \cdot 12$$

whence

$$l_1 = 122{\cdot}915$$
$$l_2 = 122{\cdot}795.$$

We may obtain from these measures an independent value of the radius of the circle in which the centre of M moves. With perfect adjustment this should be $\frac{1}{2}$L = 61·66 at 18°.

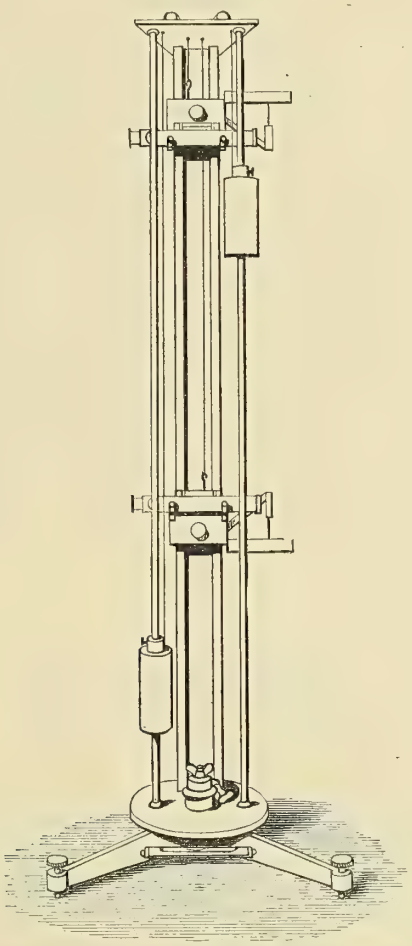
It is equal to a + radius of M + radius of m − l_2, or, by the above measures,

$$= 157{\cdot}01 + 15{\cdot}26 + 12{\cdot}115 - 122{\cdot}795$$
$$= 61{\cdot}59,$$

which is only ·07 centim. less than $\frac{1}{2}$ L.

Inasmuch as the wood probably expanded less than the cathetometer scale, while the metal expanded more, I have assumed as a rough approximation that the total expansion equalled that of the scale, so that the values of l_1 and l_2 are correct at 18° (see p. 571). No importance is, however, to be attached to this temperature correction.

Fig. 9.



Cathetometer used to measure Vertical Diameters.

The Vertical Distances.—At the conclusion of each set of weighings with the attracted masses in a given position, the vertical distances between the top of the attracting masses and the bottom or top of the attracted masses (accordingly as they were in the upper or lower position) were measured by the cathetometer already referred to.

This instrument is of the well-known design of the Cambridge Scientific Instrument Company, and is especially adapted for measuring differences of level at different

distances in different vertical planes. It reads to ·002 centim. The account of these measurements will be found in Table II. (p. 614, *et seq.*).

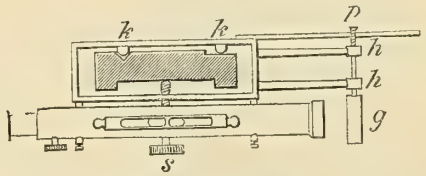
To find the distances D, *d*, H, *h* (fig. 15), it was necessary to add to the actual distances measured the sum or difference of the vertical radii of the attracting and attracted masses, and, therefore, the vertical diameters of all the masses were measured.

For this purpose I used a cathetometer which has lately been constructed for me by Messrs. Bailey, of Bennett's Hill, Birmingham. I have to thank Mr. Potts, of that firm, for his care in its construction, and also for the trouble which he has taken in the construction and alteration of much of the apparatus used throughout the work recorded in this paper. As the cathetometer is, I believe, new in design and satisfactory in its performance, it appears worthy of description.

The Cathetometer used to measure Vertical Diameters (fig. 9).—There are two telescopes, one to sight the upper the other to sight the lower of the points between which the vertical height is required. There is no scale on the instrument, but after the telescopes are fixed to sight the two points the instrument is turned round a vertical axis, so that the telescopes sight a vertical scale at the same distance from them as the points. In general, of course, the cross wire will appear to lie between two divisions, but by means of the fine adjustment, to be described below, the two nearest scale divisions are brought in succession on to the cross wire, and by interpolation the reading corresponding to the point first sighted by the telescope is determined.

The telescopes are fixed on collars running up and down the main pillar, which has a section of the form shown in fig. 10 (shaded).

Fig. 10.



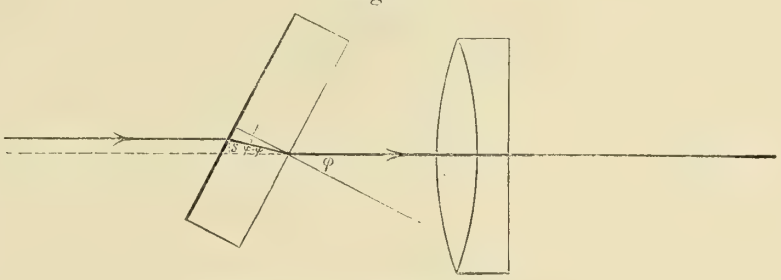
Section of pillar and collar of new Cathetometer.
s, clamping screw. *k*, *k*, guiding knobs. *g*, glass plate for fine adjustment, turning on axis *h*, *h*, with pointer at *p* perpendicular to plane of figure.

The guides consist of three knobs, *k k*, on the inside of the collar, two sliding in a vertical V-groove and one on a plane, both groove and plane being at the back of the pillar. A screw, *s*, clamps the collar in any position. Gut strings running up over pulleys and supporting counterpoises, sliding on the thinner pillars (see fig. 9), are attached to the collars so that these move easily. At first springs were used to keep the knobs always in contact, but I found it much better to remove these and trust merely to hand pressure to keep the collars in the proper position before clamping with the screw *s*.

The fine adjustment is secured by the use of a piece of plate-glass placed in the

front of each object glass (g, fig. 10), and capable of rotation about a horizontal axis, $h\ h$. A pointer is fixed on the end of this axis at p, and at its end is a small glass plate with a scratch on it moving close against a straight scale. If the plate is initially normal to the optic axis of the telescope, on turning it through ϕ, the ray which now comes along the optic axis has been shifted by transmission through the plate parallel to itself, a distance $t \sin(\phi - \psi)/\cos\psi$, where t is the thickness of the plate and ψ is the angle of refraction within it (see fig. 11).

Fig. 11.



Section of fine adjustment plate.

This shifting happens for small angles to be nearly proportional to $\tan\phi$, and, therefore, to the reading on the straight scale. To show how nearly this is the case the following table gives the shifting for angles of 5°, 10°, and 20°, with a thickness of $t = 1$ centim. and a refractive index $\mu = \frac{3}{2}$:—

Angle ϕ =	Shifting.
5°	$\frac{1}{3}\tan 5°\ (1 - \cdot00042)$
10	$\frac{1}{3}\tan 10°\ (1 - \cdot00156)$
20	$\frac{1}{3}\tan 20°\ (1 - \cdot0052)$

The error in taking the shifting as proportional to $\tan\phi$ is, up to 20°, quite negligible in ordinary telescope-cathetometer work. If it is desirable to have greater accuracy, it is probably best to use a table of corrections to the tangent; but it is possible to get an exact scale thus :—

Let OP, fig. 12, represent the pointer of length r, making ϕ with a line MN. Let a pointer PM jointed to this at P be of length μr, and let its extremity M move on the line MN. Drawing OD at right angles to MN, if s is the shifting, we have

$$\mathrm{OD} = \mathrm{OP}\,\frac{\sin(\phi - \psi)}{\cos\psi}$$

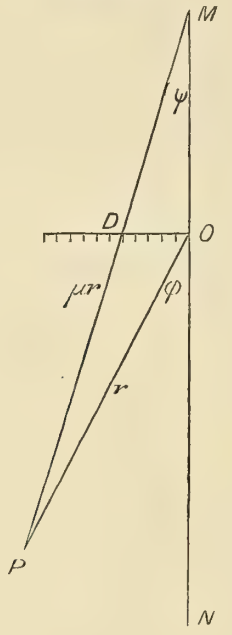
$$= \frac{rs}{t},$$

or

$$s = \frac{t}{r}\,\mathrm{OD}.$$

Probably the practical difficulties in the use of such an arrangement would render it troublesome and uncertain.

Fig. 12.



The plate is used as follows:—Adjust it normal to the optic axis of the telescope, and move the telescope till the required point is brought as near to the cross-wire as is possible by the hand. Clamp the telescope, and then turn the plate till the point is exactly on the cross-wire. Read the position of the pointer attached to the plate on its scale. Repeat these operations with the other telescope on the other point, then turn the instrument about its vertical axis till the telescopes sight the vertical scale placed at the same distance away as the two points. Looking through one of the telescopes the cross-wire is in general not exactly on a division. Turn the plate so that first the nearest division above, and next the nearest division below, is on the cross-wire. Reading the position of the pointer in each case, interpolation gives us the reading on the vertical scale corresponding to the position of the pointer when the cross-wire was between the two scale divisions. Doing this for each telescope the difference between the two points is found in terms of the vertical scale.

The plates I have used are about 9 millims. thick, and the pointers about 9 centims. long. They move over scales such that 25 to 27 divisions correspond to a shifting of 1 millim. The lower scale is graduated from 0 to 50, the upper from 50 to 100 to prevent confusion. The 50 divisions occupy a distance of 66 millims.

It will be observed that in this form of instrument the level error is practically entirely obviated. It can only come in if the scale is not at the same distance as the height to be measured, and may then be made negligible in practice by levelling the telescopes. Indeed, the uncertainty of measurement appears only to depend on the

uncertainty with which the cross-wire can be brought to the proper point, that is, it depends only on the magnifying power and definition of the telescopes used.

To illustrate the use of the instrument, a full account of the determinations of the vertical diameters is given in Table II. Below are the results, and for the sake of showing that there has certainly been no great change in shape, I give results obtained with a cathetometer more than 10 years earlier at the Cavendish Laboratory at Cambridge.

	1890.	1880.
	centims.	centims.
Large attracting mass M . .	30·526	30·5192
Small „ „ *m* . .	24·176	
Attracted mass A	15·8203	15·8166
„ „ B	15·7829	15·7842

The diameters of M and *m* in a horizontal direction parallel to a radius of the turntable measured by square callipers were

$$M = 30{\cdot}52 \text{ centims.}$$

$$m = 24{\cdot}23 \quad \text{„}$$

Temperature Correction.—Though the expansion of the masses was to be expected of an unimportant amount, I thought it advisable to attempt to measure it, in case there might be anything anomalous. One of the attracted spheres, B, was for this purpose placed between two vertical levers, in a tank through which could be run a continuous stream of cold or warm water. These levers depended from horizontal rods which could rock or slightly rotate on fine point suspensions. This was, in fact, a kind of double Lavoisier and Laplace apparatus. The motion of each lever was shown by another lever of about the same length, rising vertically up from each horizontal axis, and serving as the moving support for a double suspension mirror in which was viewed the reflection of a millimetre scale. Two telescopes and one scale were used for the two mirrors, though it would not have been difficult to arrange one telescope and two scales. The value of one scale division was determined by inserting a piece of thin glass between the sphere and each lever in turn. The method is exceedingly sensitive, but I have not been able to make it exact, owing to the warping produced in the rods due to unequal temperatures.

The measures of the expansion varied between ·0000214 and ·0000277, both vertical and horizontal diameters (in the position in the balance) being tested. The true value is probably nearly ·000025 or 1/40000. It will, therefore, lead to no appreciable error if we take the expansion as equal to that of the scale of the cathetometer, say 1/60000 (see p. 618, Table II.).

Determination of the Attraction by the Balance.

When the balance is used to measure such small forces and weights as those with which we are here concerned, it must be left swinging on its knife-edge throughout any set of weighings in which the deflections are to be compared one with another. For there is not the slightest reason to suppose that if the beam is lifted up and let down again, its new position of equilibrium will coincide with the old. And again, the beam, especially with such loads as the attracted masses, is put into a state of considerable strain, and continues to alter its shape sensibly for hours, and probably, even days, after the masses are put on to it. I have, therefore, always left the beam free for at least two or three days before commencing work with the balance, and it has of course remained free during the course of each day's work. The balance room was never entered just before any weighing, as it took many hours for the disturbance due to entrance and interference with the case to die away.

When the turn table supporting the attracting masses is moved half round, from one stop to the other, the bulk of the attraction is taken away from one attracted mass and put on to the other. The balance, being free, is slightly tilted over to the side on which is the larger attracting mass. But the deflection in the apparatus as arranged is so very small—at the most only 10 scale divisions—that errors of reading can only be neutralised by making a great number of successive measures.

Probably other errors are also largely eliminated, such as those due to the deposition of dust particles, shaking, change of ground level, and varying air currents. Of such errors I have found those due to varying air currents by far the worst. Sometimes—especially in autumn and winter—the balance will move quite irregularly through more than a scale division, and continue to move to and fro in this way for days or weeks. When in such an unsteady condition it is useless for accurate work. In spring and summer, however, it is much more steady as a rule, and frequently the scale can hardly be seen to move. I have never worked when on looking into the telescope for some time the irregular movements appeared to be more than a fraction of a tenth, *i.e.*, a fraction of one of the diagonal divisions, though, doubtless, irregularities comparable with a tenth of a whole division have often made their appearance in the work. It is perhaps not safe to ascribe these always to air currents.

I have always found the air steadiest in warm quiet weather, with a slowly rising temperature in the balance room, and most unsteady after a sudden fall of temperature. As the alteration of temperature spreads downwards, this is fully in accord with Lord Rayleigh's observation that when the air is steady the ceiling is warmer than the floor, and that when it is unsteady the floor is the warmer of the two. In the observing room I had a gas stove often kept burning day and night, in the hope that the higher temperature it produced in the ceiling of the balance room below might steady the air. But the vertical walls of the balance room interfered with the action of the ceiling, and often produced unsteadiness.

A door opening or shutting anywhere in the building had a visible though transient effect, doubtless through an air wave. In a high wind the balance was always unsteady, partly, I suspect, through rushes of air into and out of the case with sudden pressure changes, and partly through changes of ground level, with variations of wind pressure against the building.

At all times there was a march in one direction or the other of the centre of swing. This was especially marked soon after the frame was lowered and the beam left free. As already remarked, readings were not taken till changes due to change in strain of the beam had subsided. But the march was very appreciable at other times, as will be seen from the diagrams. Perhaps the change was sometimes due to tilting of the ground, with barometric variation, since the balance was a very delicate level, and sometimes due to the change in buoyancy of the air affecting the two sides unequally, though I have not been able to make out any direct connection between barometric height and position of centre of swing. I believe that the explanation is to be sought for the most part in unsymmetrical effect on the beam of slight changes of temperature, for I have frequently noticed that a rising temperature produced an upward march, and a falling one a downward march. This explanation is supported by the following table of observations of the centre of swing, extending from May 9 to May 22, 1890, the balance being free, and the balance room undisturbed meanwhile.

The relation between temperature and centre of swing is represented in Diagram IX. (Plate 19.)

Date, 1890.	Time.	Centre of swing.	Temperature.		Barometer.
			Balance room.	Observing room.	
May 9 . .	11.5 A.M.	136·0	12°·0	13°·4	739·8
	12.55 P.M.	133·0	12·0	15·0	739·2
„ 12 . .	11.15 A.M.	133·8	12·05	14·5	738·6
	1.15 P.M.	131·9	12·05	15·8	738·3
	2.40 P.M.	133·7	12·05	16·6	738·1
	Stove left on all night of 12th–13th.				
„ 13 . .	11.0 A.M.	181·7	12·6	17·5	740·2
	12.35 P.M.	181·5	12·6	18·4	740·3
	3.15 P.M.	185·0	12·7	18·6	740·3
	Stove turned off.				
	5.25 P.M.	189·4	12·7	16·5	740·5
„ 14 . .	11.20 A.M.	167·4	12·6	14·3	745·8
	1.10 P.M.	165·5	12·6	14·4	746·0
„ 15 . .	11.5 A.M.	156·8	12·4	13·8	749·5
	2.45 P.M.	160·0	12·4	14·0	749·3
„ 16 . .	1.25 P.M.	158·3	12·4	14·0	744·3
„ 17 . .	8.50 P.M.	171·0	12·55	13·7	741·9
„ 19 . .	10.30 A.M.	174·3	12·55	14·0	743·5
	6.5 P.M.	181·8	12·6	14·0	741·0
„ 20 . .	11.30 A.M.	175·2	12·75	14·0	739·8
	1.5 P.M.	173·7	12·75	14·1	740·0
	5.20 P.M.	172·3	12·7	13·8	741·7
„ 21 . .	11.10 A.M.	172·7	12·6	13·7	751·9
„ 22 . .	11.30 A.M.	192 about	12·85	14·1	757·6

Of course, after a change in the position of the attracting masses or of the riders, the balance does not at once settle in a new position of equilibrium, but oscillates about it. Inasmuch as the balance never rests in this position, it is better to term it the centre of swing rather than the equilibrium position or resting point. The dashpot used to damp the vibrations of the mirror reflecting the scale serves also to damp those of the balance beam, and they die down rapidly. Instead of waiting, however, to observe directly the point on which they are closing in, it is much more exact, and also saves much time, to find the centre of swing as with an undamped balance from the extremities of the swings. I have always observed and recorded four extremities of three successive swings, occupying in all a little more than a minute.

Notwithstanding the very considerable damping, the successive lengths of swing are still in geometrical progression, but the rate of reduction is too great to allow the ordinary approximation, in which the geometrical is assumed to be an arithmetical progression. The exact method of determining the centre of swing is as follows :—

Let a, b, c, d be four successive readings of extremities of swing, and let x be the reading of the required centre.

Let the constant ratio of each swing length to the next be λ.

Then

$$a - x = \lambda(x - b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$x - b = \lambda(c - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$c - x = \lambda(x - a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Eliminating λ from (1) and (2), we may readily obtain x in the form

$$x = c - \frac{(c-b)^2}{(a-b) + (c-b)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

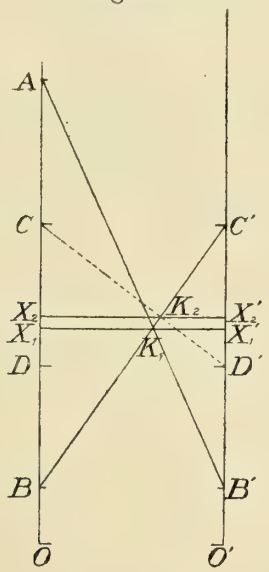
and from (2) and (3)

$$x = b + \frac{(c-b)^2}{(c-b) + (c-d)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

With no disturbances and no errors of reading the values of x in (4) and (5) will coincide; but usually there is some small difference, the result of error or disturbance, and it is better to find both and take the mean. A third value might be obtained from (1) and (3); but it appears unadvisable to combine directly observations so far separated in time.

These formulæ lend themselves to easy arithmetical treatment, especially with the aid of a slide rule; but the following graphic method of finding the centre of swing is much less tiring and quite sufficiently exact.

Fig. 13.



Let the line OA, fig. 13, represent the scale; O its zero, and A, B, C, D the points distant respectively a, b, c, d from O.

Let O′C′ be a parallel line, B′, C′, D′ being points opposite to B, C, D respectively. Let AB′ and BC′ intersect in K_1. Draw $X_1K_1X_1'$ perpendicular to OA. Then X_1 is the centre of swing given by equation (4). For

$$\frac{AX_1}{X_1B} = \frac{AX_1}{X_1'B'} = \frac{K_1X_1}{K_1X_1'} = \frac{X_1B}{X_1'C'} = \frac{X_1B}{X_1C},$$

i.e., X_1 is the point dividing AB and BC in the same ratio. Similarly if BC′ and CD′ intersect in K_2, and $X_2K_2X_2'$ be drawn perpendicular to OA, X_2 is the point given by equation (5).

The third point given by equations (1) and (3) is obtained from the intersection of AB′ and CD′, but evidently a small error in C or D′ may considerably alter the position of this point, and it is better not to use it.

The construction was carried out thus: a large opal glass plate, 10″ × 11″, was etched with cross lines 10 to the inch, so as to present the appearance of ordinary section paper. The glaze was taken off so that pencil marks could be made. A diagonal line ran at 45° across the plate through the corners of the inch squares, and this was always taken as the line BC′ in the figure. Taking any convenient horizontal line, usually, of course, far below the plate, as zero, each inch represented a scale division, each tenth a diagonal division. The values of b and c fixed the lines to be taken as OA, O′C, and on these were marked the points A, C, B′, D′. A long glass slip, with a straight scratch on it, was then laid across from A to B′ so that the scratch passed through A and B′, and its intersection K_1 with the diagonal BC′, was x_1 from the zero line. The slip was then laid with the scratch passing through C and D′, and its intersection K_2 with BC′ gave x_2. It will be observed that all the actual construction for a set of readings of the balance swings consisted in marking four points on the plate.

The following cases, the first of very regular, the second of very disturbed swing, will serve to compare the results by this exact method with those obtained from the ordinary arithmetic mean method. At the same time they will show how nearly constant is the ratio of swing decrease.

Date and Number, 1890.	Scale readings in diagonal divisions.	Length of swing.	Ratio of each to preceding.	Centre of swing, exact.	Centre of swing, approximate $\frac{a+2b+c}{4}$.
May 4, No. 3	865 939 896 921	74 43 25	·581 ·581	911·8 911·8	909·75 913·0
				Mean 911·8	Mean 911·375
Sept. 17, No. 45	1118 1053 1093 1068	65 40 25	·615 ·625	1077·8 1078·6	1079·25 1076·75
				Mean 1078·2	Mean 1078·0

In finding the attraction the observations were always made in the same order, the determination of the scale value of rider and attraction being sandwiched so that each might be equally affected by any comparatively slow changes. Starting with the initial position, the attracting masses and riders were so arranged that, on moving either, the balance was deflected in the same direction and over the same part of the scale.

The following was the order of proceeding always observed, the column headed "Centre of Swing" being supposed to contain the values of the position in each case determined from four swing extremities as just explained :—

	Centre of swing.
(1) Initial position	i_1
(2) Riders moved	r_1
(3) Riders moved back to initial position . .	i_2
(4) Masses moved round	m_1
(5) Masses moved back to initial position . .	i_3
(6) Riders moved	r_2
(7) Riders moved back to initial position . .	i_4
(8) Masses moved round	m_2
and so on.	

To minimise the effect of progressive changes these observations were always combined in threes in the following way. Denoting the scale value of rider by R, and of attraction by M :—

$$\text{From (1), (2), (3) . . . } \quad R_1 = r_1 - \frac{i_1 + i_2}{2},$$

$$\text{,, (3), (4), (5) . . . } \quad M_1 = m_1 - \frac{i_2 + i_3}{2},$$

$$\text{,, (5), (6), (7) . . . } \quad R_2 = r_2 - \frac{i_3 + i_4}{2},$$

and so on.

These again were combined in threes, so that (the notation being continued) the successive values of attraction/rider are

$$\frac{2M_1}{R_1 + R_2}, \quad \frac{M_1 + M_2}{2R_2}, \quad \frac{2M_2}{R_2 + R_3}, \text{ and so on.}$$

The successive centres of swing i_1, r_1, i_2, m_1, &c., correspond to instants of time following each other at intervals of about 2 minutes, rather more than 1 minute being taken up in making and recording the four readings for each, and the rest in making the change of position in rider or mass and waiting for the next readings. It will be seen that each value of M or R is based on three successive centres of swing, the weighings extending over about 6 minutes, while each value of M/R is based on seven successive centres of swing determined in about 14 minutes.

A series of readings was usually continued for about 2 or 3 hours. The temperature in both observing and balance room was read at the beginning and end of the series, and the barometric height was also observed. As soon as possible after the desired number of determinations was completed with the attracted masses in one of the two positions, the vertical distances between attracting and attracted masses were measured by the cathetometer in the manner explained in Table II., and the position of the attracted masses was altered.

A full account of all the weighings is given in Table III., and the results are represented in Diagrams I.–VI. The three upper rows of points in each diagram represent the centres of swing, those in the initial position being marked • . After movement of the rider they are marked ×, and after movement of the masses they are marked ○. The base lines for the different rows are altered to save space, as described on the diagrams, for on the scale adopted the rider series would always be about 10 inches above or below the initial series. In Diagram I. the rider and mass series are also brought down and superposed on the initial series, so that each of the three has the same average height. It will be seen that all three are affected by the same disturbances. The advantage of the short time of swing and the mode of combining the results in threes will be realised more easily from this superposition.

The base line may be regarded as a time scale, as the instants corresponding to successive centres of swing were almost exactly equidistant.

In each case, under the representation of the centres of swing, are plotted the resulting values of M/R, and at the side will be found a representation of the distribution of results about the mean.

Assuming that each day's mean value is correct, and that the differences for different days are to be set down to variation of distance, &c., we can find the distribution of all the values about the mean by simply superposing the marginal curves at the side of the figures. The result fairly shows the accuracy as far as the weighing alone is concerned. It is represented in Diagram VII., where A is the

mean value of the attraction in the lower, and a that in the upper position. A and a are brought near together to save space, but really they should be 40 inches apart. It will be seen that the range is about 2 per cent. of $A - a$ on each side of the mean, or taking the value of $A - a$ in milligrammes weight as about $\frac{1}{3}$ milligrm., and the load on each side as 20 kilogrms., the range is about $1/3 \times 10^9$ of this load on each side of the mean.

A comparison of the values of M/R in Diagrams I. and II., shows a very curious similarity in the fluctuations, and at first I was inclined to think there was some common external disturbance producing these fluctuations. But an analysis of the two sets of values appeared to show that the resemblance is merely accidental. When the values of M and R are set out separately, it is seen that the fluctuations depend chiefly on M, of which the fluctuations are slightly like each other for the two series, while those of R are quite different, but such that they make the fluctuations in M/R resemble each other much more closely than those in M alone. Further, it is not easy to see how fluctuations due to some external source would affect the values of M equally in the upper and lower positions and not have any effect on R. Some periodic change of level might be suspected, but this ought certainly to be traced in R. I have examined all the other diagrams and plotted out the component values of M and R, but have found no trace of resemblance, so that I think the curious likeness in I. and II. must be set down to accident.

There is a curious step by step descent of the centre of swing in the initial position on September 23, Diagram VI., which I cannot explain. It may be due to some error in the method of finding the centre of swing which comes in with a rapid march of that centre. The effect on the result is probably only small, for the value of M/R obtained with a march in the reverse direction on September 25 is very nearly the same, the two values being

September 23 . . . . . ·2112753.
,, 25 . . . . . ·2112533.

The following is a list of the weighings recorded, with the distances measured and the mean values of the attraction :—

SET I.

Date. 1890.	Position of attracted masses.	No. of values of M/R.	Mean values of M/R.	D or d in centims.	H or h in centims.
Feb. 4.	Upper . . .	50	·2142212	62·318	61·416
April 30 and May 4 .	Lower . . .	100	1·0109685	31·783	30·824
May 25	Upper . . .	50	·2157379	62·308	61·373

Set II.

Date. 1890.	Position of attracted masses.	No. of values of M/R.	Mean values of M/R.	D or d in centims.	H or h. in centims.
July 28	Lower . . .	25	·9973168	32·106	30·965
Sept. 17	Lower . . .	25	·9984148	32·116	30·954
Sept. 23 and 25 . .	Upper . . .	52	·2112647	62·708	61·566

On the completion of Set I. the four masses were inverted, and changed over from right to left or left to right, and the initial position was after this always arranged so that movement of rider or mass decreased the reading. This was done in order to lessen errors due to want of symmetry. If reversal had no effect, Set II. should, with the increased distance recorded above, give a value of M/R in the lower position of about ·990, instead of ·998. The larger value actually found is no doubt chiefly due to a want of symmetry in the large attracting mass M. The effect of this want of symmetry will be discussed after the investigation of the mathematical formula, and an account will be given of an independent method of detecting it. I think there is still outstanding a small difference, due, perhaps, to want of symmetry in the turn-table or in the attracted masses. The result of the reversal shows how necessary it was to make it. I should have liked to have in Set II. as many determinations as in Set I., so that the mean should be based on values of equal weight. During June and July, 1890, a complete set of 100 in each position, upper and lower, was made; but, owing to the pressure of other work, I was unable to calculate the results till the completion of the set. I then found that the value of M/R was still more than in Set II., and, on plotting out the results, it appeared that occasionally the rider value fell very considerably, and in an irregular way. On examination, there was little doubt that the rider came in contact occasionally with the suspending frame, when it was raised and should have been clear from it. Very likely temperature changes had brought about a displacement of the lever apparatus. Comparison with Set I. seemed to show that during that set no such contact had taken place, for there was no comparable irregularity. As it appeared dangerous to attempt to disentangle the good from the bad, the set of June and July was rejected, and Set II. was taken as recorded. When I had made the weighings giving 50 and 52 values in the two positions respectively, the balance became so irregular, through the cooler weather, that it was useless to continue work. Rather than carry over the experiment into another season, when it might be necessary to repeat the whole of the work, I have preferred to take Set II. as it stands, and give it the same weight as Set I. The final results are calculated from the means of Sets I. and II., as explained hereafter. I may here state the results obtained :—

Constant of attraction . . $G = \frac{6{\cdot}6984}{10^8}$.

Mean density of the Earth . $\Delta = 5{\cdot}4934$.

General Remarks on the Method.

Comparing the common balance with the torsion balance, there is no doubt that the former labours under the great disadvantage that the disturbances due to air currents are greatest in the vertical direction, that of the displacement to be measured. But even with this disadvantage the common balance may, I believe, be made to do much more than has hitherto been supposed possible. As an instrument in itself, apart from the external disturbances of air-currents, dust, &c., I believe its accuracy would be far beyond anything approached when these external disturbances are, as they always are, present to interfere with its action. I have always found that every precaution to ward off air currents and external disturbance has been accompanied by a corresponding increase in steadiness; and I have seen no sign of a limit of accuracy depending on the instrument itself.

Besides the protection from air currents, there are two conditions essential above all others for accurate work:—

1st. That during any set of weighings in which the deflections are to be compared with each other, the beam should be supported on its knife-edge, and should be under constant strain.

2nd. That all moving parts, such as apparatus for changing riders or weights, should be supported quite independently of the balance or its case.

With regard to the first condition, it seems impossible to make the supporting frame move so truly and with so little disturbance that the knife edge shall return exactly to the same line. Even were it possible, the beam after raising and lowering would be practically a different beam, for, as my observations show, the condition of strain changes considerably after the load is first put on, and it would be merely a chance coincidence if the mean state of strain were the same during successive weighings. I have, in my former paper ('Proceedings of the Royal Society,' No. 190, 1878), described one method of comparing weights of nearly equal value with the beam throughout on its knife-edge and equally strained,* and I should now only modify that method in having regard to the second condition, of which I have since realised the importance when working with the large balance and with increased optical sensitiveness. It is surprising to find how much disturbance is produced by having the moving parts of the apparatus connected with the balance or its case.

As to air currents there is no doubt that, as Professor Boys has shown, the greater

* I am glad that Dr. Thiesen urges the importance of this condition ('Travaux et Mémoires du Bureau International des Poids et Mesures,' vol. 5, "Études sur la Balance").

the apparatus the greater the errors produced by them. At the time my apparatus was designed I did not know this, and there seemed to be a great advantage in making it large, as riders could be used of weight large enough to be measured accurately. Were I about to start with a new design I should certainly go towards the other extreme and make the apparatus small, attempting to get over the rider difficulty by some such method as that explained on p. 582. For not only is a smaller apparatus kept more easily at a uniform temperature, and, therefore, freer from the source of air currents, but it is much more handy to adjust, and even if the adjustments are not more accurate they will at least take much less time to make.

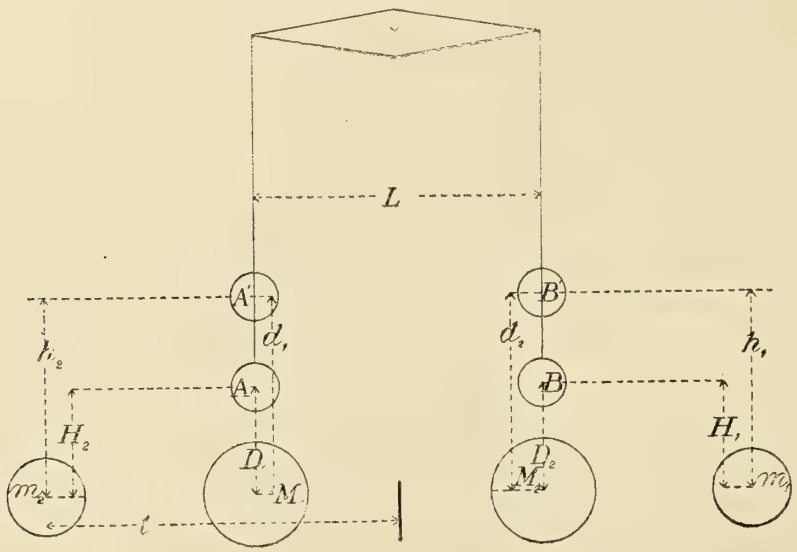
At the same time it is only fair to say, on behalf of the large apparatus, that some errors have been magnified on a like scale till they have become observable, and so could be investigated and eliminated. Starting with a small apparatus they would probably never have been detected, and would, therefore, have appeared in the final result.

II. Mathematical Investigation.

The Value of the Attraction Expressed in Terms of the Masses and Distances, and the Investigation of the Effect of Want of Symmetry in the Masses.

Let us suppose that initially the attracting masses are in the positions M_1, m_1, fig. 14, the larger on the left, the smaller on the right, and that the attracted masses

Fig. 14.



are in the lower position A, B. When the turntable is moved round so that the positions of the masses are M_2, m_2, the greater attraction is taken from the left and put on to the right. Let the centre of swing of the balance alter by an amount corresponding to a total change of vertical pull of n dynes. Assuming that a

spherical mass M attracts another spherical mass M′ when their centres are D centimetres apart with a force of GMM'/D^2 dynes, we can express the change of vertical pull due to the change of position of the masses as G × a function F of the masses and distances. There is also a change of pull on the suspending rods and the balance beam which we may denote by E.

Then

$$n = GF + E.$$

In order to eliminate E let the attracted masses be moved into their upper positions A′, B′, and let the change on moving round the attracting masses be n' dynes. If f is the function of the masses and new distances corresponding to F,

$$n' = Gf + E.$$

Subtracting

$$n - n' = G(F - f),$$

whence

$$G = \frac{n - n'}{F - f},$$

and knowing G, the mean density of the earth may be at once found in the manner shown later.

We have then to find the form of the functions F, f, and as a preliminary step it is necessary to find the effect of the holes bored through the attracted masses A, B. This may be made to take the form of a correcting factor to the attraction which would be exercised on them if they were spheres.

The piece bored out in each case has radius ·31 centim. This we denote by c. It may be taken as practically a cylinder with plane ends and length equal to 15·8 centims., the diameter $2r$ of the spheres. The intensity due to such a cylinder of mass μ at D from its centre is (Todhunter's ‘An. Stat.,’ Ed. 5, p. 292),

$$G\mu \frac{2r - \sqrt{\{(D + r)^2 + c^2\}} + \sqrt{\{(D - r)^2 + c^2\}}}{c^2 r},$$

which equals, to a sufficient approximation,

$$\frac{G\mu}{D^2 - r^2}.$$

If the mass remaining after μ is removed is A, and if the centre of the mass M is D below that of A, the attraction of M on A is

$$\frac{GM(A + \mu)}{D^2} - \frac{GM\mu}{D^2 - r^2}$$

$$= \frac{GMA}{D^2} + GM\mu\left(\frac{1}{D^2} - \frac{1}{D^2 - r^2}\right)$$

$$= \frac{GMA}{D^2}\left\{1 - \frac{\mu}{A}\left(\frac{r^2}{D^2} + \text{higher powers of } \frac{r^2}{D^2}\right)\right\}.$$

Now

$$\frac{\mu}{A} = \frac{2\pi c^2 r}{\frac{4}{3}\pi r^3 - 2\pi c^2 r} = \frac{3}{2}\frac{c^2}{r^2} \text{ nearly} = \tfrac{3}{2}\left(\tfrac{31}{790}\right)^2 = \cdot 00231,$$

and the greatest value of

$$\frac{r^2}{D^2} = \left(\tfrac{79}{320}\right)^2 = \cdot 061.$$

Then the higher powers may be neglected, and the attraction may be written

$$\frac{GMA}{D^2}\left(1 - \tfrac{3}{2}\frac{c^2}{D^2}\right) = \frac{GMA}{D^2}(1 - \theta), \text{ say.}$$

When A and B are in the lower position, D = 32, and $1 - \theta = \cdot 99986$. When they are in the upper position, D = 62 and $1 - \theta = \cdot 99996$, a value so near 1 that we shall in this position omit the correction, since it is only applied to one-fourth of the final result.

In the cross attractions we shall also omit the correction.

Referring to fig. 14 let the vertical differences of level between the centres of the various spheres be denoted as follows, the suffixes to M and m denoting their first and second positions respectively :—

$$\begin{aligned} A - M_1 &= D_1 & \qquad B - M_1 &= D_1', \\ B - M_2 &= D_2 & \qquad A - M_2 &= D_2', \\ B - m_1 &= H_1 & \qquad A - m_1 &= H_1', \\ A - m_2 &= H_2 & \qquad B - m_2 &= H_2'. \end{aligned}$$

When the masses A, B are placed in their upper positions, let the corresponding distances be denoted by small letters.

Let the horizontal distance between the centres of A and B be L, being within sensible limits equal to that between the centres of M in its two positions, and to the length of the beam, and let the radius of the circle in which m moves be l.

Then we have the following horizontal distances :—

$$\begin{aligned} A - M_2 &= B - M_1 = L, \\ A - m_1 &= B - m_2 = l + \tfrac{1}{2}L, \\ A - m_2 &= B - m_1 = l - \tfrac{1}{2}L. \end{aligned}$$

We may now write the change in vertical pull on the left by the motion of M from left to right, and of m from right to left, as follows—the first four terms representing the vertical attractions on A and B by M and m in their first position, the next four

their attractions when moved round, and the last term E representing the change in attraction on the beam and suspending rods :—

$$G\left\{\frac{MA(1-\theta)}{D_1^2}-\frac{MBD_1'}{(D_1'^2+L^2)^{\frac{3}{2}}}-\frac{mBH_1}{\left\{H_1^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}+\frac{mAH_1'}{\left\{H_1'^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}\right.$$
$$\left.+\frac{MB(1-\theta)}{D_2^2}-\frac{MAD_2'}{(D_2'^2+L^2)^{\frac{3}{2}}}-\frac{mAH_2}{\left\{H_2^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}+\frac{mBH_2'}{\left\{H_2'^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}\right\}+E.$$

We may arrange all but the last term in nearly equal pairs.

Thus the first and fifth go together, and if we put $D_1+D_2=2D$ and $D_1+\delta=D_2-\delta=D$, their sum is

$$GM(1-\theta)\left(\frac{A}{D_1^2}+\frac{B}{D_2^2}\right)$$
$$=GM(1-\theta)\left\{\frac{A}{D^2}\left(1+\frac{2\delta}{D}+\frac{3\delta^2}{D^2}+\ldots\right)+\frac{B}{D^2}\left(1-\frac{2\delta}{D}+\frac{3\delta^2}{D^2}-\ldots\right)\right\}$$
$$=GM(1-\theta)\frac{A+B}{D^2}\left\{\left(1+\frac{3\delta^2}{D^2}+\text{higher powers of }\frac{\delta^2}{D^2}\right)+\frac{A-B}{A+B}\cdot\frac{\delta}{D}\left(2+\frac{4\delta^2}{D^2}+\ldots\right)\right\}.$$

Now $(\delta/D)^2$ is negligible, as will be seen by reference to the table of distances, p. 617, and $(A-B)/(A+B)$ is less than $\frac{4}{3000}$, or of the same order as δ/D.

To a sufficiently close approximation then the sum of the two terms is

$$\frac{GM(A+B)(1-\theta)}{D^2}.$$

The second and sixth terms may also be taken together, and putting

$$D_1'+D_2'=2D' \quad \text{and} \quad D_1'+\delta'=D_2'-\delta'=D',$$

we may show that to a sufficient approximation

$$GM\left\{\frac{BD_1'}{(D_1'^2+L^2)^{\frac{3}{2}}}+\frac{AD_2'}{(D_2'^2+L^2)^{\frac{3}{2}}}\right\}=\frac{GM(A+B)D'}{(D'^2+L^2)^{\frac{3}{2}}}.$$

The two pairs with m give similar results with

$$H=\tfrac{1}{2}(H_1+H_2) \quad \text{and} \quad H'=\tfrac{1}{2}(H_1'+H_2').$$

Now

$$2D=D_1+D_2=A-M_1+B-M_2$$
$$=B-M_1+A-M_2=D_1'+D_2'=2D',$$

and similarly $2H = 2H'$, so that we may put the expression in the form

$$G\left\{\frac{M(A+B)(1-\theta)}{D^2} - \frac{M(A+B)D}{(D^2+L^2)^{\frac{3}{2}}} - \frac{m(A+B)H}{\left\{H^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}} + \frac{m(A+B)H}{\left\{H^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}\right\}$$
$$+ E = GF + E \text{ say.}$$

It is evident that we may combine experiments at different distances on different occasions in the same way by taking D and H to represent the mean values of these distances, so long as there is only a small variation from the mean.

If the attracted masses are now moved into their upper positions the expression for the change in attraction may be at once deduced from that in the lower position by replacing D and H by d and h, and omitting the factor $1 - \theta$. Let it be denoted by $Gf + E$.

Subtracting one expression from the other E is eliminated, and we have

$$G(F-f) = G\left\{\frac{M(A+B)(1-\theta)}{D^2} - \frac{M(A+B)D}{(D^2+L^2)^{\frac{3}{2}}} - \frac{m(A+B)H}{\left\{H^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}} + \frac{m(A+B)H}{\left\{H^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}\right.$$
$$\left. - \frac{M(A+B)}{d^2} + \frac{M(A+B)d}{(d^2+L^2)^{\frac{3}{2}}} + \frac{m(A+B)h}{\left\{h^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}} - \frac{m(A+B)h}{\left\{h^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}\right\}.$$

This is to be equated to the difference in the values of the change in attraction in the two positions, as determined by the rider.

Let

b = the length of the small rider beam.
w = the mass of each rider.
A = mass deflection ÷ rider deflection in lower position.
a = ,, ,, ,, ,, ,, upper ,,
g_B = acceleration of gravity, or dynes weight per unit mass at Birmingham.

Then

$$G(F-f) = \frac{(A-a)\,bwg_B}{\frac{1}{2}L}.$$

Whence we may find the gravitation constant

$$G = \frac{2bwg_B(A-a)}{L(F-f)},$$

where all the quantities on the right hand are given in the Tables at the end.

The value of g_B may be found sufficiently nearly from the formula (EVERETT'S 'Units,' p. 21); $g_B = 980{\cdot}6056 - 2{\cdot}5028 \cos 2\lambda - {\cdot}000003h$, where λ is the latitude $= 52° 28'$ at Birmingham, and h is the height above sea-level, which may be taken as 450 feet, or 13,725 centims. Whence $g_B = 981{\cdot}21$.

Since all the operations are conducted in air, the effective masses should throughout be less by the mass of air each displaces. But since they all have nearly the same densities, and w and A + B appear respectively in numerator and denominator, it is sufficient to take their true masses, and to correct for air displaced in the case of M and m only.

To obtain the mean density of the earth Δ, we must express the acceleration of gravity in terms of G and the mass and dimensions of the earth.

The ordinary formula (PRATT, 'Figure of the Earth,' 4th ed., p. 119) is based on the assumption that the earth is a spheroid. It is sufficiently correct for our purpose, the departure of the assumed spheroid from the actual shape being very small. Adding a term $-3 \times 10^{-9}h$, or approximately, -41×10^{-6}, since the balance room is taken as 13,725 centims. above sea-level (see above), the value of gravity at Birmingham may be written

$$g_B = \frac{GV\Delta}{a^2}\left\{1 + \frac{\epsilon}{3} - \frac{3}{2}m + \left(\frac{5}{2}m - \epsilon\right)\sin^2 52° 28' - 41 \times 10^{-6}\right\},$$

where

V = volume of the earth $= 1{\cdot}0832 \times 10^{27}$ (EVERETT'S 'Units,' p. 57).
a = mean radius of the earth $= 6{\cdot}3709 \times 10^8$ (*loc. cit.*).
Δ = mean density of the earth.
m = equatorial "centrifugal force" ÷ gravity $= \frac{1}{289}$.
ϵ = ellipticity of the earth $= \frac{1}{282}$.

The value of the ellipticity is taken to make the formula agree with that quoted above from EVERETT'S 'Units.' The uncertainty in the value is quite unimportant, for were ϵ as low as $\frac{1}{295}$, the error in Δ, introduced by taking it as $\frac{1}{282}$, would be less than 1 in 50,000.

Substituting for G, the value of the mean density of the earth is

$$\Delta = \frac{a^2 L(F - f)}{2bwV\left\{1 + \frac{\epsilon}{3} - \frac{3}{2}m + \left(\frac{5}{2}m - \epsilon\right)\sin^2 52° 28' - 41 \times 10^{-6}\right\}\left(A - a\right)}.$$

Here, as in the value of G, w and A + B may have their true values, M and m their values less the mass of air displaced.

In the foregoing investigation we have supposed that all the masses are homogeneous and spherical, with the exception of the borings through A and B. We have supposed, also, that the turntable is exactly symmetrical about a vertical plane

through its axis, so that its motion through two right angles is without effect. Doubtless, these suppositions and the formula based on them are not quite true. But, if we invert all the masses and change their sides, or pervert the whole arrangement of them, on taking the mean of the results obtained in the original and inverted and perverted positions we ought to greatly reduce the errors. Indeed, those due to want of symmetry in the turntable should evidently be quite eliminated, and those due to want of homogeneity in the masses should certainly be lessened.

To show this, we shall calculate the effect of a spherical " blow-hole," or gas cavity in M, in the first and most important term of F. This we shall take as being

$$\frac{GM(A+B)}{D^2},$$

on the supposition that M is homogeneous and spherical.

If the mass of metal which would fill the blowhole is λ, supposing it to be placed there, the sphere is completed and its attraction is

$$\frac{G(M+\lambda)(A+B)}{D^2};$$

but the vertical attraction is less than this in reality by the vertical component of the attraction of λ.

Let

B be the centre of the cavity,
P the centre of the attracted mass,
O the centre of the attracting mass,
δ the distance of B from the centre of M,
θ the angle BOP.

The vertical component of the attraction of λ is

$$\frac{G\lambda(A+B)\cos BPO}{PB^2},$$

but

$$BP^2 = D^2 + \delta^2 - 2D\delta\cos\theta,$$

and

$$\cos BPO = \frac{D - \delta\cos\theta}{BP},$$

whence the attraction of λ may be put

$$\frac{G\lambda(A+B)(D-\delta\cos\theta)}{(D^2+\delta^2-2D\delta\cos\theta)^{\frac{3}{2}}} = -G\lambda(A+B)\frac{d}{dD}\frac{1}{\sqrt{D^2+\delta^2-2D\delta\cos\theta}},$$

$$= \frac{G\lambda(A+B)}{D^2}\left(1 + 2P_1\frac{\delta}{D} + 3P_2\frac{\delta^2}{D^2} + \ldots\right),$$

where P_1, P_2 . . . are zonal harmonics. The attraction of the sphere with the cavity is therefore

$$\frac{GM(A+B)}{D^2}\left\{1 - \frac{\lambda}{M}\left(2P_1\frac{\delta}{D} + 3P_2\frac{\delta^2}{D^2} + 4P_3\frac{\delta^3}{D^3} + \ldots\right)\right\}.$$

If the mass is inverted, the vertical component is obtained by changing the sign of δ, and the mean of the two values is

$$\frac{GM(A+B)}{D^2}\left\{1-\frac{\lambda}{M}\left(3P_2\frac{\delta^2}{D^2}+5P_4\frac{\delta^4}{D^4}+\ldots\right)\right\},$$

the first power of δ/D being eliminated.

If $\theta = 0$, P_2 and all the other harmonics $= 1$.

If $\theta = 90°$, $P_2 = -\frac{1}{2}$, $P_4 = \frac{3}{8}$, &c.

Now, with the actual dimensions of the apparatus, $(\delta/D)^2$ cannot be so great as $(\frac{1}{2})^2$ or $\frac{1}{4}$, and may, of course, be much smaller. The first term of those involving λ, therefore, is the most important, and it lies between $+\frac{3}{2}(\lambda/M)(\delta^2/D^2)$ and $-3(\lambda/M)(\delta^2/D^2)$ changing sign for the value of θ given by $P_2 = 0$.

The second set of experiments recorded in this paper was taken after inversion and change of side of all the masses, and the final result obtained from this set differs by a little more than 1 per cent. from that obtained from the first set, the observed attraction being slightly greater at the same distance. The difference may be due to irregularities in any or all of the masses and in the turntable, and to other undetected effects, such as change of level on rotating the turntable. It would be a very long task to disentangle these, and I have contented myself with trying to find how much must be set down to irregularity in the large mass M, by taking a set of weighings with it alone inverted.

After the weighings on July 28, and the subsequent measures of distances, M was inverted only, and the other masses remained as in Set II. Some weeks later on, September 14, 25 values of $M/R = A$ were obtained, the mean being ·9926. The distances were $D = 32{\cdot}118$, $H = 30{\cdot}978$. The mass M was then put in its original position, as in Set II., and on September 17, on referring to the tables, it will be seen that the value of M/R obtained was ·9984, the distances being $D = 32{\cdot}117$ and $H = 30{\cdot}955$, practically the same as on September 14.

Assuming that the difference in attraction is due to cavities in various places, and that, for each, the term $3P_2\delta^2/D^2$ is negligible, we have, approximately,

$$\frac{1-2\frac{\Sigma\lambda P_1\delta}{MD}}{1+2\frac{\Sigma\lambda P_1\delta}{MD}}=\frac{9926}{9984}.$$

Whence, approximately, since $D = 32$,

$$\frac{\Sigma\lambda P_1\delta}{M} = {\cdot}0464 \text{ centim.}$$

This result may be tested by independent experiment. For, let the centre of gravity be x below the horizontal plane through the point bisecting the vertical

diameter (*i.e.*, the centre of figure), in the position of September 14. The distance of any missing particle λ from the horizontal plane is $\delta \cos\theta = P_1\delta$. Completing the sphere by the addition of all such particles, the centre of gravity is brought to the centre of figure, so that we have

$$Mx = \Sigma\lambda P_1\delta,$$

and

$$x = \frac{\Sigma\lambda P_1\delta}{D}.$$

We have, therefore, to determine the vertical distance of the centre of gravity from the centre of figure.

In order to do this, a large flat-bottomed scale-pan (one belonging to the balance used in the gravitation experiment) was suspended by two parallel wires about 8 centims. apart and 3 metres long. In the middle of the pan was a shallow cup about 7·5 centims. internal diameter, arranged so that it could turn freely but truly about a vertical axis. The mass, M, was placed on this cup with the diameter, which had been vertical, arranged horizontal, and perpendicular to the plane of the suspending wires. A vertical flat plate, worked by a horizontal micrometer screw, could be brought just in contact with the end of the diameter, and the reading of the micrometer gave the position of the point of contact. The position of the scale-pan was determined by a plumb line hanging over one edge in front of a horizontal scale. On turning the cup and mass through 180°, and repeating the readings, knowing the weight of the scale-pan and the position of its centre of gravity, x could at once be found.

Two separate experiments gave

$$x = \cdot0536 \text{ centim.},$$

and

$$x = \cdot0516 \text{ centim.},$$

not very different from the value ·0464, obtained from the attraction experiments. The agreement is, I think, very close when it is noted that a difference in the attraction in one of the sets of weighings of 1 in 1000 would make x either ·038 or ·054.

This result appears to justify the rejection of all terms in the expansion above the first, and so supports the belief that the reversal largely eliminates errors due to irregularity of shape. For it is in the case of M that there is the greatest danger of large value for δ/D, and the above experiments seem to indicate that even in this case it is small.

It is, perhaps, noteworthy that the largest term rejected in the attraction of M, viz., $3\lambda P_2\delta^2/MD^2$ is, if we give P_2 its maximum value 1,

$$\frac{3\lambda\delta}{MD}\cdot\frac{\delta}{D} = \frac{3x}{D}\cdot\frac{\delta}{D},$$

which is not greater than

$$\frac{3 \times \cdot 0464}{32} \times \frac{15}{32} = \cdot 0020,$$

since the radius of the mass is 15.

This is in a term about 5/4 of the final result, so that the greatest error which can be introduced by neglecting this term is ·0025, or 1 in 400.

In calculating the results of the experiments the means of Sets I. and II. have been taken. Equal weights have been given to each set. It would have been more satisfactory if the number of experiments had been the same in each set; but I should have had to wait for another season to obtain more, and then it would, probably, have been necessary to repeat the whole series in both arrangements, as it is not safe to assume that the various disturbing causes remain the same over a wide interval of time. The second set, though fewer in number, are, in some respects, I believe, better; partly owing to the additional experience gained when they were taken.

In order that the various terms in $F - f$ may be compared, I give below their numerical values, as determined from the values of the masses and distances given in the tables. The meaning of each term in the first column will be seen on referring to fig. 14. The second column contains the actual values; the third column the values in terms of the fourth, the lowest term.

	Value of $F - f$.	
$\frac{M(A+B)(1-\theta)}{D^2}$	$= 6483938 \cdot 8$	416
$-\frac{M(A+B)D}{(D^2+L^2)^{\frac{3}{2}}}$	$- 102416 \cdot 3$	6·6
$-\frac{m(A+B)H}{\left\{H^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}$	$- 316243 \cdot 3$	20
$+\frac{m(A+B)H}{\left\{H^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}$	$+ 15579 \cdot 9$	1
$-\frac{M(A+B)}{d^2}$	$- 1693687 \cdot 2$	109
$+\frac{M(A+B)d}{(d^2+L^2)^{\frac{3}{2}}}$	$+ 156728 \cdot 0$	10
$+\frac{m(A+B)h}{\left\{h^2+\left(l-\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}$	$+ 310695 \cdot 0$	20
$-\frac{m(A+B)h}{\left\{h^2+\left(l+\frac{L}{2}\right)^2\right\}^{\frac{3}{2}}}$	$- 27597 \cdot 7$	1·7

Whence $F - f = 4826997 \cdot 2$.

The mean value of $A - a$ (see Table III.), is

$$A - a = \cdot 791295;$$

substituting these values of $F - f$ and $A - a$ in the formula for G (p. 606), we obtain

$$G = \frac{6 \cdot 6984}{10^8};$$

substituting them in the formula for Δ we obtain

$$\Delta = 5 \cdot 4934.$$

The values given by Sets I. and II., treated separately, are to two figures of decimals,

Set I. $\Delta = 5 \cdot 52$

Set II. $\Delta = 5 \cdot 46$.

III.—Tables.

Table I.—Constants of the Apparatus and Dimensions of the Earth.

Masses.

	grms.
Attracting mass M, in vacuo	= 153407·26
Less air displaced, say	= 153388·85
Attracting mass m, in vacuo	= 76497·4
Less air displaced, say	= 76488·2
Attracted mass A, in vacuo	= 21582·33
,, ,, B, ,,	= 21566·21
,, ,, A + B, in vacuo	= 43148·54
Riders each, in vacuo	= 0·010119

Vertical Diameters of Masses in terms of Cathetometer Scale correct at 18°.

The masses are taken as having the same coefficient of expansion as the scale.

	centims.
M	= 30·526
m	= 24·176
A	= 15·8203
B	= 15·7829.

The diameters of the masses A and B are taken between the nuts securing them on the suspending wires.

	centims.
Balance beam at 0°, L	= 123·232
Rider beam at 0°, b	= 2·53575
L/b (as occurring explicitly in G and Δ, independent of temperature, assuming them to have the same coefficient of expansion)	= 48·59775

Latitude of Birmingham	= 52° 28′
Height of balance room above sea-level	= 13725 centims.
Gravity at Birmingham, g_B	= 981·21 centims./sec.^{2}
Mean radius of earth	= $6{\cdot}3709 \times 10^8$ centims.
Volume of earth	= $1{\cdot}0832 \times 10^{27}$ cub. centims.
Equatorial "centrifugal force"/gravity	= $\frac{1}{289}$
Ellipticity of Earth	= $\frac{1}{289}$

$$1 + \frac{\epsilon}{3} - \frac{3m}{2} + \left(\frac{5}{2}m - 2\right)\sin^2 52^\circ\,28' - 41 \times 10^{-6} = {\cdot}999161.$$

TABLE II.—Vertical and Horizontal Distances.

Vertical Diameters of Masses taken by the Cathetometer, described p. 588.

In the tables below P.S. signifies divisions on the scale over which moves the pointer, which is attached to the small adjustment plate. V.S. signifies divisions on the vertical millimetre scale.

DIAMETER of Large Attracting Mass M.

	Reading on pointer scale.	Mean.
Upper telescope sighting top of mass . .	73·2, 73·4, 73·2	73·27 P.S.
Lower „ „ bottom „ . .	23·0, 23·0, 23·2	23·07 P.S.

TURNING round to the Vertical Scale.

	Reading on pointer scale.	Mean.
Upper telescope sighting 459 millims. V.S.	94·6, 94·9, 94·4, 95·0, 94·0	94·58 P.S.
„ „ „ 458 „	68·8, 68·8, 69·7, 69·0, 68·7, 70·0, 69·6, 69·4	69·25 P.S.

Therefore 25·33 P.S. divisions = 1 millim. V.S.,

$$\text{and scale reading for top of mass} = 458 + \frac{73{\cdot}27 - 69{\cdot}25}{25{\cdot}33}$$

$$= 458{\cdot}158 \text{ millims. V.S.}$$

	Reading on pointer scale.	Mean.
Lower telescope sighting 153 millims. V.S.	27·3, 27·4, 27·7, 27·0	27·35 P.S.
„ „ „ 152 „	0·0, 0·3, − 0·5, 0·0	− 0·07 P.S.

Therefore 27·42 P.S. divisions = 1 millim. V.S.,

$$\text{and scale reading for bottom of mass} = 152 + \frac{23{\cdot}07 + 0{\cdot}07}{27{\cdot}42}$$

$$= 152{\cdot}844 \text{ millims. V.S.}$$

The difference = 30·5314 centims.

This is rather greater than the diameter of the mass, as the cross wire was made to touch the image of the mass in each case. A series of measures of 1 millim. on the

scale, in which the cross wire was on the centre of each division, and of 1 millim. between the jaws of a wire gauge, in which the wire touched the images of the jaws, showed that at the distance at which the scale was, ·005 centim. must be subtracted, leaving

Diameter of M = 30·526 centims.

VERTICAL Diameter of Small Attracting Mass *m*.

	Reading on pointer scale.	Mean.
Upper telescope sighting top of mass . .	75·6, 75·6, 75·0	75·40 P.S.
Lower „ „ bottom „ . .	26·5, 26·3, 26·8	26·53 P.S.

TURNING round to the Vertical Scale.

	Reading on pointer scale.	Mean.
Upper telescope sighting 388 millims. V.S.	100, 99·9, 99·7	99·87 P.S.
„ „ „ 387 „	73·9, 73·4, 74·0	73·77 P.S.

Therefore 26·10 P.S. divisions = 1 millim. V.S.,

and scale reading for top of mass $= 387 + \frac{75{\cdot}40 - 73{\cdot}77}{26{\cdot}10}$

= 387·062 millims. V.S.

	Reading on pointer scale.	Mean.
Lower telescope sighting 146 millims. V.S.	45·9, 45·9, 45·0	45·60 P.S.
„ „ „ 145 „	20·4, 19·4, 20·0	19·93 P.S.

Therefore 25·77 P.S. divisions = 1 millim. V.S.,

and scale reading for bottom of mass $= 145 + \frac{26{\cdot}53 - 19{\cdot}93}{25{\cdot}77}$

= 145·256 millims. V.S.

The difference = 24·1806 centims.

Subtracting the same correction as in the last case for the cross wire,

Diameter of *m* = 24·176 centims.

Vertical Diameters of Attracted Masses A and B taken between the Junctions of the Securing Nuts with the Sphere.

A.

	Reading on pointer scale.	Mean.
Upper telescope sighting top of mass . .	82·7, 83·0, 82·9	82·9 P.S.
Lower „ „ bottom „ . .	31·0, 31·5, 31·3	31·3 P.S.

Turning round to the Vertical Scale.

	Reading on pointer scale.	Mean.
Upper telescope sighting 429 millims. V.S.	95·0, 95·2, 95·3	95·3 P.S.
„ „ „ 428 „	69·0, 69·0, 68·9	69·0 P.S.

Therefore 26·3 P.S. divisions = 1 millim. V.S.,

and scale reading for top of mass $= 428 + \frac{82{\cdot}9 - 69{\cdot}0}{26{\cdot}3}$

$= 428{\cdot}531$ millims. V.S.

	Reading on pointer scale.	Mean.
Lower telescope sighting 271 millims. V.S.	48·0, 47·8, 48·4	48·1 P.S.
„ „ „ 270 „	23·4, 23·0, 22·8	23·1 P.S.

Therefore 25·0 P.S. divisions = 1 millim. V.S.,

and scale reading for bottom of mass $= 270 + \frac{31{\cdot}3 - 23{\cdot}1}{25{\cdot}0}$

$= 270{\cdot}328$ millims. V.S.

The difference gives the diameter since the middle of the cross wire was used, so that

Diameter of A = 15·8203 centims.

B.

	Reading on pointer scale.	Mean.
Upper telescope sighting top of mass . .	72·0, 71·0, 71·0	71·3 P.S.
Lower „ „ bottom „ . .	24·6, 25·0, 25·2	24·9 P.S.

Turning round to the Vertical Scale.

	Reading on pointer scale.	Mean.
Upper telescope sighting 430 millims. V.S.	94·0, 94·6, 94·0	94·2 P.S.
„ „ „ 429 „	68·0, 68·1, 68·1	68·1 P.S.

Therefore 26·1 P.S. divisions = 1 millim. V.S.,

and scale reading for top of mass $= 429 + \frac{71{\cdot}3 - 68{\cdot}1}{26{\cdot}1}$

$= 429{\cdot}123$ millims. V.S.

	Reading on pointer scale.	Mean.
Lower telescope sighting 272 millims. V.S.	43·3, 43·0, 43·0	43·1 P.S.
„ „ „ 271 „	17·1, 17·3, 17·4	17·3 P.S.

Therefore 25·8 P.S. divisions = 1 millim. V.S.,

and scale reading for bottom of mass $= 271 + \frac{24{\cdot}9 - 17{\cdot}3}{25{\cdot}8}$

$= 271{\cdot}294$ millims. V.S.

And diameter of B = 15·7829 centims.

Vertical Distances between the Levels of the Centres of the Attracting and Attracted Masses Measured by Cathetometer.

The measurements were made as soon as possible after the completion of a set of weighings, usually on the following day.

It was necessary to fix the attracted masses in the position occupied during the weighings, and with the beam of the balance in the same strained condition. This was done in some cases by gripping the left suspending wire by a pair of jaws; in others, by adding a small weight to one side, and placing a block of the right thickness under the mass on that side.

The cathetometer was placed in front of the left side of the balance case, from which position all the masses could be viewed by turning the telescope round the central pillar (fig. 2). It was read when sighting the top of each attracting mass and the top of each attracted mass when in the lower position, the bottom of each when in the upper position, the top and bottom being taken at the junctions of the securing nuts with the masses. It is therefore necessary to add to the distances

measured by the cathetometer the difference of the radii of attracting and attracted masses in the lower position, and their sum in the upper position (see p. 613). The work is shown in full for February 5 and May 5.

Tests were made at various times, showing that there was no change in the distances (at least within errors of reading), either through moving the turn-table or in the course of a few days (see February 5 and May 5 for examples).

Temperature Correction.—The cathetometer scale is taken as correct at 18°, and its coefficient of expansion is assumed to be 1/60000. That of the masses is probably about 1/40000, but, for simplicity, is taken as equal to that of the scale, the difference, 1/120000, never amounting to as much as the errors of reading, since the greatest length concerned is 23 centims.

The temperature was estimated to be about 1° above that observed during the immediately preceding weighings, the presence of the observer and the lights used tending to raise it.

The cathetometer rested always on the brick floor of the room. Its vernier reads to ·002 centim.

Set I.

Attracted masses A on the left, B on the right. Attracting mass M moving round from left to right in front of the balance case.

February 5, 1890.—Attracted masses in upper position. Assumed temperature 11°.

Half way through the measurements the cathetometer was accidentally moved, and could not be exactly replaced. Repeating the reading of A it was found that ·197 must be added to the previous readings to compare with the following ones. This addition is made where the numbers have an asterisk.

	A. 64·999* 65·001*	B. 65·284 65·282	
m_2 23·448	M_1 25·895* 25·889*	M_2 26·070 26·064	m_1 23·947*
Differences : A − m_2 = 41·552	A − M_1 = 39·108	B − M_2 = 39·216	B − m_1 = 41·336

From Table I., p. 613, the sums of the radii of the masses are

$$R_M + R_A = 23{\cdot}173,$$
$$R_M + R_B = 23{\cdot}154,$$
$$R_m + R_A = 19{\cdot}998,$$
$$R_m + R_B = 19{\cdot}979,$$

whence

$$d = \tfrac{1}{2}\{39{\cdot}108 + 23{\cdot}173 + 39{\cdot}216 + 23{\cdot}154\}$$
$$= 62{\cdot}326,$$

and

$$h = \tfrac{1}{2}\{41{\cdot}336 + 19{\cdot}979 + 41{\cdot}552 + 19{\cdot}998\}$$
$$= 61{\cdot}433.$$

These are in terms of a scale correct at 18°, so that the value is too great by about 7/60000. We take as true values

Corrected

$$d = 62{\cdot}318,$$
$$h = 61{\cdot}425.$$

Test experiment.—At the conclusion, the distance A − M_1 was measured again and found to be 39·110.

May 28, 1890.—Attracted masses in upper position. Assumed temperature 14°.

	A.	B.	
	64·674 64·674	65·286 65·288	
m_2	M_1	M_2	m_1
23·422 23·424	25·726 25·724	25·920 25·920	23·766 23·756
Differences: A − m_2 = 41·552	A − M_1 = 38·949	B − M_2 = 39·367	B − m_1 = 41·526

whence

$$d = 62{\cdot}312,$$
$$h = 61{\cdot}377.$$

Subtracting temperature correction ·004,

Corrected

$$d = 62{\cdot}308,$$
$$h = 61{\cdot}373.$$

Mean values in Set I.,

$$d = 62{\cdot}313,$$
$$h = 61{\cdot}399.$$

May 5, 1890.—Attracted masses in lower position. Assumed temperature 13°.

	A.	B.	
	50·324	50·622	
	50·324	50·634	
	50·328	50·630	
m_2	M_1	M_2	m_1
23·672	25·972	26·138	23·998
23·674	25·970	26·138	24·008
	25·972	26·132	
Differences: A − m_2 = 26·652	A − M_1 = 24·354	B − M_2 = 24·493	B − m_1 = 26·626

From Table I., p. 613,

$$R_M - R_A = 7{\cdot}353,$$
$$R_M - R_B = 7{\cdot}372,$$
$$R_m - R_A = 4{\cdot}178,$$
$$R_m - R_B = 4{\cdot}197,$$

whence

$$D = \tfrac{1}{2}\{24{\cdot}354 + 7{\cdot}353 + 24{\cdot}493 + 7{\cdot}372\}$$
$$= 31{\cdot}786,$$

and

$$H = \tfrac{1}{2}\{26{\cdot}626 + 4{\cdot}197 + 26{\cdot}652 + 4{\cdot}178\}$$
$$= 30{\cdot}827.$$

Subtracting temperature correction ·0025,

Corrected values for Set I.,

$$D = 31{\cdot}783,$$
$$H = 30{\cdot}824.$$

Test Experiment.—The balance was set free at the end of these measures, and two days later, on May 7, it was again fixed, and the distance D was determined by the cathetometer described on p. 588. The value obtained was D = 31·786.

Note.—If the apparatus were perfectly rigid and constant in its dimensions we should expect D − H = $d - h$ = constant. The values actually given by the above experiments are

February 5 . . . . . ·892,
May 5 . . . . . ·959,
May 28 . . . . . ·935.

There is apparently a slight increase during the course of the spring, probably due to the warping of the wood supporting the mass m. But there was some uncertainty in sighting the top of the mass m, especially when in the distant position on the right.

Set II.

Attracted masses A on the right, B on the left. Attracting mass M moving round from left to right behind the balance case. All the masses inverted.

July 29, 1890.—Attracted masses in lower position. Assumed temperature 16°.

	B.	A.	
	49·014 49·014	49·846 49·844	
m_1	M_2	M_1	m_2
22·434 22·436	24·584 24·586	24·788 24·782	22·868 22·864
Differences: B − m_1 = 26·579	B − M_2 = 24·429	A − M_1 = 25·060	A − m_2 = 26·979

whence

$$D = 32{\cdot}107,$$
$$H = 30{\cdot}967.$$

Subtracting temperature correction ·001,

Corrected

$$D = 32{\cdot}106,$$
$$H = 30{\cdot}966.$$

September 18, 1890.—Attracted masses in lower position. Assumed temperature 16°.

	B.	A.	
	49·076 49·074	49·768 49·766	
m_1	M_2	M_1	m_2
22·467	24·576 24·576	24·756 24·758	22·840
Differences: B − m_1 = 26·608	B − M_2 = 24·499	A − M_1 = 25·010	A − m_2 = 26·927

whence

$$D = 32{\cdot}117,$$
$$H = 30{\cdot}955.$$

Subtracting temperature correction ·001,

Corrected

$$D = 32{\cdot}116,$$
$$H = 30{\cdot}954.$$

Mean values in Set II.,

$$D = 32{\cdot}111,$$
$$H = 30{\cdot}960.$$

September 27, 1890.—Attracted masses in upper position. Assumed temperature 16°.

	B.	A.	
	63·880 63·876	64·540 64·544	
m_1 22·450 22·448	M_2 24·570 24·572	M_1 24·756 24·758	m_2 22·810 22·816
Differences: B − m_1 = 41·429	B − M_2 = 39·307	A − M_1 = 39·785	A − m_2 = 41·729

whence

$$d = 62{\cdot}710,$$
$$h = 61{\cdot}568.$$

Subtracting temperature correction ·002,

Corrected values for Set II.,

$$d = 62{\cdot}708,$$
$$h = 61{\cdot}566.$$

Note.—The values of D − H and $d - h$, which should be constant, are from the above, and from another set of measures (not here recorded, see p. 609) on September 15, as follows. (We have no reason to expect the same value as in Set I., as the masses M, m, have changed sides.)

July 29	1·140,
September 15	1·140,
September 18	1·162,
September 27	1·142.

From July 29 to September 15 inclusive, the balance was swinging freely without alteration. The values of H should, therefore, be the same on those dates. They were

July 29 . . . . . 30·967,
September 15 . . . . . 30·978,

equal almost within errors of reading for the top of m.

Means of Sets I. and II. :—

$$D = \tfrac{1}{2}(31{\cdot}783 + 32{\cdot}111)$$
$$= 31{\cdot}947.$$
$$H = \tfrac{1}{2}(30{\cdot}824 + 30{\cdot}960)$$
$$= 30{\cdot}892.$$
$$d = \tfrac{1}{2}(62{\cdot}313 + 62{\cdot}708)$$
$$= 62{\cdot}511.$$
$$h = \tfrac{1}{2}(61{\cdot}399 + 61{\cdot}566)$$
$$= 61{\cdot}483.$$

Horizontal Distances.

Set I.

$$L = 123{\cdot}269 \text{ centims.}$$

At 18°

$$l_1 = 122{\cdot}915 \quad ,,$$

and

$$\frac{L}{2} = 61{\cdot}635 \quad ,,$$

whence

$$l_1 + \frac{L}{2} = 184{\cdot}550 \quad ,,$$

$$l_1 - \frac{L}{2} = 61{\cdot}280 \quad ,,$$

Taking the mean temperature of the Set as 12°, and assuming 1/60000 as the coefficient of expansion, on correcting to 12°,

$$l_1 + \frac{L}{2} = 184{\cdot}532 \text{ centims.}$$

$$l_1 - \frac{L}{2} = 61{\cdot}274 \quad ,,$$

SET II.

At 18°

$$l_2 = 122{\cdot}795 \text{ centims.}$$
$$\frac{L}{2} = 61{\cdot}635 \quad ,,$$

Whence

$$l_2 + \frac{L}{2} = 184{\cdot}430 \quad ,,$$
$$l_2 - \frac{L}{2} = 61{\cdot}160 \quad ,,$$

Taking the mean temperature of the Set as 15°, and correcting to 15°,

$$l_2 + \frac{L}{2} = 184{\cdot}421 \text{ centims.}$$
$$l_2 - \frac{L}{2} = 61{\cdot}157 \quad ,,$$

Mean values for the two Sets

$$L = 123{\cdot}260 \quad ,,$$
$$l + \frac{L}{2} = 184{\cdot}477 \quad ,,$$
$$l - \frac{L}{2} = 61{\cdot}216 \quad ,,$$

TABLE III.—DETERMINATION OF ATTRACTION BY THE BALANCE.

Determinations of the Attraction in terms of the Riders by the Balance.

In each case four turning points of three successive swings are recorded in tenths of a division, *i.e.*, in divisions on the diagonal lines. In the columns headed *i* the masses and riders are in the initial position, in those headed *r* the riders are moved, and in those headed *m* the masses are moved. Under each set of four readings is the calculated centre of swing (see p. 595). In the next line are the deflections due to movements of riders and masses, each placed under the middle one of the three centres of swing from which it is calculated. In the next line are the values of deflection due to mass ÷ deflection due to rider, or M/R (see p. 598).

SET I.

I. ATTRACTED Masses in Upper Position. Feb. 4, 1890, 7.59 P.M. to 10.49 P.M. Temperature: in Observing Room, 15°·7–16°·5; in Balance Room, 10°·05. Barometer 752·2–752·0 millims. Weather mild and still, after slight frost on the two previous nights. Time between successive passages of centre about 20 seconds.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	725 798 759 780	912 838 878 856	725 799 759 781	804 787 796 791	764 779 771 776	913 838 879 857	726 800 759 781	804 787 797 791
Centre of swing . . .	772·55	863·85	773·00	792·80	773·90	864·60	773·40	793·20
Deflection due to rider or mass	..	91·075	..	19·350	..	90·950	..	19·700
Mass deflection ÷ rider deflection	..	..	..	·212608	..	·214688	..	·217110

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	763 779 771 775	913 837 879 857	724 801 759 782	804 787 796 792	764 779 771 776	914 838 880 857	725 801 760 783	805 789 796 792
Centre of swing . . .	773·60	864·25	773·85	793·05	773·90	865·05	774·50	793·65
Deflection due to rider or mass	..	90·525	..	19·175	..	90·850	..	18·950
Mass deflection ÷ rider deflection	..	·214720	..	·211440	..	·209824	..	·208758

Table III. (continued)

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	765	916	726	803	765	914	725	805
	780	839	803	788	779	838	800	787
	772	881	762	797	771	879	759	796
	777	859	784	792	776	857	782	791
Centre of swing . . .	774·90	866·30	776·30	793·65	774·00	864·50	773·60	792·85
Deflection due to rider or mass	..	90·700	..	18·500	..	90·700	..	19·225
Mass deflection ÷ rider deflection	..	·206174	..	·203966	..	·207966	..	·211438

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	763	914	727	805	764	913	725	804
	780	838	800	789	779	838	800	786
	770	880	760	797	771	879	760	796
	776	857	782	792	775	857	781	791
Centre of swing . . .	773·65	865·05	774·15	793·90	773·65	864·65	773·80	792·50
Deflection due to rider or mass	..	91·150	..	20·000	..	90·925	..	19·275
Mass deflection ÷ rider deflection	..	·215167	..	·219690	..	·215975	..	·211494

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	763	913	724	803	763	912	726	802
	778	838	801	789	778	837	799	787
	770	879	759	796	769	879	759	797
	774	857	782	792	774	857	780	792
Centre of swing . . .	772·65	864·60	773·85	793·50	772·30	864·20	772·95	793·30
Deflection due to rider or mass	..	91·350	..	20·425	..	91·575	..	19·875
Mass deflection ÷ rider deflection	..	·217296	..	·223316	..	·220038	..	·216503

TABLE III. (continued).

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings . . .	764 779 771 776	915 839 880 858	725 800 759 780	803 786 795 791	763 778 770 774	913 838 879 857	726 799 759 781	803 787 796 792
Centre of swing . . .	773·90	86·555	773·15	792·25	772·70	864·60	773·15	792·95
Deflection due to rider or mass	..	92·025	..	19·325	..	91·675	..	19·200
Mass deflection ÷ rider deflection	..	·212986	..	·210397	..	·210117	..	·209693

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)	*m.* (56)
Scale readings	764 779 772 776	914 839 880 858	724 802 759 781	803 787 795 790	762 778 769 774	912 836 877 855	721 798 755 779	801 785 794 788
Centre of swing . . .	774·35	865·55	773·85	792·10	772·20	862·55	770·35	790·55
Deflection due to rider or mass	..	91·450	..	19·075	..	91·275	..	20·275
Mass deflection ÷ rider deflection	..	·209267	..	·208784	..	·215557	..	·222161

	i. (57)	*r.* (58)	*i.* (59)	*m.* (60)	*i.* (61)	*r.* (62)	*i.* (63)*	*i.* (63*a*)	*m.* (64)
Scale readings .	760 776 767 772	911 836 877 855	724 799 758 779	803 785 795 789	762 777 769 773	911 836 877 855	722 800 758 780	725 799 759 780	803 786 796 790
Centre of swing	770·20	862·50	772·20	791·35	771·70	862·50	772·50	772·90	792·30
Deflection due to rider or mass . . .	..	91·250	..	19·425	..	90·400	..	..	19·900
Mass deflection ÷ rider deflection . .	..	·217534	..	·213873	..	·217506	..	..	·217873

*After 63 the riders were moved by mistake instead of the masses, therefore it was necessary to return to the initial position, and take the readings in (63*a*).

Table III. (continued).

	i. (65)	*r.* (66)	*i.* (67)	*m.* (68)	*i.* (69)	*r.* (70)	*i.* (71)	*m.* (72)
Scale readings	762	913	725	802	759	909	722	802
	777	838	800	785	776	834	797	784
	769	879	758	794	767	875	757	794
	774	857	780	789	772	854	779	789
Centre of swing . . .	771·90	864·60	772·75	790·80	770·15	860·80	771·05	790·55
Deflection due to rider or mass	..	92·275	..	19·350	..	90·200	..	19·450
Mass deflection ÷ rider deflection	..	·213170	..	·212084	..	·215078	..	·214947

	i. (73)	*r.* (74)	*i.* (75)	*m.* (76)	*i.* (77)	*r.* (78)	*i.* (79)	*m.* (80)
Scale readings	760	911	724	803	762	911	721	800
	777	835	798	785	777	835	797	784
	768	877	757	795	769	877	756	793
	773	854	779	790	773	854	778	788
Centre of swing . . .	771·15	862·05	771·40	791·50	771·70	862·05	770·30	789·75
Deflection due to rider or mass	..	90·775	..	19·950	..	91·050	..	20·150
Mass deflection ÷ rider deflection	..	·217020	..	·219442	..	·220209	..	·221064

	i. (81)	*r.* (82)	*i.* (83)	*m.* (84)	*i.* (85)	*r.* (86)	*i.* (87)	*m.* (88)
Scale readings	759	910	722	801	759	910	723	802
	774	833	796	783	775	834	797	783
	766	876	757	793	767	876	757	793
	771	854	778	787	771	854	778	787
Centre of swing . . .	768·90	860·95	770·50	789·30	769·60	861·25	770·90	789·35
Deflection due to rider or mass	..	91·250	..	19·250	..	91·000	..	19·300
Mass deflection ÷ rider deflection	..	·215990	..	·211248	..	·211813	..	·212995

TABLE III. (continued).

	i. (89)	*r.* (90)	*i.* (91)	*m.* (92)	*i.* (93)	*r.* (94)	*i.* (95)	*m.* (96)
Scale readings	759 775 766 771	908 831 874 852	719 795 754 776	800 783 792 788	760 775 767 772	910 835 876 854	723 798 756 779	800 785 793 789
Centre of swing . . .	769·20	859·00	768·35	789·05	769·90	861·60	770·90	790·30
Deflection due to rider or mass	..	90·225	..	19·925	..	91·200	..	19·350
Mass deflection ÷ rider deflection	..	·217373	..	·219650	..	·215323	..	·212462

	i. (97)	*r.* (98)	*i.* (99)	*m.* (100)	*i.* (101)	*r.* (102)	*i.* (103)	*m.* (104)	*i.* (105)
Scale readings .	761 777 768 772	910 835 876 853	721 796 756 777	798 783 791 787	759 775 765 770	909 833 874 852	721 796 756 777	800 783 793 787	759 775 767 771
Centre of swing	771·00	861·35	769·80	788·35	768·60	859·65	769·80	789·35	769·60
Deflection due to rider or mass . . .	..	90·950	..	19·150	..	90·450	..	19·650	
Mass deflection ÷ rider deflection . .	..	·211655	..	·211136	..	·214483			

Feb. 4, 1890. Mean of 50 determinations of $M/R = a$ } ·21422122.
Attracted masses in upper position }

TABLE III. (continued).

II.—ATTRACTED Masses in Lower Position. April 30, 1890, 7.45 P.M. to 10.32 P.M. Temperature: in Observing Room, 17°–16°·1; in Balance Room, 11°·1; Barometer, 748·6–749·2 millims. Weather clear; S.E. wind; sunny during day. Time between successive passages of centre not quite 20 seconds.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	1046	1133	951	1127	955	1134	952	1123
	969	1055	1024	1062	1025	1059	1028	1069
	1012	1098	984	1099	986	1102	985	1099
	988	1075	1007	1078	1007	1077	1009	1082
Centre of swing . . .	996·60	1082·85	998·35	1085·50	999·80	1086·25	1000·40	1088·25
Deflection due to rider or mass	..	85·375	..	86·425	..	86·150	..	87·350
Mass deflection ÷ rider deflection	..	..	..	1·00772	..	1·00856	..	1·01437

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	962	1136	955	1129	961	1136	956	1133
	1023	1060	1029	1067	1027	1064	1031	1069
	989	1104	987	1102	989	1105	989	1103
	1009	1079	1012	1083	1012	1081	1013	1085
Centre of swing . . .	1001·40	1088·00	1002·45	1089·55	1003·15	1090·00	1004·15	1091·25
Deflection due to rider or mass	..	86·075	..	86·750	..	86·350	..	86·350
Mass deflection ÷ rider deflection	..	1·01133	..	1·00623	..	1·00232	..	1·00101

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	967	1141	957	1135	965	1143	958	1134
	1027	1064	1034	1070	1031	1066	1036	1073
	994	1108	990	1106	994	1110	993	1108
	1012	1083	1015	1086	1015	1085	1017	1089
Centre of swing . . .	1005·65	1092·00	1006·00	1093·20	1007·40	1094·00	1008·35	1095·40
Deflection due to rider or mass	..	86·175	..	86·500	..	86·125	..	86·825
Mass deflection ÷ rider deflection . . .	..	1·00290	..	1·00406	..	1·00624	..	1·01106

TABLE III. (continued).

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	976 1027 998 1016	1143 1069 1110 1087	963 1037 996 1019	1141 1073 1112 1090	966 1037 996 1019	1145 1070 1112 1089	960 1040 995 1020	1141 1075 1113 1092
Centre of swing . . .	1008·80	1095·35	1010·65	1097·90	1010·85	1097·05	1011·15	1099·30
Deflection due to rider or mass	..	85·625	..	87·150	..	86·050	..	87·575
Mass deflection ÷ rider deflection	..	1·01591	..	1·01529	..	1·01264	..	1·01713

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	973 1034 1000 1020	1147 1071 1114 1090	962 1041 996 1022	1137 1079 1112 1094	975 1034 1002 1020	1146 1075 1114 1091	965 1042 999 1021	1138 1080 1113 1094
Centre of swing . . .	1012·30	1098·55	1012·50	1100·20	1013·40	1099·80	1014·00	1101·00
Deflection due to rider or mass	..	86·150	..	87·250	..	86·100	..	86·650
Mass deflection ÷ rider deflection	..	1·01465	..	1·01306	..	1·00987	..	1·00858

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	977 1035 1003 1022	1150 1073 1116 1093	964 1043 1000 1025	1089 1110 1098 1104	976 1038 1004 1023	1153 1074 1118 1094	968 1044 1001 1025	1148 1078 1118 1095
Centre of swing . . .	1014·70	1100·80	1015·45	1102·20	1016·15	1102·35	1016·45	1103·40
Deflection due to rider or mass	..	85·725	..	86·400	..	86·050	..	86·475
Mass deflection ÷ rider deflection	..	1·00933	..	1·00597	..	1·00450	..	1·00625

TABLE III. (continued).

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)	*m.* (56)
Scale readings	978 1039 1005 1025	1153 1076 1119 1094	969 1045 1002 1027	1149 1080 1118 1097	976 1043 1005 1026	1153 1079 1121 1096	968 1047 1004 1028	1145 1085 1118 1099
Centre of swing . . .	1017·40	1103·35	1017·65	1104·45	1018·60	1105·55	1019·25	1106·15
Deflection due to rider or mass	..	85·825	..	86·325	..	86·625	..	86·875
Mass deflection ÷ rider deflection	..	1·00670	..	1·00116	..	·99971	..	1·00973

	i. (57)	*r.* (58)	*i.* (59)	*m.* (60)	*i.* (61)	*r.* (62)	*i.* (63)	*m.* (64)
Scale readings	984 1039 1008 1026	1155 1078 1120 1097	971 1048 1004 1029	1151 1083 1122 1100	977 1046 1007 1030	1157 1081 1123 1100	972 1051 1007 1031	1152 1087 1123 1102
Centre of swing . . .	1019·30	1105·10	1020·00	1107·85	1021·25	1108·10	1022·60	1109·95
Deflection due to rider or mass	..	85·450	..	87·225	..	86·175	..	86·850
Mass deflection ÷ rider deflection	..	1·01872	..	1·01646	..	1·01011	..	1·00798

	i. (65)	*r.* (66)	*i.* (67)	*m.* (68)	*i.* (69)	*r.* (70)	*i.* (71)	*m.* (72)
Scale readings	983 1046 1011 1031	1159 1082 1125 1102	976 1051 1008 1033	1153 1088 1126 1104	983 1049 1012 1032	1161 1083 1127 1102	978 1053 1011 1034	1158 1088 1127 1106
Centre of swing . . .	1023·60	1109·80	1023·70	1112·05	1025·15	1111·05	1025·95	1113·15
Deflection due to rider or mass	..	86·150	..	87·625	..	85·500	..	86·700
Mass deflection ÷ rider deflection	..	1·01262	..	1·02097	..	1·01944	..	1·01226

TABLE III. (continued).

	i. (73)	*r.* (74)	*i.* (75)	*m.* (76)	*i.* (77)	*r.* (78)	*i.* (79)	*m.* (80)
Scale readings	985	1163	980	1156	990	1163	982	1161
	1051	1086	1056	1092	1051	1087	1057	1093
	1014	1129	1013	1129	1017	1131	1015	1132
	1033	1104	1036	1108	1036	1107	1039	1109
Centre of swing . . .	1026·95	1113·40	1028·25	1115·50	1029·20	1115·20	1030·15	1117·65
Deflection due to rider or mass	..	85·880	..	86·775	..	85·525	..	87·100
Mass deflection ÷ rider deflection	..	1·01093	..	1·01299	..	1·01652	..	1·01782

	i. (81)	*r.* (82)	*i.* (83)	*m.* (84)	*i.* (85)	*r.* (86)	*i.* (87)	*m.* (88)
Scale readings	991	1167	984	1158	992	1169	984	1161
	1054	1090	1059	1098	1056	1092	1063	1100
	1018	1133	1018	1132	1021	1136	1018	1135
	1038	1108	1041	1112	1041	1111	1042	1115
Centre of swing . . .	1030·95	1117·40	1032·60	1119·55	1033·55	1120·00	1034·00	1122·20
Deflection due to rider or mass	..	85·625	..	86·475	..	86·225	..	87·600
Mass deflection ÷ rider deflection	..	1·01358	..	1·00640	..	1·00942	..	1·01890

	i. (89)	*r.* (90)	*i.* (91)	*m.* (92)	*i.* (93)	*r.* (94)	*i.* (95)	*m.* (96)
Scale readings	996	1171	987	1165	995	1172	989	1169
	1058	1094	1064	1100	1061	1097	1066	1099
	1022	1137	1022	1137	1024	1137	1023	1140
	1043	1114	1045	1117	1045	1116	1046	1117
Centre of swing . . .	1035·20	1121·75	1036·85	1123·75	1037·35	1123·15	1038·20	1125·05
Deflection due to rider or mass	..	85·725	..	86·650	..	85·375	..	86·575
Mass deflection ÷ rider deflection	..	1·01633	..	1·01286	..	1·01450	..	1·01065

Table III. (continued).

	i. (97)	*r.* (98)	*i.* (99)	*m.* (100)	*i.* (101)	*r.* (102)	*i.* (103)	*m.* (104)	*i.* (105)
Scale readings	998 1062 1026 1045	1175 1097 1141 1116	992 1066 1025 1047	1174 1102 1141 1119	995 1064 1027 1048	1176 1098 1143 1118	995 1067 1026 1049	1169 1105 1143 1121	1001 1066 1029 1049
Centre of swing	1038·75	1125·05	1039·45	1127·15	1040·15	1126·70	1040·75	1128·90	1042·20
Deflection due to rider or mass . . .	..	85·950	..	87·350	..	86·250	..	87·425	
Mass deflection ÷ rider deflection . .	..	1·01178	..	1·01452	..	1·01319			

April 30. Mean of 50 determinations of M/R = A } 1·010905.
Attracted masses in lower position }

May 4, 1890, 11.11 to 11.50 A.M. Temperature: in Observing Room, 13°·5 to 13°·8; in Balance Room, 11°·7; Barometer, 742·0 to 741·7 millims. Weather inclined to rain; a little cooler than previous day; wind S. to S.W.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	875 936 900 920	1045 969 1013 988	865 939 896 921	1044 970 1014 989	865 938 897 920	1045 967 1013 986	861 940 894 921	1041 971 1012 989
Centre of swing . . .	913·10	996·95	911·80	997·75	911·60	996·00	910·95	997·10
Deflection due to rider or mass	..	84·500	..	86·050	..	84·725	..	86·275
Mass deflection ÷ rider deflection	..	..	..	1·01699	..	1·01697	..	1·01950

TABLE III. (continued).

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	869 936 896 919	1046 966 1012 985	862 939 894 920	1036 972 1009 988	867 936 896 918	1045 964 1011 984	860 937 893 919	1040 968 1009 986
Centre of swing . . .	910·700	995·10	910·45	995·55	910·40	993·80	909·20	994·20
Deflection due to rider or mass	..	84·525	..	85·125	..	84·000	..	85·450
Mass deflection ÷ rider deflection	..	1·01390	..	1·01024	..	1·01533	..	1·01454

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)	*i.* (25)
Scale readings	863 934 894 916	1043 964 1009 983	859 936 892 917	1033 971 1006 985	863 932 892 915	1040 963 1007 982	857 936 889 915	1035 966 1006 983	862 930 891 914
Centre of swing	908·30	992·60	908·00	993·15	906·60	991·00	906·10	991·40	905·25
Deflection due to rider or mass . . .	..	84·450	..	85·850	..	84·650	..	85·725	
Mass deflection ÷ rider deflection . .	..	1·01421	..	1·01538	..	1·01344			

May 4. Morning. Mean of 10 determinations of M/R = A } 1·015050.
Attracted masses in lower position }

TABLE III. (continued).

MAY 4, 1890.—*Same day.* 2.40 to 4.54 P.M. Temperature: in Observing Room, 13°·9–14°·1; in Balance Room, 11°·7–11°·75; Barometer, 740·3–739·7.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings . . .	847	1035	853	1031	864	1035	853	1026
	933	957	930	961	925	960	931	965
	883	1003	886	1002	890	1003	885	1001
	912	977	911	979	909	977	912	980
Centre of swing . . .	901·40	986·20	902·00	987·05	902·55	987·10	902·00	987·65
Deflection due to rider or mass	..	84·500	..	84·775	..	84·825	..	85·300
Mass deflection ÷ rider deflection	..	..	..	1·00133	..	1·00251	..	1·00783

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	866	1035	853	1023	864	1036	854	1024
	924	959	932	968	925	960	931	968
	891	1004	886	1000	889	1004	886	1000
	909	977	912	982	910	978	912	982
Centre of swing . . .	902·70	987·20	902·80	988·35	902·30	987·75	902·50	988·45
Deflection due to rider or mass	..	84·450	..	85·800	..	85·350	..	85·925
Mass deflection ÷ rider deflection	..	1·01303	..	1·01060	..	1·00601	..	1·00940

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings . . .	864	1039	855	1025	866	1040	855	1011
	925	958	931	969	925	958	932	977
	890	1005	887	1001	891	1005	888	996
	909	978	913	982	911	978	913	985
Centre of swing . . .	902·55	987·80	903·25	989·20	903·50	987·90	904·0	989·10
Deflection due to rider or mass	..	84·900	..	85·825	..	84·150	..	85·225
Mass deflection ÷ rider deflection	..	1·01148	..	1·01538	..	1·01634	..	1·01232

TABLE III. (continued).

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings . . .	864 927 890 912	1036 961 1004 979	854 933 887 914	1024 970 1001 984	865 926 892 912	1037 962 1005 980	854 934 888 915	1031 967 1004 983
Centre of swing . . .	903·75	988·10	904·00	989·85	904·35	989·25	904·90	990·55
Deflection due to rider or mass	..	84·225	..	85·675	..	84·625	..	85·625
Mass deflection ÷ rider deflection	..	1·01454	..	1·01481	..	1·01211	..	1·01182

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	864 928 892 912	1039 961 1006 980	855 934 888 914	1024 972 1002 985	864 929 891 913	1041 961 1006 980	856 934 888 915	1025 971 1002 984
Centre of swing . . .	904·95	989·50	904·80	991·10	905·00	989·65	905·00	990·65
Deflection due to rider or mass	..	84·625	..	86·200	..	84·650	..	85·500
Mass deflection ÷ rider deflection	..	1·01521	..	1·01846	..	1·01418	..	1·00796

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	866 927 893 913	1038 962 1007 981	857 934 889 915	1030 960 1004 984	865 930 893 915	1043 962 1008 981	858 934 891 915	1035 966 1006 984
Centre of swing . . .	905·30	990·40	905·50	991·30	906·60	991·15	906·55	991·55
Deflection due to rider or mass	..	85·000	..	85·250	..	84·575	..	84·95
Mass deflection ÷ rider deflection	..	1·00441	..	1·00546	..	1·00621	..	1·00741

TABLE III. (continued).

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)	*m.* (56)
Scale readings	869 927 895 914	1041 961 1008 982	858 935 891 916	1037 965 1007 984	867 929 894 914	1041 963 1007 982	856 936 890 916	1029 972 1004 986
Centre of swing . . .	906·65	990·90	907·00	991·80	906·60	991·05	906·70	992·50
Deflection due to rider or mass	..	84·075	..	85·000	..	84·440	..	85·750
Mass deflection ÷ rider deflection	..	1·01070	..	1·00905	..	1·01155	..	1·01509

	i. (57)	*r.* (58)	*i.* (59)	*m.* (60)	*i.* (61)	*r.* (62)	*i.* (63)	*m.* (64)
Scale readings	871 928 895 913	1041 963 1008 982	859 936 890 917	1035 969 1008 985	869 931 895 916	1044 963 1010 983	860 937 892 917	1030 973 1005 986
Centre of swing . . .	906·80	991·50	907·10	993·50	908·20	992·85	908·25	993·30
Deflection due to rider or mass	..	84·550	..	85·850	..	84·625	..	84·775
Mass deflection ÷ rider deflection	..	1·01478	..	1·01493	..	1·00812	..	1·00162

	i. (65)	*r.* (66)	*i.* (67)	*m.* (68)	*i.* (69)	*r.* (70)	*i.* (71)	*m.* (72)
Scale readings	863 935 894 917	1042 965 1010 984	863 935 894 917	1039 969 1008 985	840 949 886 923	1042 965 1009 985	861 937 893 918	1037 971 1009 987
Centre of swing . . .	908·80	993·45	908·80	993·75	909·15	993·25	909·05	995·05
Deflection due to rider or mass	..	84·650	..	84·775	..	84·150	..	85·750
Mass deflection ÷ rider deflection	..	1·00148	..	1·00444	..	1·01322	..	1·01449

TABLE III. (continued).

	i. (73)	*r.* (74)	*i.* (75)	*m.* (76)	*i.* (77)	*r.* (78)	*i.* (79)	*m.* (80)
Scale readings	865 935 895 918	1045 965 1011 985	860 938 893 919	1038 969 1010 987	868 934 895 918	1045 965 1012 985	863 937 894 919	1041 969 1011 988
Centre of swing . . .	909·55	994·30	909·25	995·00	909·50	994·75	909·80	995·80
Deflection due to rider or mass	..	84·900	..	85·625	..	85·100	..	85·625
Mass deflection ÷ rider deflection	..	1·00928	..	1·00735	..	1·00617	..	1·00765

	i. (81)	*r.* (82)	*i.* (83)	*m.* (84)	*i.* (85)
Scale readings	864 938 895 919	1044 967 1012 986	860 940 894 920	1036 974 1010 989	867 936 896 919
Centre of swing . . .	910·55	995·45	910·65	996·80	910·60
Deflection due to rider or mass	..	84·850	..	86·175	
Mass deflection ÷ rider deflection	..	1·01238			

May 4, afternoon. Mean of 40 determinations of M/R = A, Attracted masses in lower position } 1·0100278.

April 30 and May 4. Mean of 100 determinations of M/R = A, Attracted masses in upper position } 1·0109685.

Table III. (continued).

III.—Attracted Masses in Upper Position. May 25, 1890 ; 11.20 to 12.53 noon ; Temperature : in Observing Room, 15°·4–16° ; in Balance Room, 13°·3 ; Barometer, 748·5–748·1 millims. Weather, E. wind, warm, very bright. Time of swing not recorded.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	1071	1175	960	1049	1003	1173	956	1049
	986	1085	1047	1028	1021	1085	1047	1028
	1033	1134	998	1041	1010	1134	996	1040
	1005	1108	1025	1034	1017	1107	1024	1034
Centre of swing . . .	1015·90	1116·90	1015·50	1036·20	1014·25	1116·55	1014·25	1035·80
Deflection due to rider or mass	..	101·200	..	21·325	..	102·300	..	21·500
Mass deflection ÷ rider deflection	..	..	..	·209582	..	·209311	..	·210320

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	1001	1173	958	1049	1002	1173	957	1049
	1021	1085	1046	1028	1020	1084	1046	1028
	1011	1134	996	1038	1010	1134	995	1040
	1016	1106	1024	1033	1015	1105	1023	1032
Centre of swing . . .	1014·35	1116·35	1014·05	1034·70	1013·50	1115·80	1013·25	1035·90
Deflection due to rider or mass	..	102·150	..	20·925	..	102·425	..	22·850
Mass deflection ÷ rider deflection	..	·207660	..	·204571	..	·213688	..	·223297

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	1003	1173	956	1048	1001	1172	958	1048
	1019	1082	1043	1025	1019	1081	1042	1026
	1009	1133	994	1038	1008	1131	994	1038
	1015	1104	1023	1032	1014	1103	1021	1030
Centre of swing . . .	1012·85	1114·60	1011·90	1033·60	1012·05	1113·10	1011·35	1033·50
Deflection due to rider or mass	..	102·225	..	21·625	..	101·400	..	22·375
Mass deflection ÷ rider deflection	..	·217535	..	·212400	..	·216962	..	·220172

TABLE III. (continued).

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	1000 1017 1007 1014	1171 1081 1130 1103	953 1044 992 1021	1047 1025 1037 1031	1000 1016 1007 1013	1171 1080 1130 1103	955 1041 992 1021	1046 1025 1036 1030
Centre of swing . . .	1010·90	1112·70	1010·80	1032·90	1010·45	1112·40	1010·00	1032·15
Deflection due to rider or mass	..	101·850	..	22·275	..	102·175	..	22 050
Mass deflection ÷ rider deflection	..	·219195	..	·218356	..	·216907	..	·215885

	i. (33)	*r.* (34)	*i* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	999 1017 1006 1013	1168 1080 1131 1102	952 1043 992 1021	1046 1024 1037 1030	999 1018 1007 1012	1170 1082 1130 1102	955 1043 993 1021	1046 1026 1039 1030
Centre of swing . . .	1010·20	1112·40	1010·40	1032·35	1010·70	1112·65	1011·05	1033·75
Deflection due to rider or mass	..	102·100	..	21·800	..	101·775	..	22·100
Mass deflection ÷ rider deflection	..	·214740	..	·213857	..	·215672	..	·216858

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	998 1019 1009 1014	1171 1082 1132 1104	955 1043 994 1022	1046 1027 1038 1031	1000 1019 1009 1014	1173 1082 1131 1104	956 1046 995 1023	1048 1028 1039 1032
Centre of swing . . .	1012·25	1114·00	1011·65	1033·85	1012·35	1113·80	1013·20	1034·85
Deflection due to rider or mass	..	102·050	..	21·850	..	101·025	..	21·900
Mass deflection ÷ rider deflection	..	·215388	..	·215197	..	·216531	..	·216350

TABLE III. (continued).

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)
Scale readings	1001 1019 1009 1015	1172 1083 1132 1105	956 1046 996 1023	1048 1027 1039 1032	1000 1020 1009 1016	1173 1082 1133 1105	956 1044 995 1023
Centre of swing . . .	1012·70	1114·60	1013·65	1034·60	1013·15	1114·70	1012·65
Deflection due to rider or mass	..	101·425	..	21·200	..	101·800	
Mass deflection ÷ rider deflection	..	·212472	..	·208636			

May 25. Morning. Mean of 25 determinations of $M/R = a$ } ·21446168.
Attracted masses in upper position }

Same Day. 3.15 to 4.50 P.M. Temperature: in Observing Room, 16°·0 to 16°·25; in Balance Room, 13°·3 to 13°·35; Barometer, 747·7–747·4 millims.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	1001 1069 1031 1052	1205 1116 1165 1138	990 1077 1029 1057	1081 1061 1073 1066	1034 1055 1044 1049	1207 1120 1168 1139	991 1080 1030 1059	1083 1062 1076 1068
Centre of swing . . .	1044·55	1147·60	1046·30	1068·55	1047·60	1150·50	1048·35	1070·65
Deflection due to rider or mass	..	102·175	..	21·600	..	102·525	..	21·675
Mass deflection ÷ rider deflection	..	..	..	·211041	..	·211046	..	·212162

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	1037 1056 1046 1052	1209 1119 1169 1141	995 1082 1030 1059	1086 1066 1078 1071	1039 1058 1048 1054	1213 1121 1172 1145	994 1086 1034 1064	1088 1069 1078 1073
Centre of swing . . .	1049·60	1151·10	1049·00	1073·50	1051·60	1154·10	1052·90	1074·90
Deflection due to rider or mass	..	101·800	..	23·200	..	101·850	..	21·675
Mass deflection ÷ rider deflection	..	·220408	..	·227842	..	·220299	..	·216738

TABLE III. (continued).

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	1045 1059 1050 1056	1212 1126 1175 1147	996 1088 1037 1066	1088 1071 1080 1076	1043 1063 1053 1058	1215 1126 1177 1148	1002 1088 1039 1066	1094 1072 1083 1077
Centre of swing . . .	1053·55	1157·20	1055·30	1077·10	1056·30	1158·50	1056·55	1079·25
Deflection due to rider or mass	..	102·775	..	21·300	..	102·075	..	22·350
Mass deflection ÷ rider deflection	..	·209073	..	·207957	..	·213813	..	·219575

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	1044 1064 1053 1061	1217 1127 1178 1150	999 1093 1041 1069	1095 1073 1086 1079	1048 1067 1056 1061	1221 1131 1180 1152	1002 1094 1042 1071	1096 1076 1088 1081
Centre of swing . . .	1057·25	1159·80	1059·35	1081·35	1059·70	1162·50	1060·70	1083·55
Deflection due to rider or mass	..	101·500	..	21·825	..	102·300	..	22·800
Mass deflection ÷ rider deflection	..	·217611	..	·214181	..	·218112	..	·223147

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	1049 1067 1057 1064	1221 1131 1182 1154	1004 1095 1045 1072	1097 1079 1089 1082	1053 1069 1059 1066	1224 1134 1183 1156	1007 1096 1047 1075	1101 1079 1091 1085
Centre of swing . . .	1060·80	1163·75	1062·60	1085·20	1062·95	1165·70	1064·65	1086·90
Deflection due to rider or mass	..	102·050	..	22·425	..	101·900	..	22·000
Mass deflection ÷ rider deflection	..	·221583	..	·219907	..	·217983	..	·215898

TABLE III. (continued).

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	1054 1072 1061 1068	1226 1135 1185 1157	1009 1096 1048 1076	1101 1081 1093 1086	1054 1073 1064 1068	1225 1135 1187 1159	1008 1099 1048 1080	1102 1081 1094 1089
Centre of swing . . .	1065·15	1167·15	1065·35	1088·55	1066·85	1168·40	1067·00	1089·70
Deflection due to rider or mass	..	101·90	..	22·450	..	101·475	..	21·625
Mass deflection ÷ rider deflection	..	·218106	..	·220774	..	·217172	..	·213607

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)
Scale readings	1058 1075 1066 1072	1228 1138 1189 1161	1014 1102 1053 1080	1104 1086 1097 1091	1059 1076 1068 1074	1229 1141 1192 1163	1019 1103 1055 1081
Centre of swing . . .	1069·15	1170·80	1070·45	1093·00	1070·95	1173·40	1072·15
Deflection due to rider or mass	..	101·000	..	22·300	..	101·850	
Mass deflection ÷ rider deflection	..	·217458	..	·219867			

May 25, afternoon. Mean of 25 determinations of M/R = a } ·21701412
Attracted masses in upper position }

Mean of 50 determinations, morning and afternoon, ·2157379.

SUMMARY of Set I.

February 4 . . . . . a = ·2142212
May 25 . . . . . a = ·2157379
Mean value of . a = ·2149791

April 30 . . . . . A = 1·010905
May 4 . . . . . A = 1·011032
Mean value of . A = 1·0109685

therefore

$$A - a = ·7959894.$$

TABLE III. (continued).

SET II.

All Attracting and Attracted Masses inverted and changed over, each to the other side. The Suspending Rods also reversed and Riders interchanged. The initial position always the higher reading on the scale.

I. ATTRACTED Masses in Lower Position. July 28, 1890, 8.10 to 9.43 P.M. Temperature: in Observing Room, 17°–16°·9; in Balance Room, 15°·4; Barometer, 747·6–748 millims. Weather fine and calm; wind W.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	1099	912	1130	917	1126	914	1131	922
	1051	1007	1034	1005	1036	1008	1035	1005
	1081	951	1093	952	1091	951	1093	954
	1063	985	1057	985	1058	986	1057	984
Centre of swing . . .	1069·65	971·95	1070·55	972·10	1070·20	972·60	1070·95	973·15
Deflection due to rider or mass	..	98·150	..	98·275	..	97·975	..	97·575
Mass deflection ÷ rider deflection	..	..	..	1·00217	..	·99949	..	·99541

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	1128	913	1134	924	1130	915	1137	919
	1035	1010	1035	1006	1038	1013	1034	1012
	1092	951	1095	956	1094	953	1098	955
	1058	987	1058	987	1061	989	1060	989
Centre of swing . . .	1070·50	973·30	1072·25	975·00	1073·05	975·65	1073·80	976·40
Deflection due to rider or mass	..	98·075	..	97·650	..	97·775	..	97·700
Mass deflection ÷ rider deflection	..	·99528	..	·99719	..	99898	..	·99719

TABLE III. (continued).

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	1132 1040 1095 1062	917 1014 954 989	1140 1036 1099 1060	924 1009 958 990	1134 1042 1098 1064	916 1016 955 993	1136 1041 1098 1064	960 991 972 983
Centre of swing . . .	1074·40	976·55	1075·05	977·40	1076·90	978·20	1076·70	979·05
Deflection due to rider or mass	..	98·175	..	98·575	..	98·600	..	98·100
Mass deflection ÷ rider deflection	..	·99962	..	1·00191	..	·99734	..	·99506

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings . . .	1133 1044 1098 1065	916 1018 956 994	1142 1039 1103 1064	925 1013 960 994	1134 1045 1101 1068	918 1019 957 997	1143 1042 1103 1066	925 1018 961 996
Centre of swing . . .	1077·60	979·55	1078·65	980·30	1079·80	981·00	1080·00	982·65
Deflection due to rider or mass	..	98·575	..	98·925	..	98·900	..	97·075
Mass deflection ÷ rider deflection	..	·99937	..	1·00190	..	·99090	..	·99031

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings . . .	1136 1046 1099 1068	924 1019 961 996	1145 1042 1104 1067	930 1016 964 995	1140 1046 1104 1069	918 1022 959 997	1143 1045 1104 1068	928 1018 962 996
Centre of swing . . .	1079·45	982·95	1080·75	983·50	1082·00	982·80	1081·80	983·30
Deflection due to rider or mass	..	97·150	..	97·875	..	99·100	..	98·875
Mass deflection ÷ rider deflection	..	1·00335	..	·99745	..	·99268	..	1·00051

TABLE III. (continued).

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings . . .	1138 1047 1104 1071	926 1020 963 997	1146 1045 1107 1069	928 1022 964 998	1142 1047 1106 1071	927 1021 963 999	1144 1047 1107 1070	937 1015 967 996
Centre of swing . . .	1082·55	984·45	1083·45	985·75	1083·70	985·10	1084·05	985·10
Deflection due to rider or mass	..	98·550	..	97·825	..	98·775	..	98·900
Mass deflection ÷ rider deflection	..	·99797	..	·99151	..	·99582	..	1·00139

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)
Scale readings . . .	1140 1049 1105 1072	923 1024 962 999	1148 1045 1108 1071	932 1021 966 998	1141 1050 1106 1072	924 1024 963 1001	1144 1048 1107 1072
Centre of swing . . .	1083·95	985·40	1084·35	986·60	1084·80	986·25	1084·75
Deflection due to rider or mass	..	98·750	..	97·975	..	98·525	
Mass deflection ÷ rider deflection	..	·99684	..	·99328			

July 28, 1890. Mean of 25 determinations of M/R = A } ·9973168.
Attracted masses in lower position }

Table III. (continued).

September 17, 1890, 8.0 to 9.31 p.m. Temperature : in Observing Room, 17°–17°·5 ; in Balance Room, 15°·8. Barometer, 746·2–746·4 millims. Weather warm, cloudy.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	1085 1051 1073 1058	908 1004 945 981	1118 1029 1085 1050	921 995 949 978	1109 1036 1081 1053	905 1006 944 981	1126 1026 1087 1050	921 996 951 978
Centre of swing . . .	1064·20	967·35	106·34	966·70	1063·75	967·35	1063·95	967·90
Deflection due to rider or mass	..	96·450	..	96·875	..	96·500	..	96·450
Mass deflection ÷ rider deflection	..	..	..	1·00415	..	1·00168	..	·99613

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	1113 1034 1084 1053	907 1006 944 982	1126 1027 1088 1052	929 993 953 978	1110 1038 1083 1056	910 1007 947 984	1131 1027 1092 1052	934 993 956 979
Centre of swing . . .	1064·75	967·75	1065·05	968·40	1065·90	969·90	1067·10	970·20
Deflection due to rider or mass	..	97·150	..	97·075	..	96·600	..	96·850
Mass deflection ÷ rider deflection	..	·99601	..	1·00206	..	1·00375	..	1·00026

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	1104 1044 1081 1059	910 1008 947 985	1129 1030 1091 1054	924 1001 953 983	1116 1040 1086 1057	909 1009 947 986	1121 1036 1088 1056	927 1000 956 984
Centre of swing . . .	1067·00	970·44	1067·90	971·45	1066·70	970·85	1067·05	972·80
Deflection due to rider or mass	..	97·050	..	958·50	..	96·025	..	95·550
Mass deflection ÷ rider deflection	..	·99279	..	·99288	..	·99662	..	·99105

TABLE III. (continued).

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	1112 1043 1086 1060	914 1009 951 987	1131 1033 1093 1056	929 1002 957 984	1114 1044 1087 1061	916 1011 952 988	1132 1035 1094 1058	934 1002 960 986
Centre of swing . . .	1069·65	973·10	1070·15	974·00	1070·70	974·50	1071·70	976·00
Deflection due to rider or mass	..	96·800	..	96·425	..	96·700	..	96·550
Mass deflection ÷ rider deflection	..	99161	..	·99664	..	·99780	..	·99690

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	1097 1058 1083 1067	916 1015 954 991	1135 1037 1098 1061	942 999 965 986	1119 1048 1093 1066	919 1017 957 993	1136 1039 1099 1063	935 1006 963 989
Centre of swing . . .	1073·40	977·10	1074·80	977·85	1075·80	979·70	1076·25	979·10
Deflection due to rider or mass	..	97·000	..	97·450	..	96·325	..	97·025
Mass deflection ÷ rider deflection	..	1·00000	..	1·00815	..	1·00947	..	1·00362

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	1122 1048 1093 1065	917 1018 956 994	1141 1038 1101 1062	929 1011 962 993	1118 1053 1093 1068	921 1019 958 996	1141 1041 1103 1065	941 1009 966 993
Centre of swing . . .	1076·00	979·45	1076·95	980·65	1078·20	981·40	1079·40	982·65
Deflection due to rider or mass	..	97·025	..	96·925	..	97·400	..	96·975
Mass deflection ÷ rider deflection	..	·99948	..	·99704	..	·99538	..	·99628

TABLE III. (continued).

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)
Scale readings	1134 1047 1100 1067	920 1022 958 998	1143 1041 1104 1065	932 1016 964 996	1134 1048 1102 1068	925 1021 962 997	1140 1045 1104 1068
Centre of swing . . .	1079·85	982·65	1080·00	983·85	1081·15	984·20	1081·55
Deflection due to rider or mass	..	97·275	..	96·725	..	97·150	
Mass deflection ÷ rider or mass	..	·99563	..	·99499			

September 1890. Mean of 25 determinations of M/R = A, Attracted masses in lower position } ·9984148.

July 28 and September 17. Mean of 50 determinations of M/R = A, ·9978658.

II. ATTRACTED Masses in Upper Position. September 23, 1890, 7.52 to 9.30 P.M. Temperature: in Observing Room, 15°·3–15°·4; in Balance Room, 15°·05. Barometer, 749·8–750·2 millims. Weather, light S.W. wind and clear after heavy showers. Scale readings between about 1100 and 1300; 1000 omitted.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	307 248 285 263	113 210 151 186	329 235 293 257	235 257 243 251	281 261 273 265	112 208 149 185	326 232 290 256	233 256 241 249
Centre of swing . . .	271·00	173·10	270·95	248·25	268·35	171·45	268·25	246·60
Deflection due to rider or mass	..	97·875	..	21·400	..	96·850	..	21·175
Mass deflection ÷ rider deflection.	..	..	..	·219797	..	·219799	..	·218581

TABLE III. (continued).

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	279 260 272 264	110 207 148 183	331 228 290 253	232 255 239 248	277 258 271 262	110 205 147 182	324 229 288 252	230 254 239 247
Centre of swing . . .	267·30	170·15	266·80	245·15	265·80	168·90	265·55	244·50
Deflection due to rider or mass	..	96·900	..	21·150	..	96·775	..	20·225
Mass deflection ÷ rider deflection	..	·218395	..	·218407	..	·213769	..	·209179

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	275 256 269 261	108 204 145 181	323 228 286 251	228 253 237 247	276 255 268 260	107 203 145 179	328 224 287 249	226 252 236 245
Centre of swing . . .	263·90	167·40	264·10	243·15	263·00	166·65	263·25	241·85
Deflection due to rider or mass	..	96·600	..	20·400	..	96·475	..	20·225
Mass deflection ÷ rider deflection	..	·210274	..	·211317	..	·210547	..	·209652

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	274 254 265 258	106 199 143 176	320 224 283 246	232 245 237 241	271 252 262 255	100 197 138 175	317 221 281 243	222 247 232 241
Centre of swing . . .	260·90	163·90	260·40	239·85	258·25	160·50	257·90	237·60
Deflection due to rider or mass	..	96·750	..	19·475	..	97·575	..	19·100
Mass deflection ÷ rider deflection	..	·205323	..	·200437	..	·197668	..	·196730

TABLE III. (continued).

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	266	98	317	221	265	97	311	218
	249	197	219	244	246	193	218	242
	259	136	279	228	257	135	275	226
	254	173	241	237	251	171	240	235
Centre of swing . . .	255·50	159·10	255·90	234·20	253·05	157·00	253·30	232·10
Deflection due to rider or mass	..	96·600	..	20·275	..	96·175	..	20·150
Mass deflection ÷ rider deflection	..	·203804	..	·210351	..	·210164	..	·209271

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	264	95	310	215	261	91	311	212
	243	191	216	239	241	188	210	236
	256	133	273	223	253	128	272	220
	249	168	238	232	246	166	234	229
Centre of swing . . .	251·20	154·90	251·40	229·10	248·55	151·10	248·40	226·10
Deflection due to rider or mass	..	96·400	..	20·875	..	97·375	..	20·675
Mass deflection ÷ rider deflection	..	·212785	..	·215456	..	·213350	..	·213585

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)	*m.* (56)
Scale readings	257	90	306	211	256	88	303	208
	237	186	208	234	236	184	209	232
	250	127	269	218	248	125	265	216
	243	162	232	227	242	160	231	225
Centre of swing . . .	245·15	149·25	245·80	224·10	244·25	147·21	243·95	222·10
Deflection due to rider or mass	..	96·225	..	20·925	..	96·900	..	20·350
Mass deflection ÷ rider deflection	..	·216160	..	·216699	..	·212977	..	·209956

TABLE III. (continued).

	i. (57)	*r.* (58)	*i.* (59)	*m.* (60)	*i.* (61)
Scale readings	253 233 246 238	86 180 122 157	301 205 263 227	203 228 213 222	293 204 280* 245
Centre of swing . . .	240·95	144·00	240·95		
Deflection due to rider or mass	..	96·950			

September 23, 1890. Mean of 27 determinations of $M/R = a$ } ·2112753.
Attracted masses in upper position }

SEPTEMBER 25, 1890, 7.10–8.43 P.M. Temperature: in Observing Room, 15°–15°·2; in Balance Room, 15°. Barometer, 760·8, steady. Weather cloudy, with westerly airs. Time of swing 21 seconds. 1000 omitted in scale readings.

	i. (1)	*r.* (2)	*i.* (3)	*m.* (4)	*i.* (5)	*r.* (6)	*i.* (7)	*m.* (8)
Scale readings	246 238 243 239	84 179 121 156	301 205 263 228	206 229 215 224	248 233 243 236	82 178 119 156	297 204 260 226	202 228 212 222
Centre of swing . . .	240·90	142·90	240·95	220·40	248·95	141·60	238·95	218·10
Deflection due to rider or mass	..	98·025	..	19·550	..	97·350	..	20·850
Mass deflection ÷ rider deflection	..	..	..	·200128	..	·207499	..	·213163

* This is a considerable rise, showing either a sudden disturbance or a displacement of the apparatus; possibly the telescope was touched. The rise was maintained and therefore the observations were discontinued.

Table III. (continued).

	i. (9)	*r.* (10)	*i.* (11)	*m.* (12)	*i.* (13)	*r.* (14)	*i.* (15)	*m.* (16)
Scale readings	248 233 243 236	83 176 119 155	300 203 261 226	204 228 214 224	252 233 245 239	84 182 122 158	303 206 265 228	207 232 217 226
Centre of swing . . .	238·95	140·80	239·20	219·50	240·70	144·60	242·45	222·60
Deflection due to rider or mass	..	98·275	..	20·450	..	96·975	..	20·825
Mass deflection ÷ rider deflection	..	·210125	..	·209475	..	·212813	..	·214718

	i. (17)	*r.* (18)	*i.* (19)	*m.* (20)	*i.* (21)	*r.* (22)	*i.* (23)	*m.* (24)
Scale readings	255 237 249 242	87 184 125 162	307 207 267 232	210 234 219 228	257 238 251 244	90 184 127 163	271 233 255 241	215 233 222 229
Centre of swing . . .	244·40	147·55	244·70	224·65	246·05	148·75	246·70	226·15
Deflection due to rider or mass	..	97·000	..	20·725	..	97·625	..	20·725
Mass deflection ÷ rider deflection	..	·214175	..	·212974	..	·212292	..	·212564

	i. (25)	*r.* (26)	*i.* (27)	*m.* (28)	*i.* (29)	*r.* (30)	*i.* (31)	*m.* (32)
Scale readings	258 241 251 244	90 186 129 164	307 213 270 236	213 237 223 232	262 242 253 246	93 189 131 167	307 215 272 237	215 239 225 233
Centre of swing . . .	247·05	150·45	248·60	228·30	248·85	153·00	250·25	230·05
Deflection due to rider or mass	..	97·375	..	20·425	..	96·550	..	19·750
Mass deflection ÷ rider deflection	..	·211297	..	·210649	..	·208053	..	·204425

TABLE III. (continued).

	i. (33)	*r.* (34)	*i.* (35)	*m.* (36)	*i.* (37)	*r.* (38)	*i.* (39)	*m.* (40)
Scale readings	261 243 253 247	93 189 132 167	312 214 273 237	215 241 225 235	263 245 257 250	96 192 135 168	312 217 275 240	217 243 227 237
Centre of swing . . .	249·35	153·40	250·80	231·10	252·40	156·05	253·05	233·05
Deflection due to rider or mass	..	96·675	..	20·500	..	96·675	..	20·525
Mass deflection ÷ rider deflection	..	·208172	..	·212051	..	·212180	..	·212254

	i. (41)	*r.* (42)	*i.* (43)	*m.* (44)	*i.* (45)	*r.* (46)	*i.* (47)	*m.* (48)
Scale readings	264 247 259 251	98 194 136 171	314 219 277 242	220 243 231 238	267 249 260 255	100 197 138 174	321 223 281 246	224 246 233 242
Centre of swing . . .	254·10	157·85	255·05	235·20	256·25	160·35	259·30	238·05
Deflection due to rider or mass	..	96·725	..	20·450	..	97·425	..	21·250
Mass deflection ÷ rider deflection	..	·211812	..	·210662	..	·214011	..	·218650

	i. (49)	*r.* (50)	*i.* (51)	*m.* (52)	*i.* (53)	*r.* (54)	*i.* (55)
Scale readings	271 252 264 256	102 200 139 176	321 221 282 245	224 247 233 242	271 251 264 256	102 198 140 174	314 226 280 247
Centre of swing . . .	259·30	162·20	259·00	238·40	258·95	161·60	259·50
Deflection due to rider or mass	..	96·950	..	20·575	..	97·625	
Mass deflection ÷ rider deflection	..	·215704	..	·211487			

September 25, 1890. Mean of 25 determinations of M/R = α, Attracted masses in upper position } ·21125332.

September 23 and September 25 } Mean of 52 determinations of M/R = α, ·2112647.

TABLE III. (continued).

SUMMARY of Set II.

July 28	A = ·9973168
September 17	A = ·9984148
Mean value of .	A = ·9978658
September 23	a = ·2112753
,, 25	a = ·2112533
Mean value of .	a = ·2112647

Therefore

$$A - a = \cdot 7866011.$$

Mean value, giving equal weights to Sets I. and II.,

$$A - a = \cdot 791295.$$

INDEX

TO THE

PHILOSOPHICAL TRANSACTIONS (A)

FOR THE YEAR 1891.

A.

Air, carbon dioxide, and hydrogen, on the specific heats of, 73 (see Joly).
Ayrton (W. E.), Perry (J.), and Sumpner (W. E.) Quadrant Electrometers, 519.

B.

Bakerian Lecture.—On Tidal Prediction, 159 (see Darwin).
Basset (A. B.). On the Reflection and Refraction of Light at the Surface of a Magnetized Medium, 371.
Boiling and freezing points, on the determination of some, by means of the platinum thermometer, 43 (see Griffiths).
Burch (G. J.) and Veley (V. H.). The Variations of Electromotive Force of Cells consisting of certain Metals, Platinum, and Nitric Acid, 319.

C.

Callendar (H. L.) and Griffiths (E. H.). On the Determination of the Boiling-Point of Sulphur, and on a Method of Standardising Platinum Resistance Thermometers by reference to it.—Experiments made at the Cavendish Laboratory, Cambridge, 119.
Carbon dioxide, air, and hydrogen, on the specific heats of, 73 (see Joly).
Chemical change between nitric acid and certain metals, the conditions of, 279 (see Veley).
Cobalt, iron, and nickel, the internal friction of, studied by means of magnetic cycles of very minute range, 341 (see Tomlinson).

D.

DARWIN (G. H.). On Tidal Prediction.—BAKERIAN LECTURE, 159.

E.

Earth, on a determination of the mean density of the, and the gravitation constant by means of the common balance, 565 (see POYNTING).
Electrometers, quadrant, 519 (see AYRTON, PERRY, and SUMPNER).
Electromotive force of cells consisting of certain metals, platinum, and nitric acid, the variations of, 319 (see BURCH and VELEY).
Ellipsoidal harmonics, on, 231 (see NIVEN).

F.

Freezing and boiling-points, on the determination of some, by means of the platinum thermometer, 43 (see GRIFFITHS).

G.

Gases at constant volume, on the specific heats of.—Part I. Air, carbon dioxide, and hydrogen, 73 (see JOLY).
Gravitation constant, on a determination of the mean density of the earth and the, by means of the common balance, 565 (see POYNTING).
GRIFFITHS (E. H.). On the Determination of some Boiling and Freezing Points by means of the Platinum Thermometer, 43.
GRIFFITHS (E. H.) (see CALLENDAR and GRIFFITHS).

H.

Harmonics, on ellipsoidal, 231 (see NIVEN).
Hydrogen, air, and carbon dioxide, on the specific heats of, 73 (see JOLY).

I.

Iron, nickel, and cobalt, the internal friction of, studied by means of magnetic cycles of very minute range, 341 (see TOMLINSON).

J.

JOLY (J.). On the Specific Heats of Gases at Constant Volume.—Part I. Air, Carbon Dioxide, and Hydrogen, 73.
JONES (J. V.). On the Determination of the Specific Resistance of Mercury in Absolute Measure, 1.

L.

Light, on the reflection and refraction of, at the surface of a magnetized medium, 371 (see Basset).
Lockyer (J. N.). On the Causes which produce the Phenomena of New Stars, 397.

M.

Magnetic induction, 341 (see Tomlinson).
Mercury, on the determination of the specific resistance of, in absolute measure, 1 (see Jones).
Metals, platinum, and nitric acid, the variations of electromotive force of cells consisting of certain, 319 (see Burch and Veley).
Metals, the conditions of chemical change between nitric acid and certain, 279 (see Veley).

N.

New stars, on the causes which produce the phenomena of, 397 (see Lockyer).
Nickel, cobalt, and iron, the internal friction of, studied by means of magnetic cycles of very minute range, 341 (see Tomlinson).
Nitric acid and certain metals, the conditions of chemical change between, 279 (see Veley).
Nitric acid, certain metals, and platinum, the variations of electromotive force of cells consisting of, 319 (see Burch and Veley).
Niven (W. D.). On Ellipsoidal Harmonics, 231.

P.

Perry (J.) (see Ayrton, Perry, and Sumpner).
Physical properties of matter, the influence of stress and strain on the.—Part III. Magnetic induction (continued), 341 (see Tomlinson).
Platinum, certain metals, and nitric acid, the variations of electromotive force of cells consisting of, 319 (see Burch and Veley).
Platinum resistance thermometers, on a method of standardising, 119 (see Callendar and Griffiths).
Platinum thermometer, on the determination of some boiling and freezing points by means of the, 43 (see Griffiths).
Poynting (J. H.). On a Determination of the Mean Density of the Earth and the Gravitation Constant by means of the Common Balance, 565.

Q.

Quadrant electrometers, 519 (see Ayrton, Perry, and Sumpner).

R.

Reflection and refraction of light at the surface of a magnetized medium, on the, 371 (see Basset).

S.

Sampson (R. A.). On Stokes's Current Function, 449.

Specific heats of gases at constant volume, on the.—Part I. Air, carbon dioxide, and hydrogen, 73 (see Joly).

Specific resistance of mercury in absolute measure, on the determination of the, 1 (see Jones).

Stars, on the causes which produce the phenomena of new, 397 (see Lockyer).

Stokes's *current function*, on, 449 (see Sampson).

Stress and strain, the influence of, on the physical properties of matter.—Part III. Magnetic induction (continued), 341 (see Tomlinson).

Sulphur, on the determination of the boiling-point of, 119 (see Callendar and Griffiths).

Sumpner (W. E.) (see Ayrton, Perry, and Sumpner).

T.

Tidal prediction, on.—Bakerian Lecture, 159 (see Darwin).

Tomlinson (H.). The Influence of Stress and Strain on the Physical Properties of Matter.—Part III. Magnetic Induction (continued).—The Internal Friction of Iron, Nickel, and Cobalt, studied by means of Magnetic Cycles of very minute range, 341.

V.

Veley (V. H.). The Conditions of Chemical Change between Nitric Acid and certain Metals, 279.

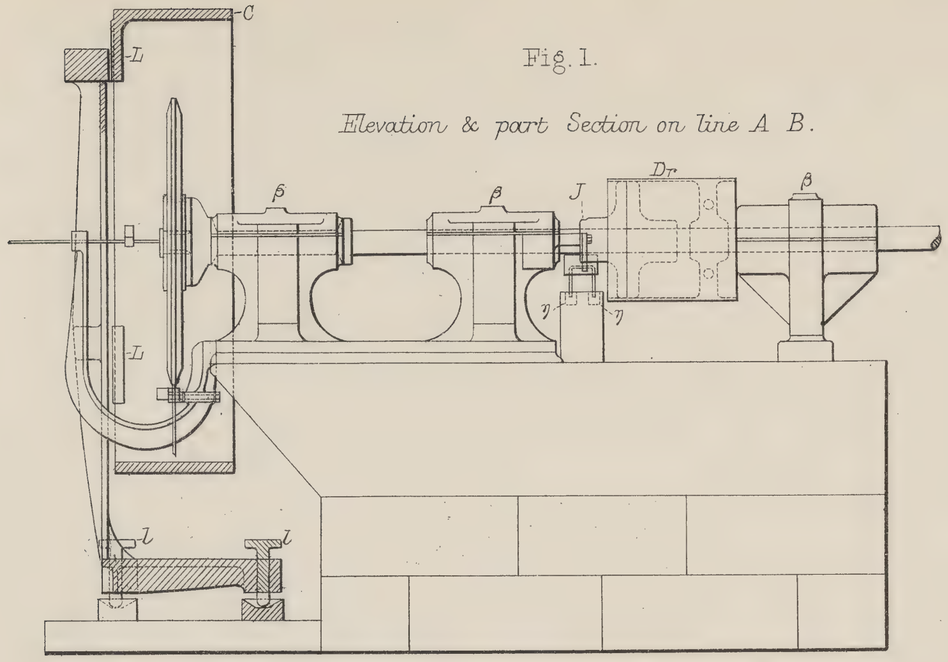
Veley (V. H.) (see Burch and Veley).

HARRISON AND SONS, PRINTERS IN ORDINARY TO HER MAJESTY, ST. MARTIN'S LANE.

Disc and Coil.
Scale, 1/8th full size.

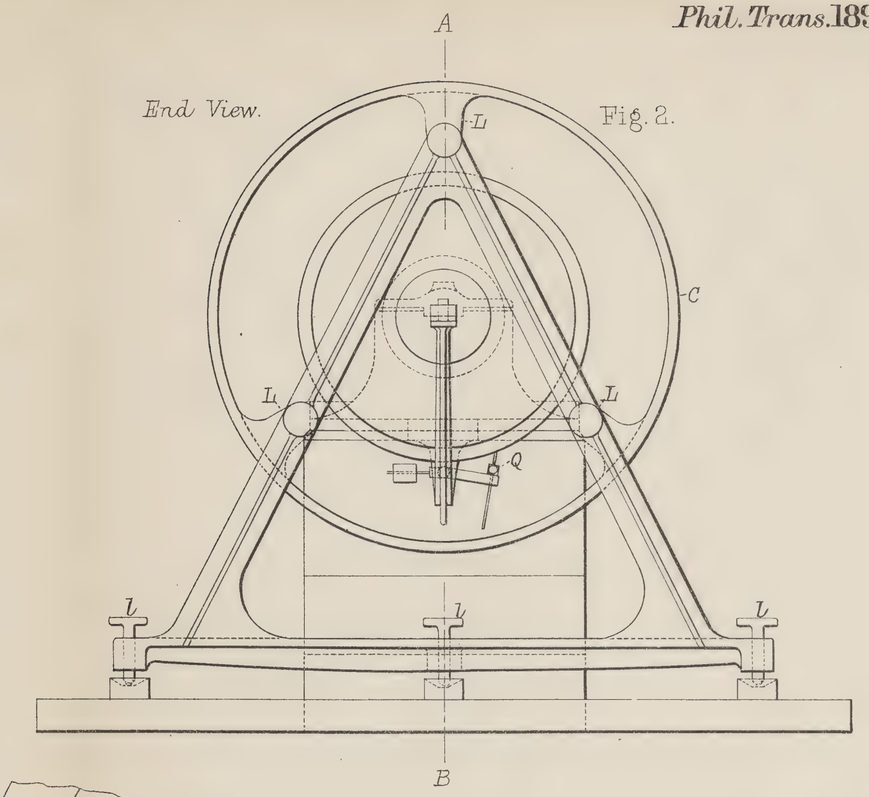
Fig. 1.

Elevation & part Section on line A B.



End View.

Fig. 2.



Details of Disc
Part Elevation & part

Insulation & Brushes.
Section on lines ab, bc.

Scale, Half full size

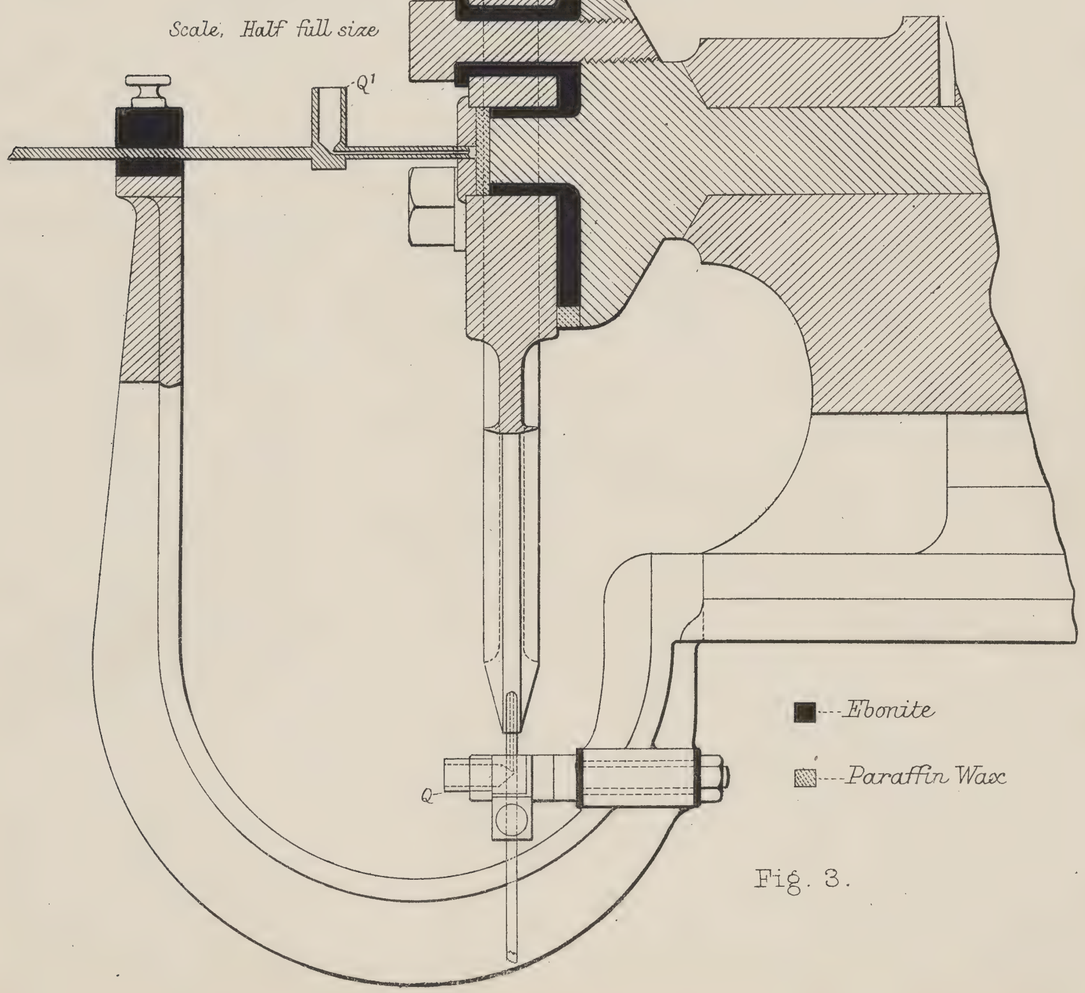


Fig. 3.

Part End View.

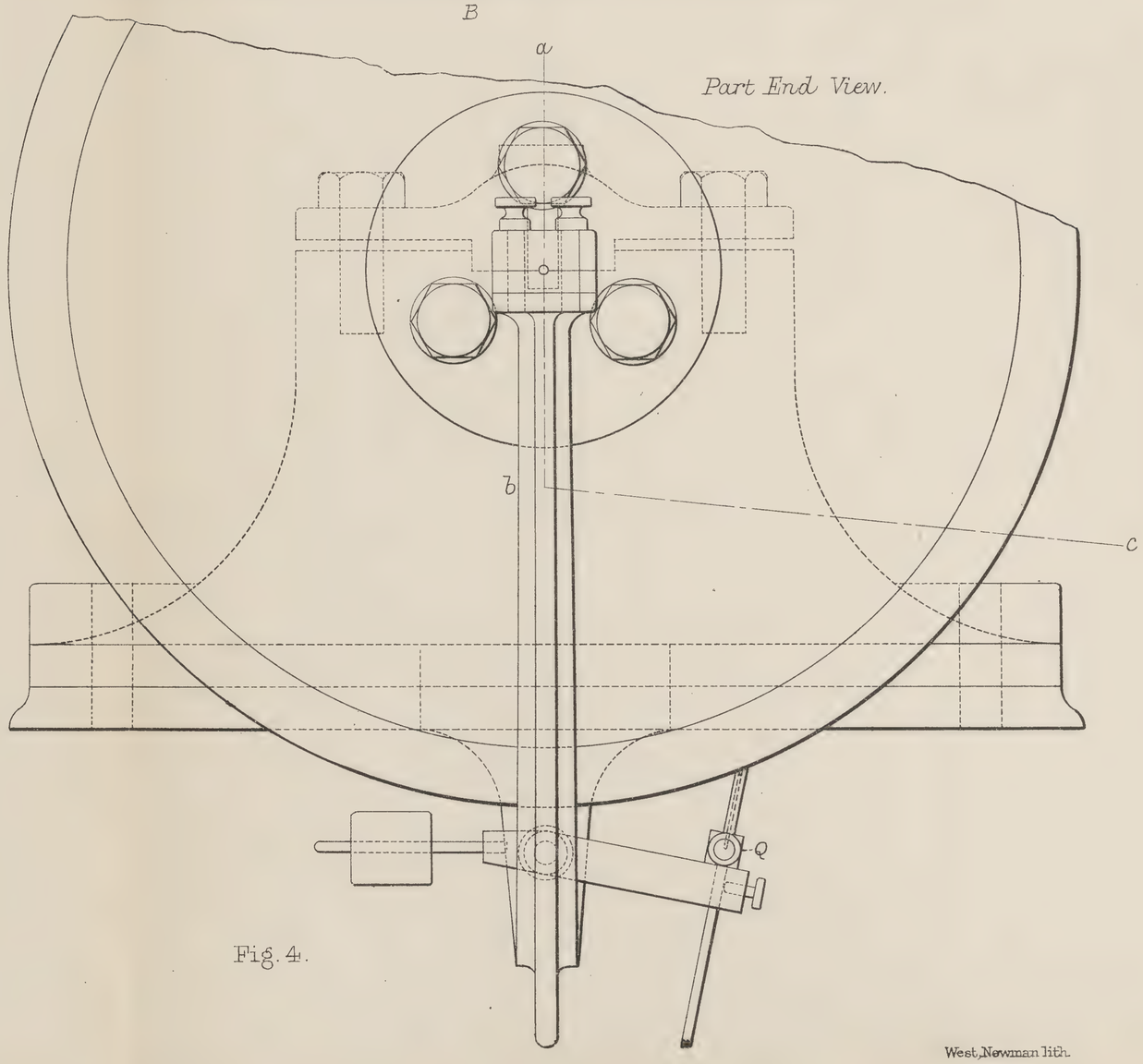
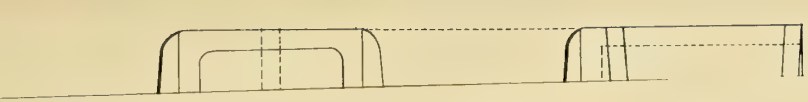


Fig. 4.

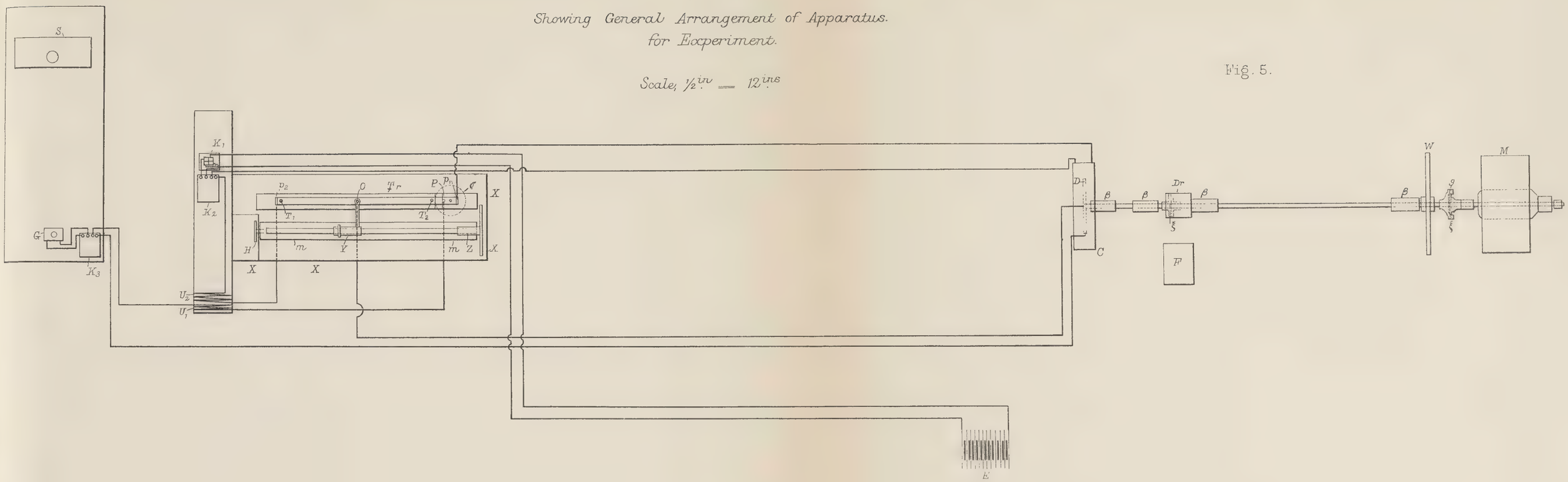
West, Newman lith.



Plan
Showing General Arrangement of Apparatus.
for Experiment.

Scale, ½ in. = 12 ins.

Fig. 5.



Mercury Trough.
Scale, ¼ Full Size.

Fig. 6. *Elevation.*

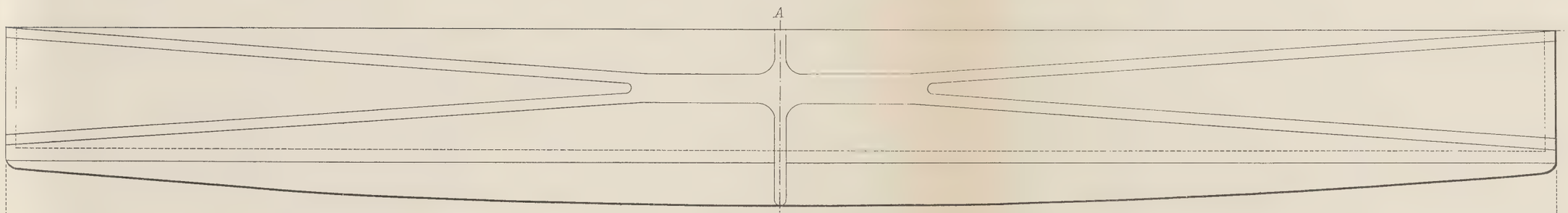


Fig. 7. *Plan.*

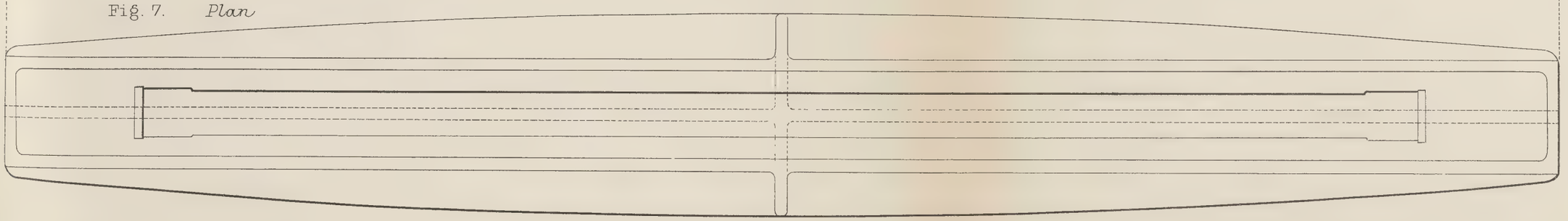
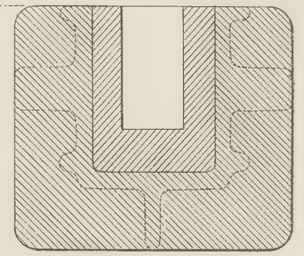


Fig. 8.
Cross Section on line A.B.



West, Newman lith.

Spherometer.

Scale, ¼ Full size

Elevation.

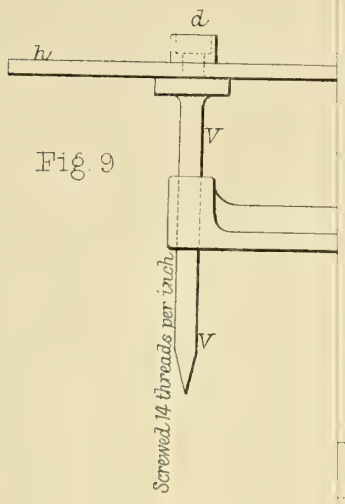


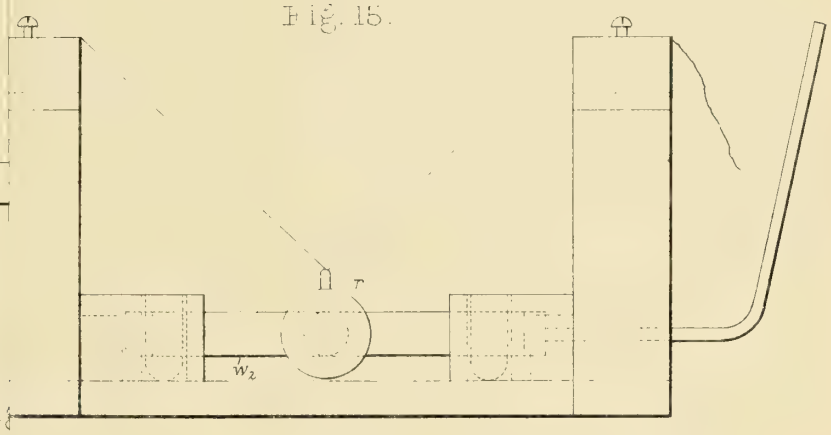
Fig. 9

Callipers.

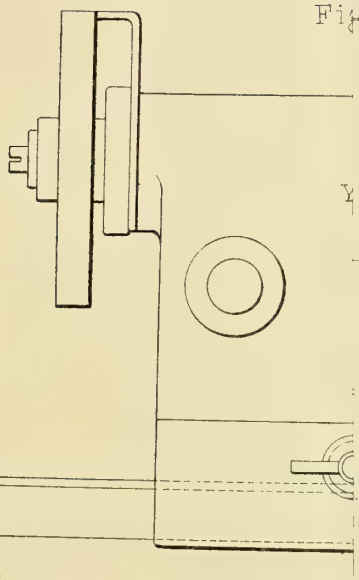
Scale, ½ Full size

Elevation.

Fig. 15.

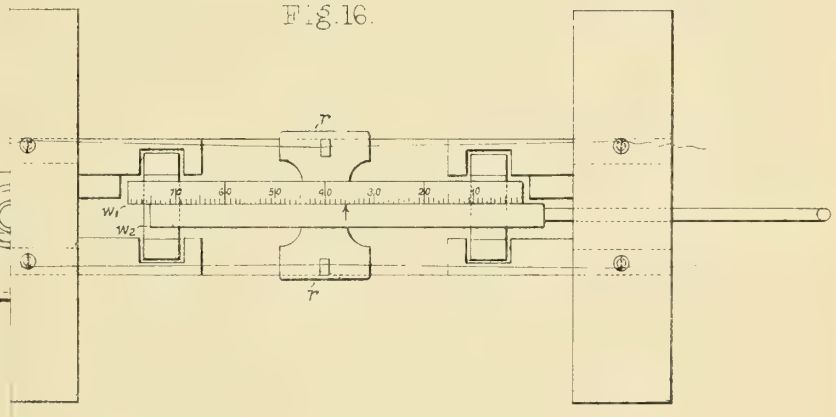


Fig



Plan.

Fig. 16.



Side

West, Newman lith

Spherometer.

Scale, ¼ Full size.

Elevation.

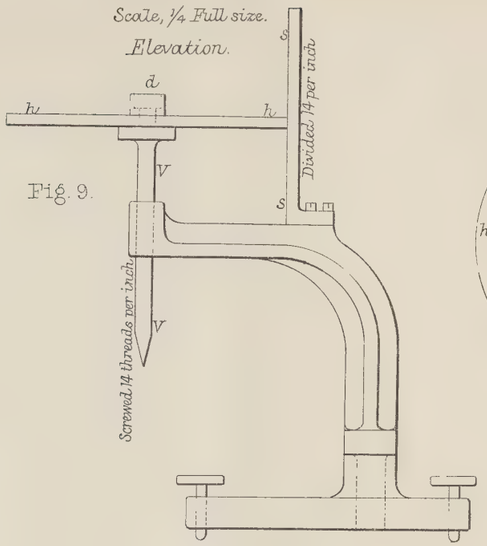
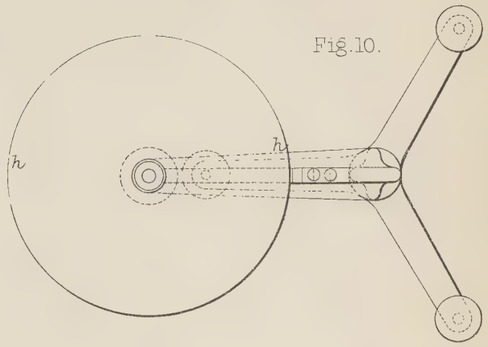


Fig. 9.

Plan.

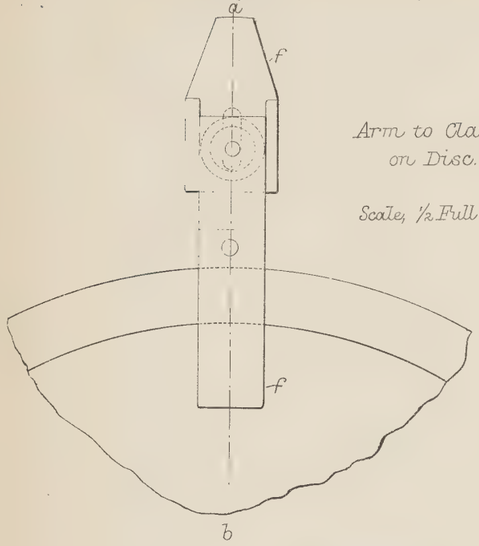
Divided into 360 divisions.

Fig. 10.



Elevation.

Fig. 13.



Arm to Clamp on Disc.

Scale, ½ Full size.

Cross Section on line a. b.

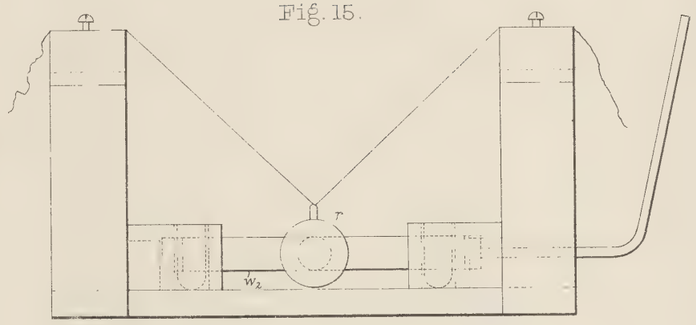
Fig. 14.

Callipers.

Scale, ½ Full size.

Elevation.

Fig. 15.



Plan.

Fig. 16.

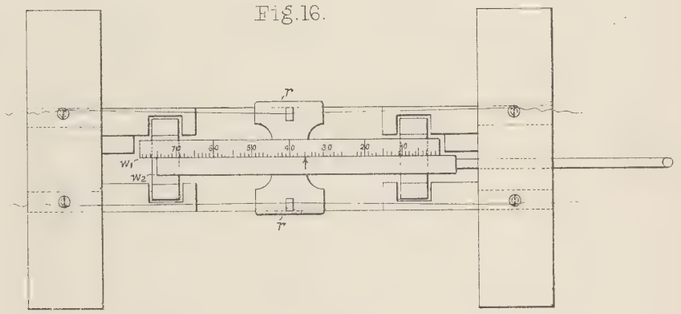
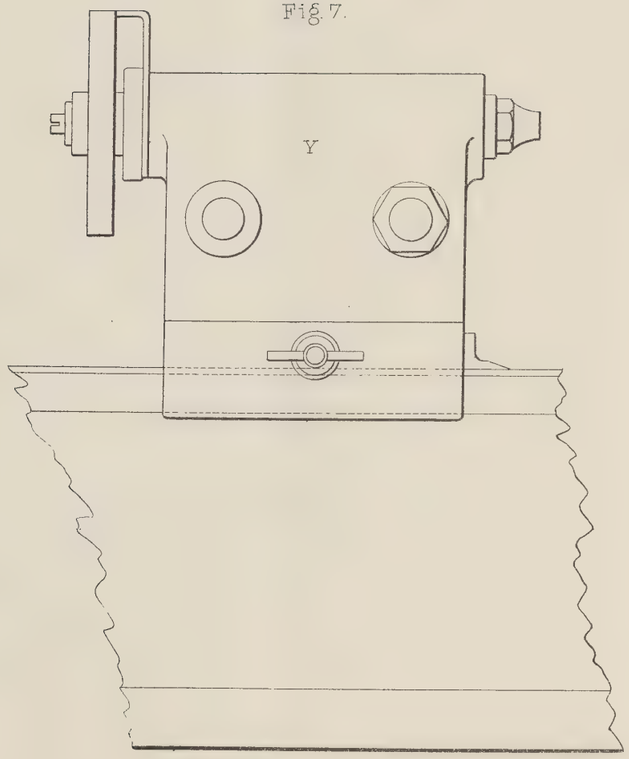
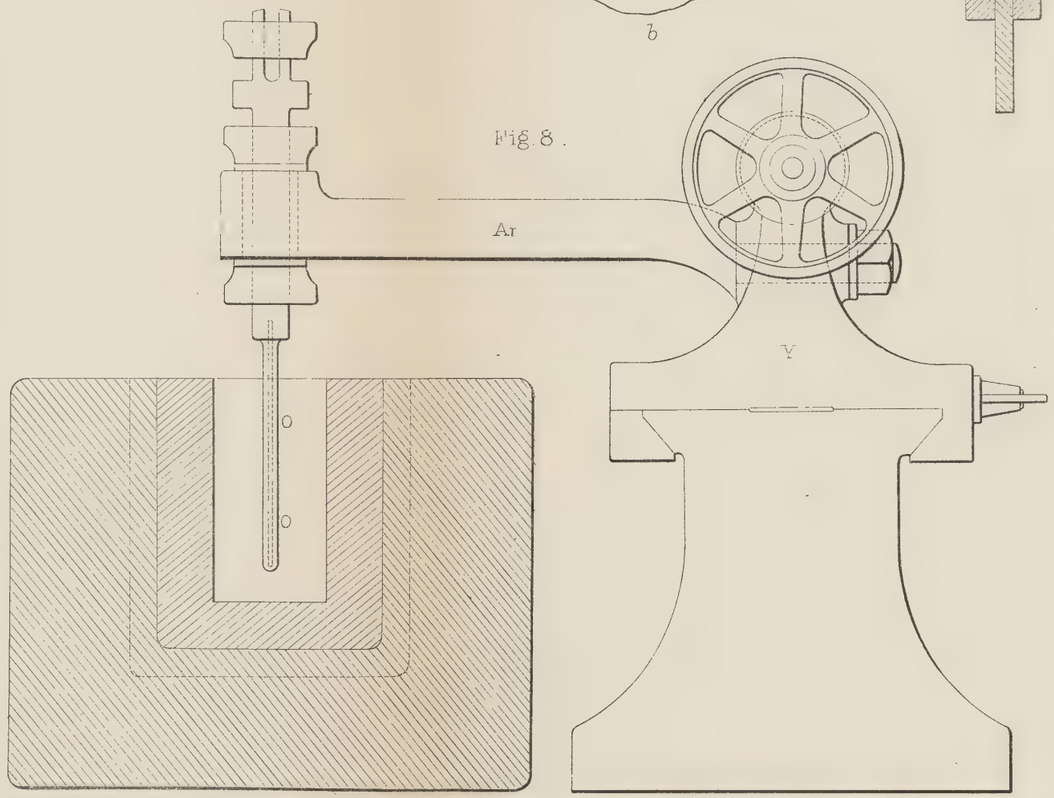


Fig. 7.



Side View

Fig. 8.

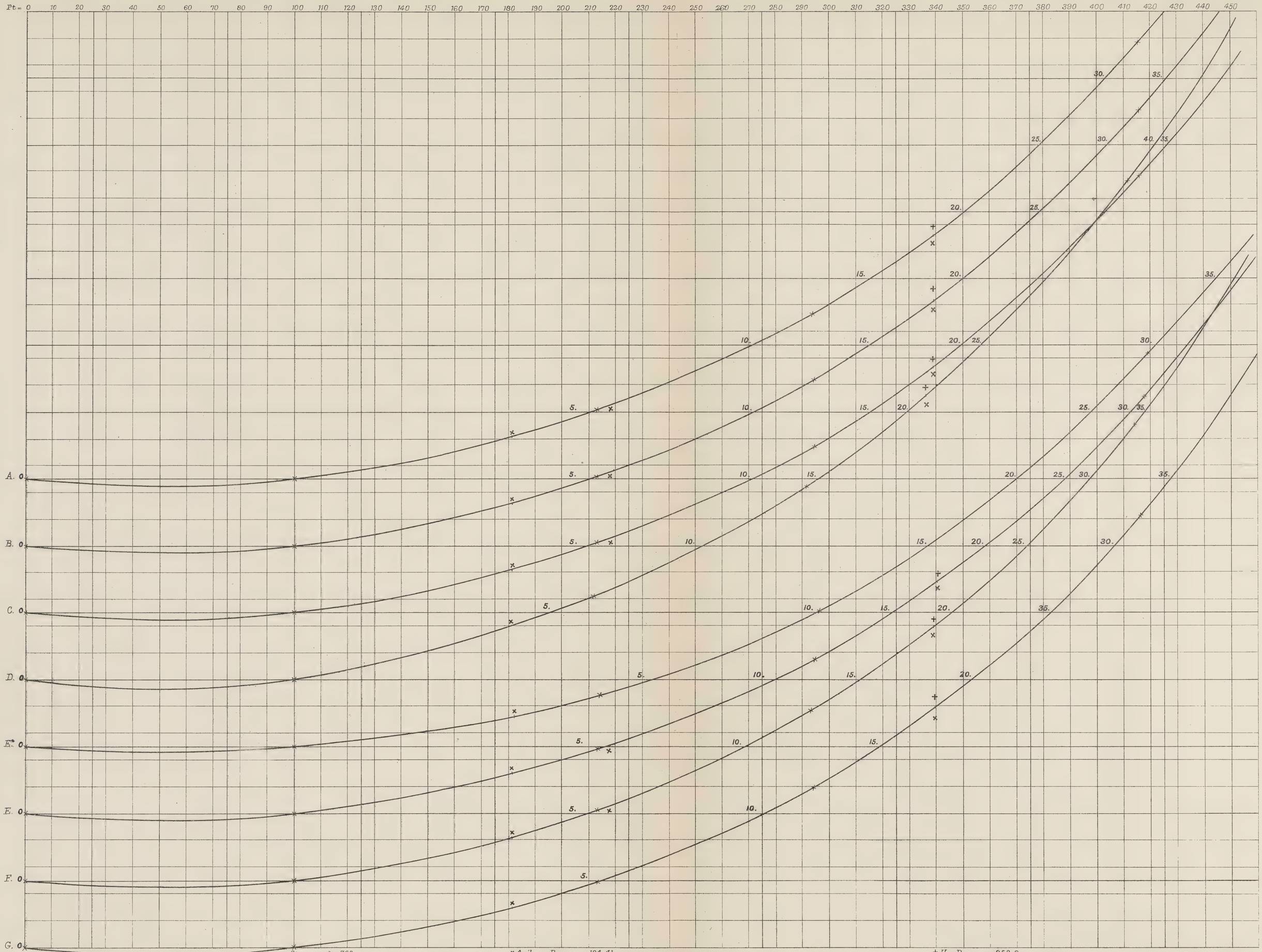


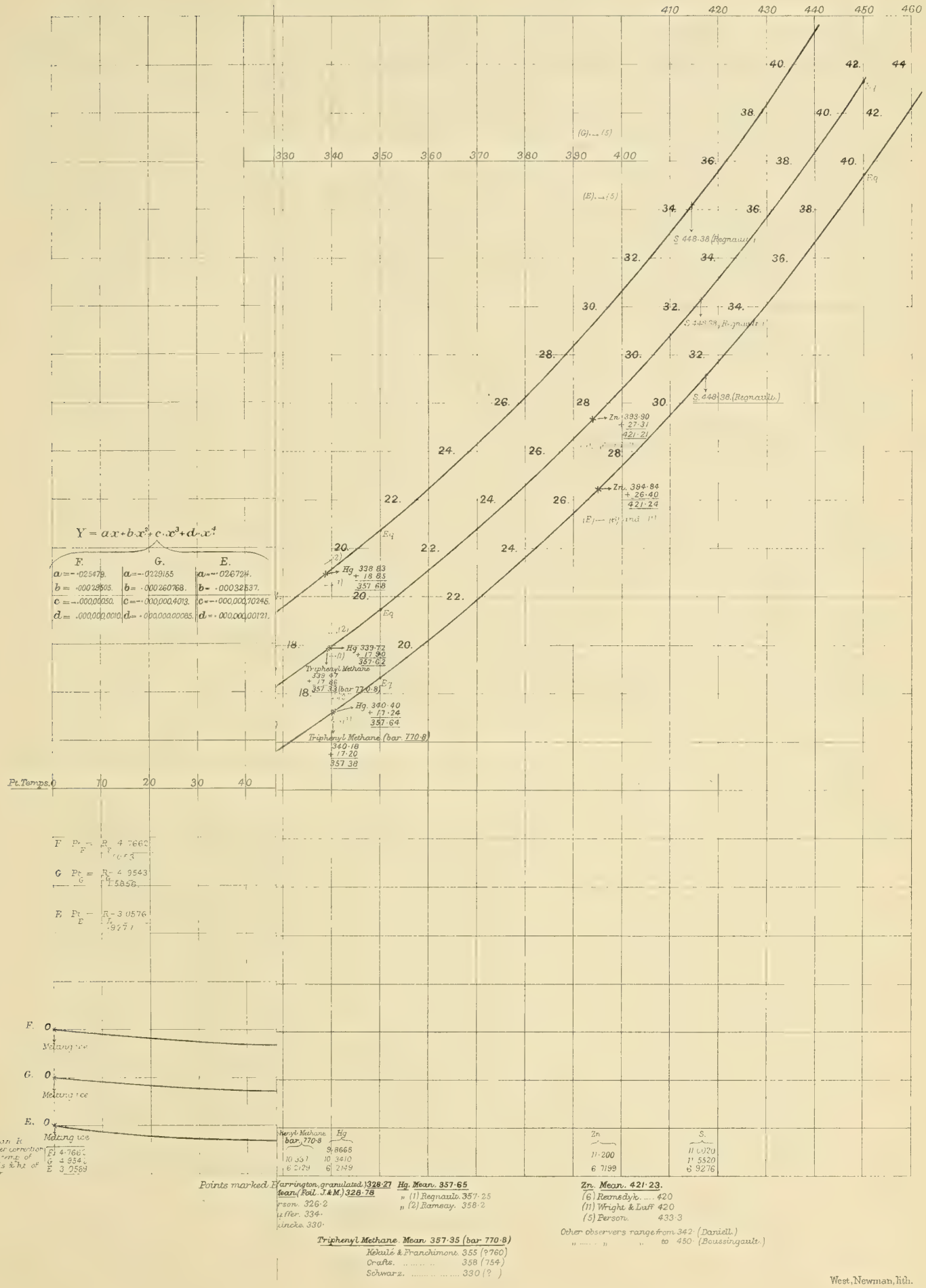
End View with Trough in Section.

Movable Headstock of Whitworth Machine and Arm carrying Movable Electrode.

Scale, ½ Full Size.

West, Newman lith.


Pt= 0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300 310 320 330 340 350 360 370 380 390 400 410 420 430 440 450
A. 0
B. 0
C. 0
D. 0
E. 0
E. 0
F. 0
G. 0
5. 10. 15. 20. 25. 30. 35. 40.
Melting ice
Steam, under 760 m.m.
Ordinate = T – pt.
Abscissa = pt.
× Aniline, . Ramsay. – 184.41
× Methyl-Salicylate, . Ramsay. 222.88
× Naphthalene. – Crafts. 218.06
+ Hg. Ramsay. 358.2
× „ Regnault. 357.25
× Benzophenone. – Crafts. 306.08
× Sulphur. Regnault. 448.38
West, Newman, lith.



Mean R after correction for zero of F 4·7662 G 4·9543 E 3·0569

Points marked Harrington, granulated) 328·27 Mean (Foil. J. & M.) 328·78
Person. 326·2
Heffer. 334·
Hincke. 330·

Hg. Mean. 357·65
(1) Regnault. 357·25
(2) Ramsay. 358·2

Zn. Mean. 421·23.
(6) Reynolds. 420
(11) Wright & Luff 420
(5) Person. 433·3
Other observers range from 342 (Daniell.)
" " " to 450 (Boussingault.)

Triphenyl Methane. Mean 357·35 (bar 770·8)
Kekulé & Franchimont. 355 (?760)
Crafts. ..... 358 (754·)
Schwarz. ..... 330 (?)

West, Newman, lith.

Fixed points assumed, Ice, Steam, Naphthalene, Benzophenone, Sulphur.

Dots marked thus ·→(6) indicate a point through which the curve whose letter is given should pass, according to the observer noted in margin.

$Y = ax + bx^2 + cx^3 + dx^4$

F.	G.	E.
a = ·025479.	a = ·0229155.	a = ·026724.
b = ·00012905.	b = ·000260768.	b = ·00032537.
c = ·00000250.	c = ·000004013.	c = ·000,000,70245.
d = ·000,000,0010.	d = ·000,000,00065.	d = ·000,000,00121.

Ordinate = T − Pt.
Abscissa = Pt.

F. $Pt_F = \frac{R - 4{\cdot}7662}{1{\cdot}6053}$.

G. $Pt_G = \frac{R - 4{\cdot}9543}{1{\cdot}5856}$.

E. $Pt_E = \frac{R - 3{\cdot}0576}{{\cdot}9277}$.

Pt. Temps. 0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300 310 320 330 340 350 360 370 380 390 400 410 420 430 440 450 460

F. 0 Melting ice — Eq. — Steam

G. 0 Melting ice — Eq. — Steam

E. 0 Melting ice — Eq. — Steam

Aniline 181·08 + 3·15 = 184·23
Aniline 181·28 + 2·95 = 184·23
Aniline 181·24 + 2·96 = 184·20

Naphthalene 218·06 (Crafts) + Methyl-Salicylate 223·21 = Sn. 225·87 + 6·13 = 232·00
Naphthalene 218·06 (Crafts) + Methyl-Salicylate 218·08 = Sn. 226·30 + 5·72 = 232·02
Naphthalene 218·06 (Crafts) + Methyl-Salicylate 223·18 = Sn. 226·45 + 5·63 = 232·08

Bi. 261·16 + 8·53 = 269·69
Bi. 261·38 + 8·30 = 269·68

Benzophenone 306·08 (Crafts) + Cd. 307·53 = (Spec. A) 321·43
Benzophenone 306·08 (Crafts) + Cd. 308·43 + 13·08 = (Spec. A) 321·51
Benzophenone 306·08 (Crafts)

Hg. 338·83 + 18·85 = 357·68
Triphenyl Methane 339·47 + 17·86 = 357·33 (bar 770·8)
Hg. 339·72 + 17·90 = 357·62
Triphenyl Methane (bar. 770·8)
Hg. 340·40 + 17·24 = 357·64
340·18 + 17·20 = 357·38

Pb. 313·14 (A) + 15·13 = 328·27
Pb. 314·40 (B) + 14·39 = 328·72
Cd. 308·04 + 13·60 = (Spec. A) 321·64
Pb. 314·91 + 13·86 = 328·77
Cd. 308·58 + 13·12 = (Spec. B) 321·70

Zn. 393·90 + 27·31 = 421·21
Zn. 394·84 + 26·40 = 421·24

S. 448·38 (Regnault.)

Mean R. after correction for temp. of coils & bc. of bar.

	Aniline	Naphthalene	Methyl-Salicylate	Sn.	Bi (A)	Bi (B)	Benzophenone	Cd. (A)	Cd. (B)	Pb	Triphenyl Methane bar. 770·8	Hg	Zn.	S.
F. 4·7662	7·4922	7·6703	8·0933	8·1638	9·0953	9·0351	9·1633	9·8344	9·8395 (2)	9·4789 (A)	10·337	10·3410	11·200	11·0020
G. 4·9542	7·8288	8·3351	5 0801	8·5426	5·4618	5·4320	9·6735	5·9180	5·8197	5·9784 (B)	6 2129	6·2149	6·7199	11·5520
E. 3·0569	4·7384	5·0253		5·1578			5·7886							6·9276

Points marked Eq. were deduced from equation.

Aniline. Mean. 184·22
(2) Ramsay. 184·41
(3) Thorpe. 183·7

Methyl-Salicylate. Sn. Mean. 232·03
Mean. 223·19
(3) Person. 232·7
(6) Reimedyk. 228·5
(7) Kupffer. 230
(8) Grichton. 238
(2) Ramsay. 222·88
(4) Cahours. 222.

Bi. Mean. 269·68 (A & B)
(5) Person. 270·5
(6) Reimedyk. 268·3

(Spec. A. Harrington's) Cd. Mean. 321·50.
(5) Person. 320·7
(6) Reimedyk. 320
(9) Van der Wyde. 335.

(Spec. B. J. & M.) Cd. Mean. 321·67

Pb. Spec. A. (Harrington, granulated.) 328·27
Spec. B. Mean (Roll. J. & M.) 328·78
(5) Person. 326·2
(7) Kupffer. 334·
(10) Quincke. 330·

Hg. Mean. 357·65
" (1) Regnault. 357·25
" (2) Ramsay. 356·2

Triphenyl Methane. Mean. 357·35 (bar. 770·8)
Kekulé & Franchimont. 355 (?760)
Crafts. 358 (?54)
Schwarz. 330 (?)

Zn. Mean. 421·23.
(6) Reimedyk. 420
(7) Wright & Luff. 420
(5) Person. 433·3

Other observers range from 342· (Daniell.)
" " " " to 450· (Boussingault.)

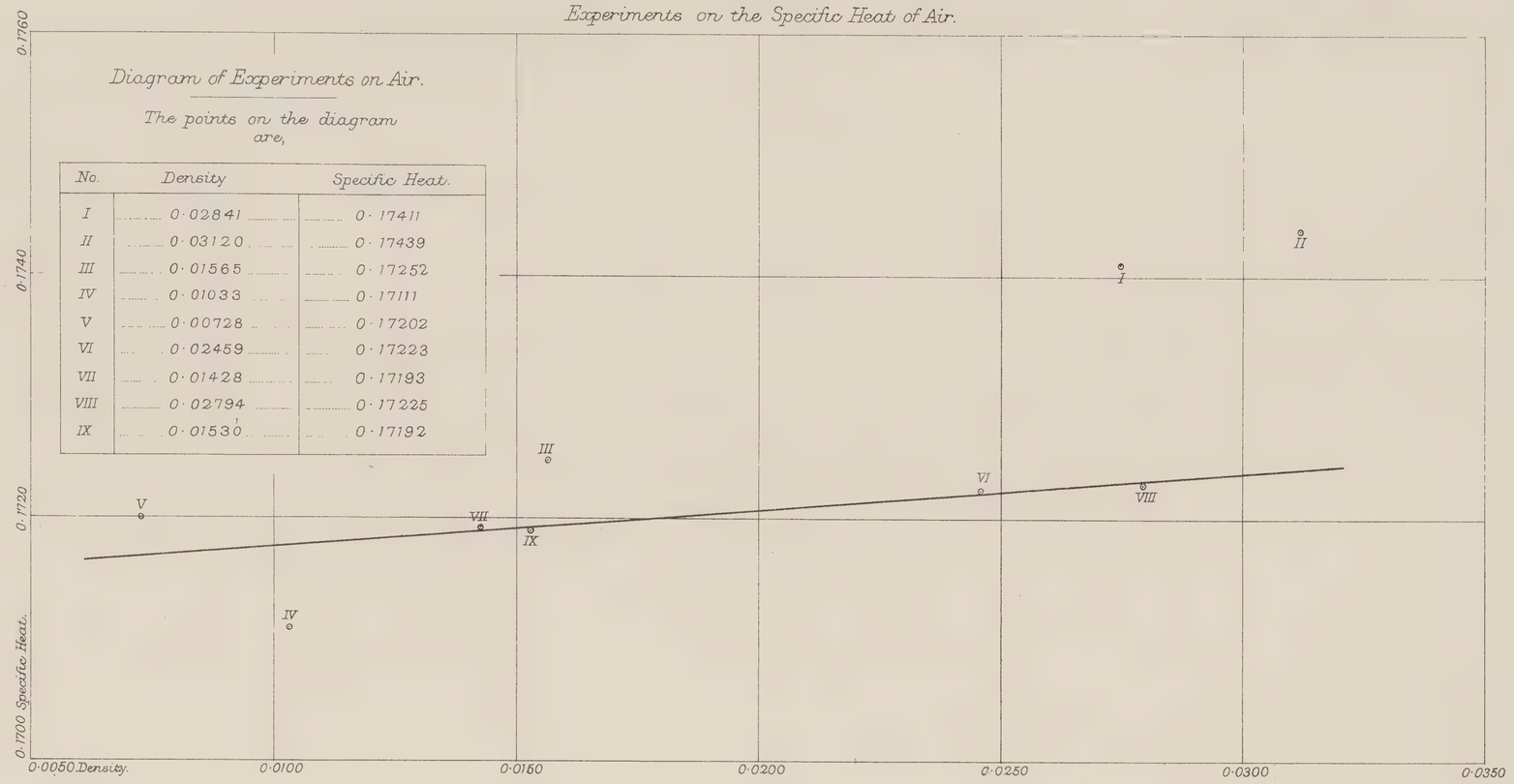
West, Newman, lith.

Experiments on the Specific Heat of Air.

Diagram of Experiments on Air.

The points on the diagram are,

No.	Density	Specific Heat.
I	0·02841	0·17411
II	0·03120	0·17439
III	0·01565	0·17252
IV	0·01033	0·17111
V	0·00728	0·17202
VI	0·02459	0·17223
VII	0·01428	0·17193
VIII	0·02794	0·17225
IX	0·01530	0·17192

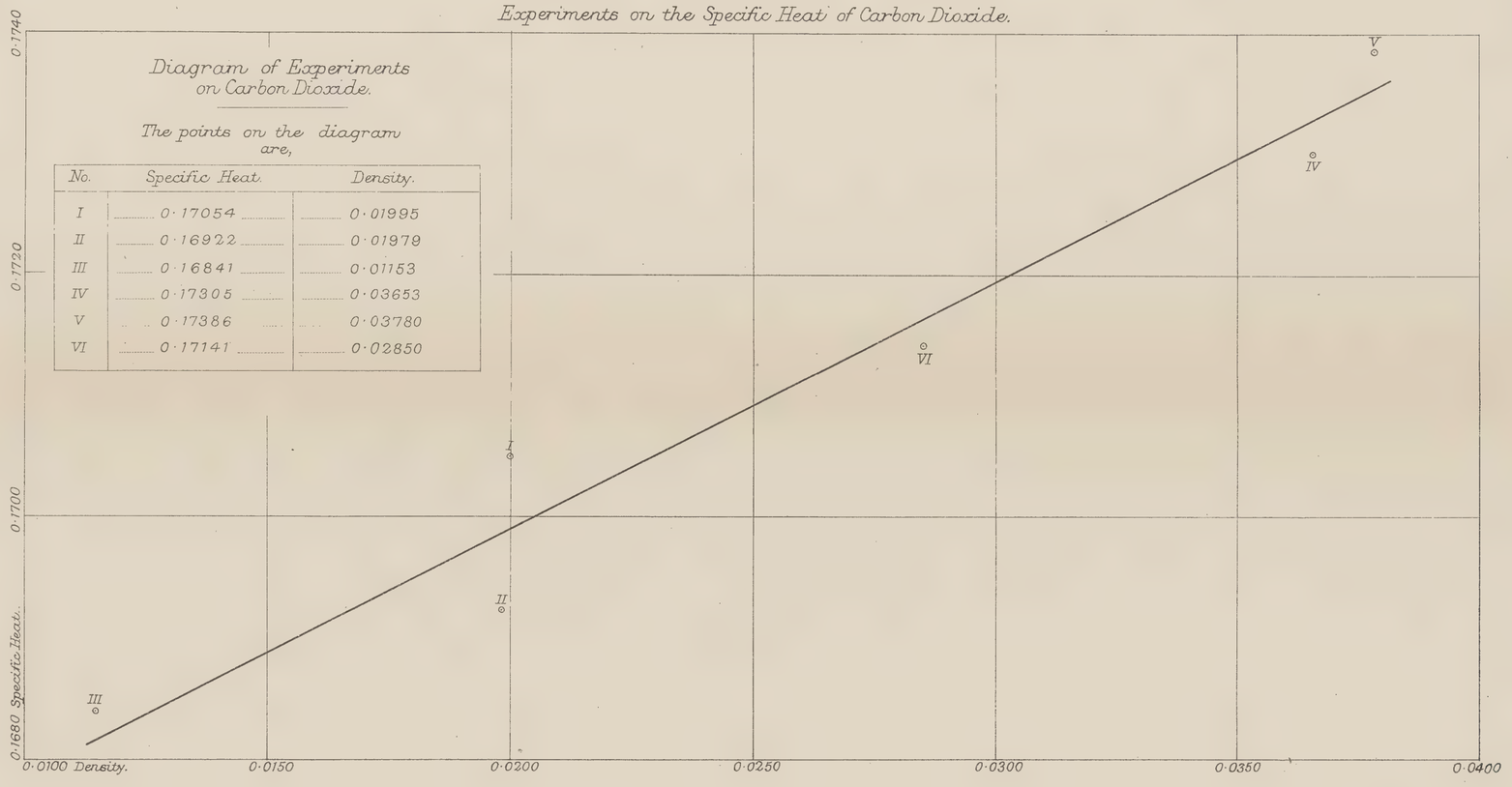


Experiments on the Specific Heat of Carbon Dioxide.

Diagram of Experiments on Carbon Dioxide.

The points on the diagram are,

No.	Specific Heat.	Density.
I	0·17054	0·01995
II	0·16922	0·01979
III	0·16841	0·01153
IV	0·17305	0·03653
V	0·17386	0·03780
VI	0·17141	0·02850

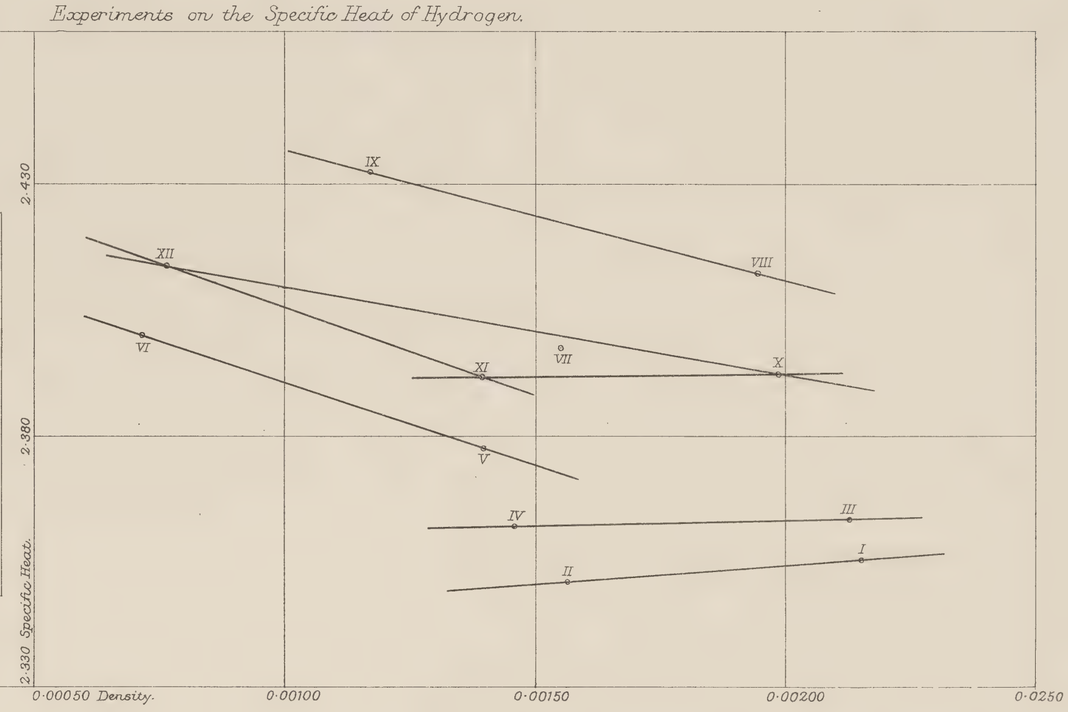


Experiments on the Specific Heat of Hydrogen.

The Experiments on Hydrogen.

The points on the diagram are,

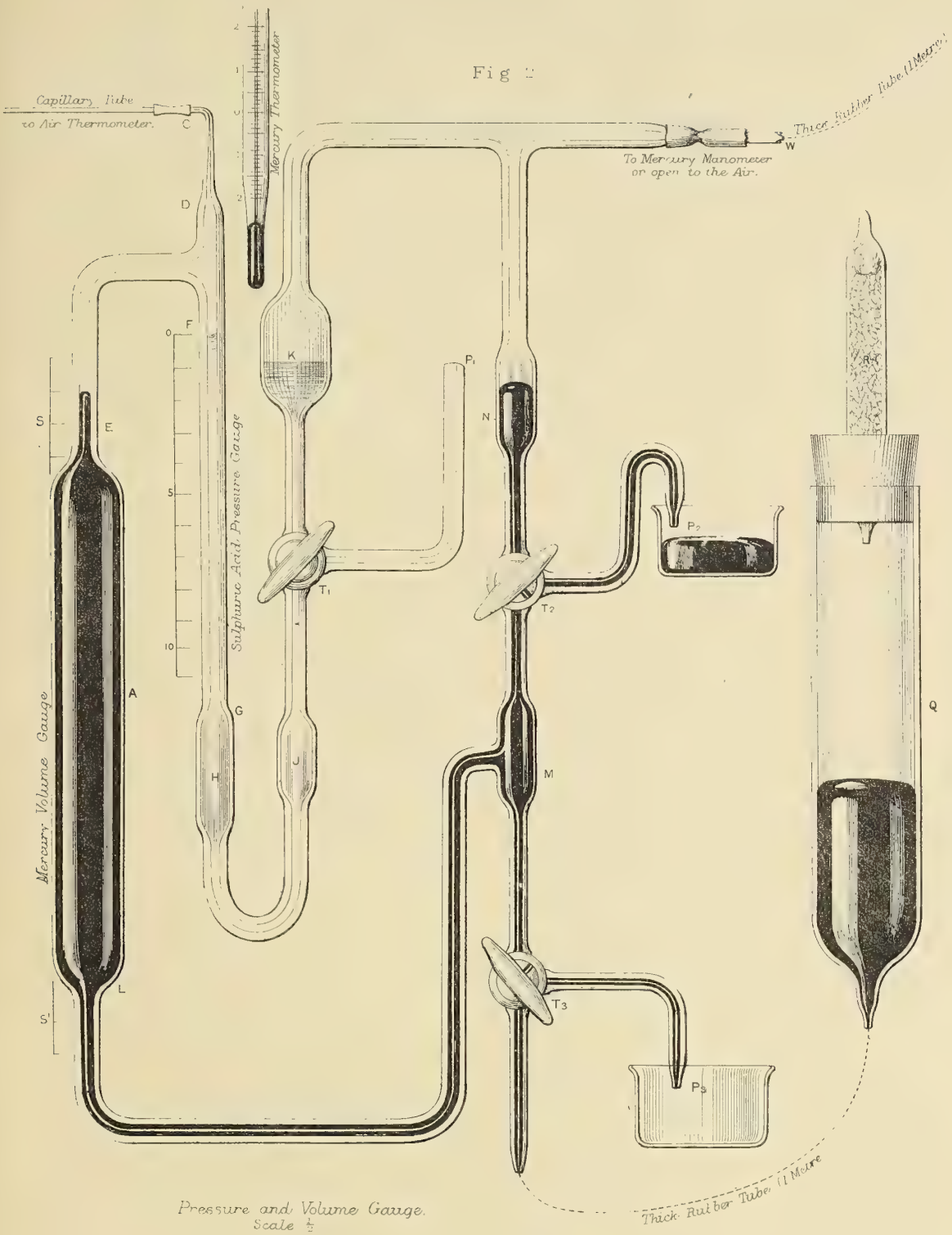
No.	Specific Heat.	Density.
I	2·3559	0·002151
II	2·3512	0·001565
III	2·3633	0·002127
IV	2·3623	0·001452
V	2·3779	0·001392
VI	2·4012	0·000741
VII	2·3975	0·001551
VIII	2·4120	0·001881
IX	2·4320	0·001169
X	2·3928	0·001987
XI	2·3917	0·001395
XII	2·4135	0·000763



Phil. Trans. 1891. A. Plate 6.

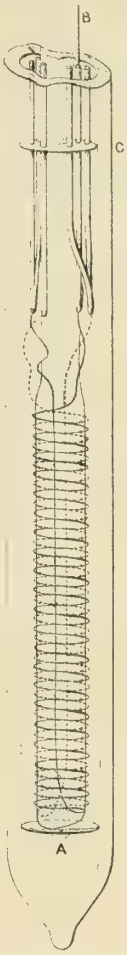
West, Newman, lith.

Fig 2



Pressure and Volume Gauge.
Scale $\frac{1}{2}$

Fig 3



Spiral of Thermometer M
Enlarged by $\frac{1}{3}$.

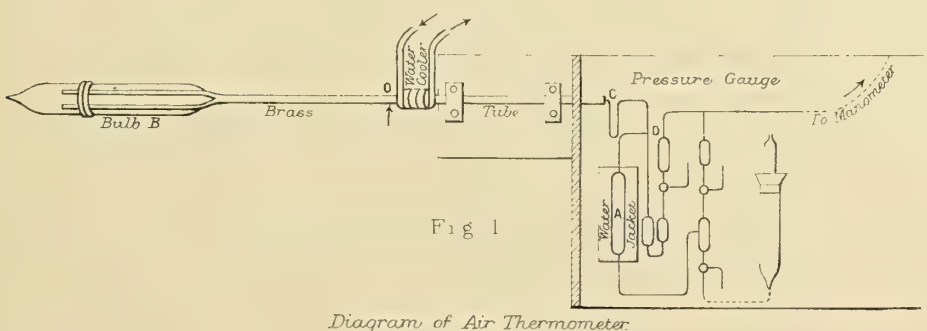
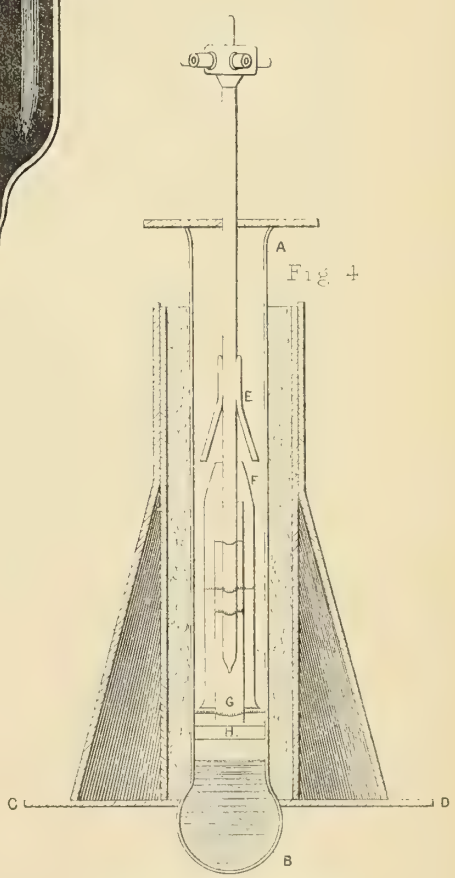


Fig 1

Diagram of Air Thermometer.
Scale $\frac{1}{10}$*th.*

Fig 4



Meyer Tube Apparatus
Scale $\frac{1}{4}$

West, Newman, lith.

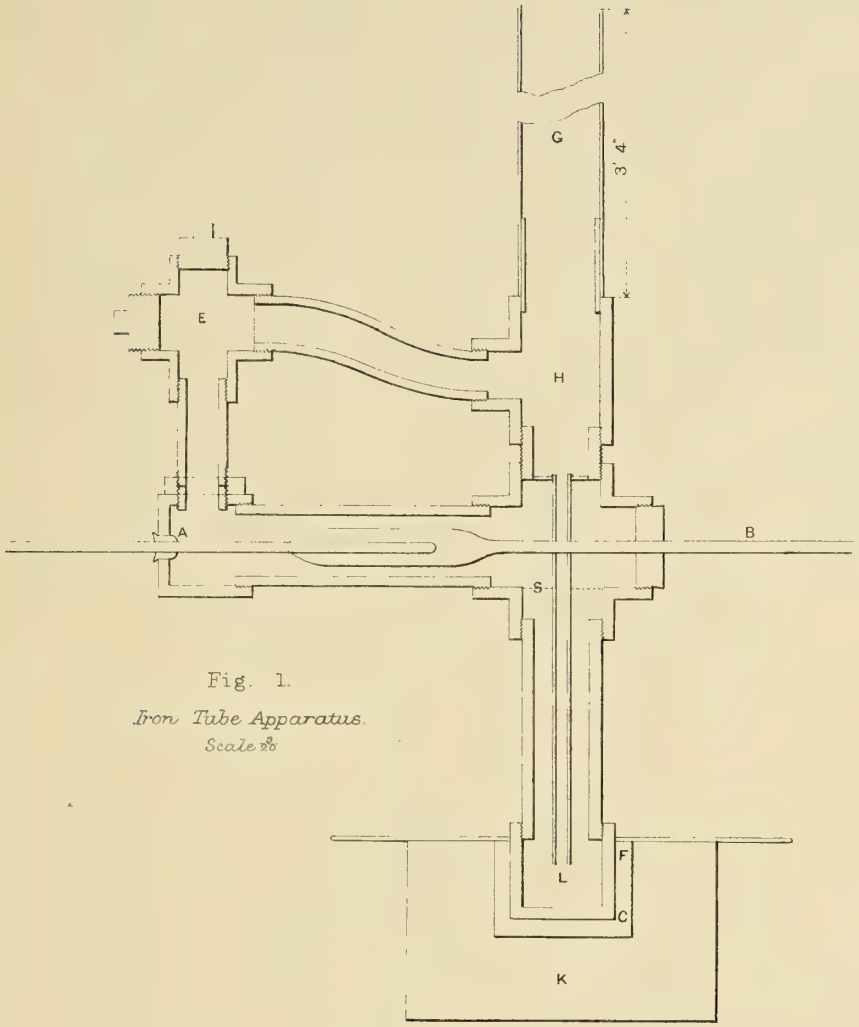
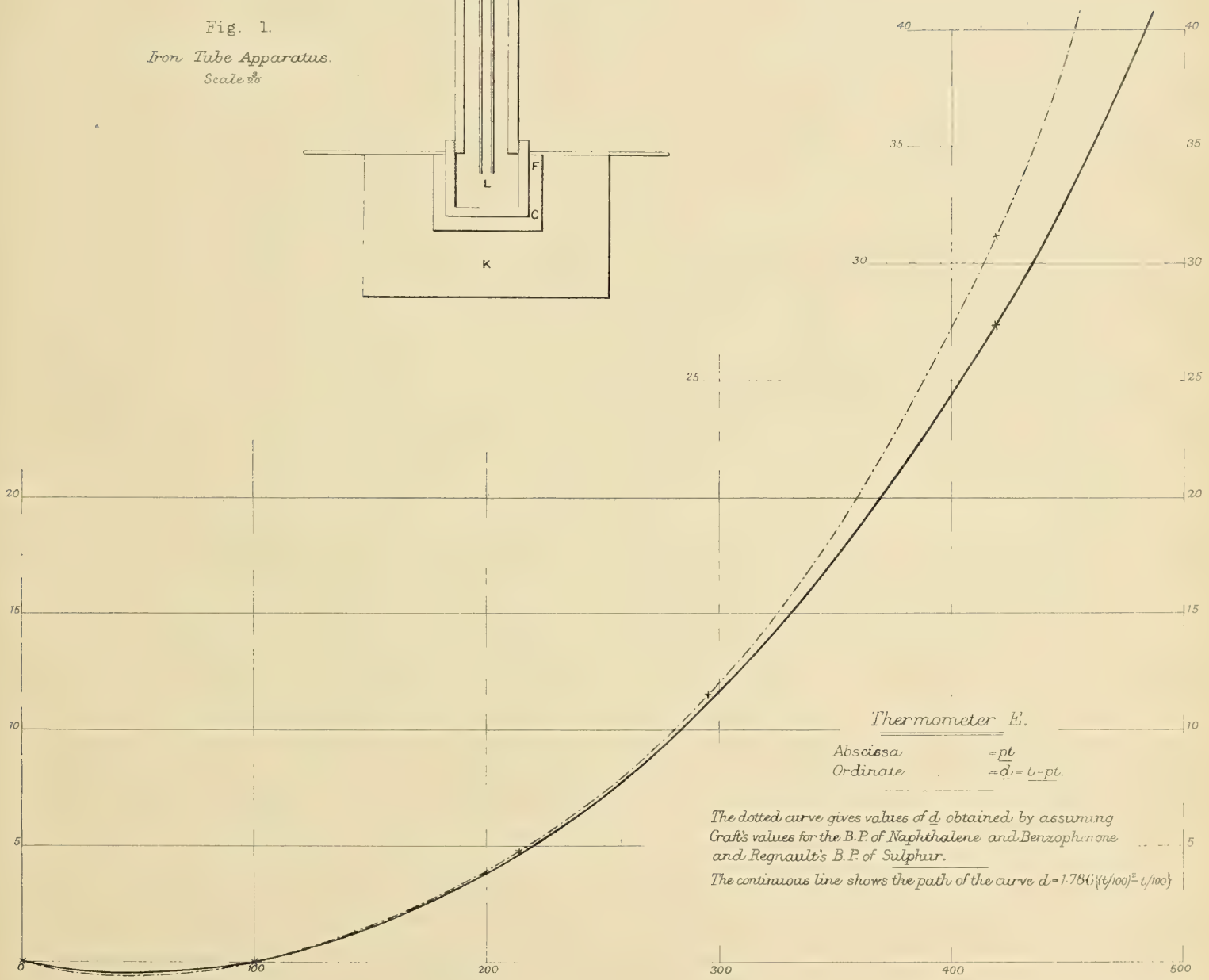


Fig. 1.

Iron Tube Apparatus.
Scale ⅜

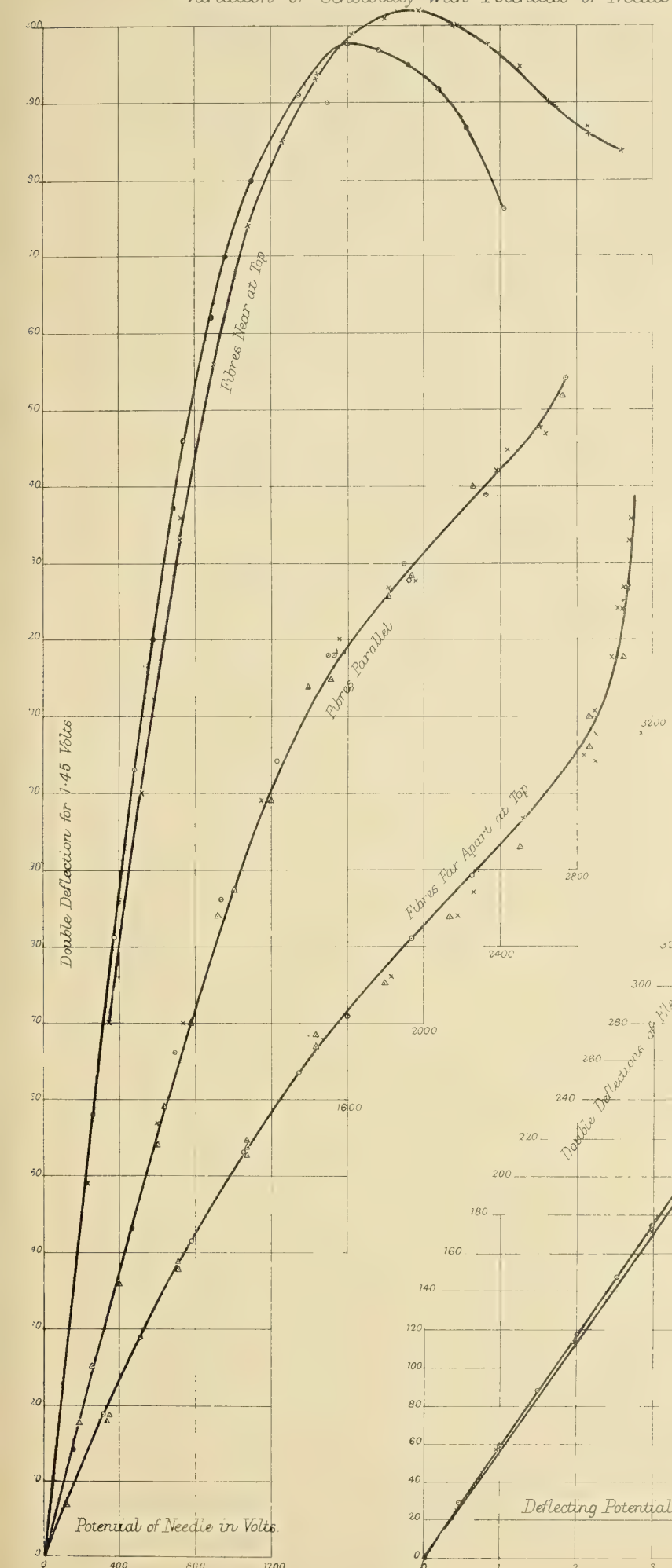


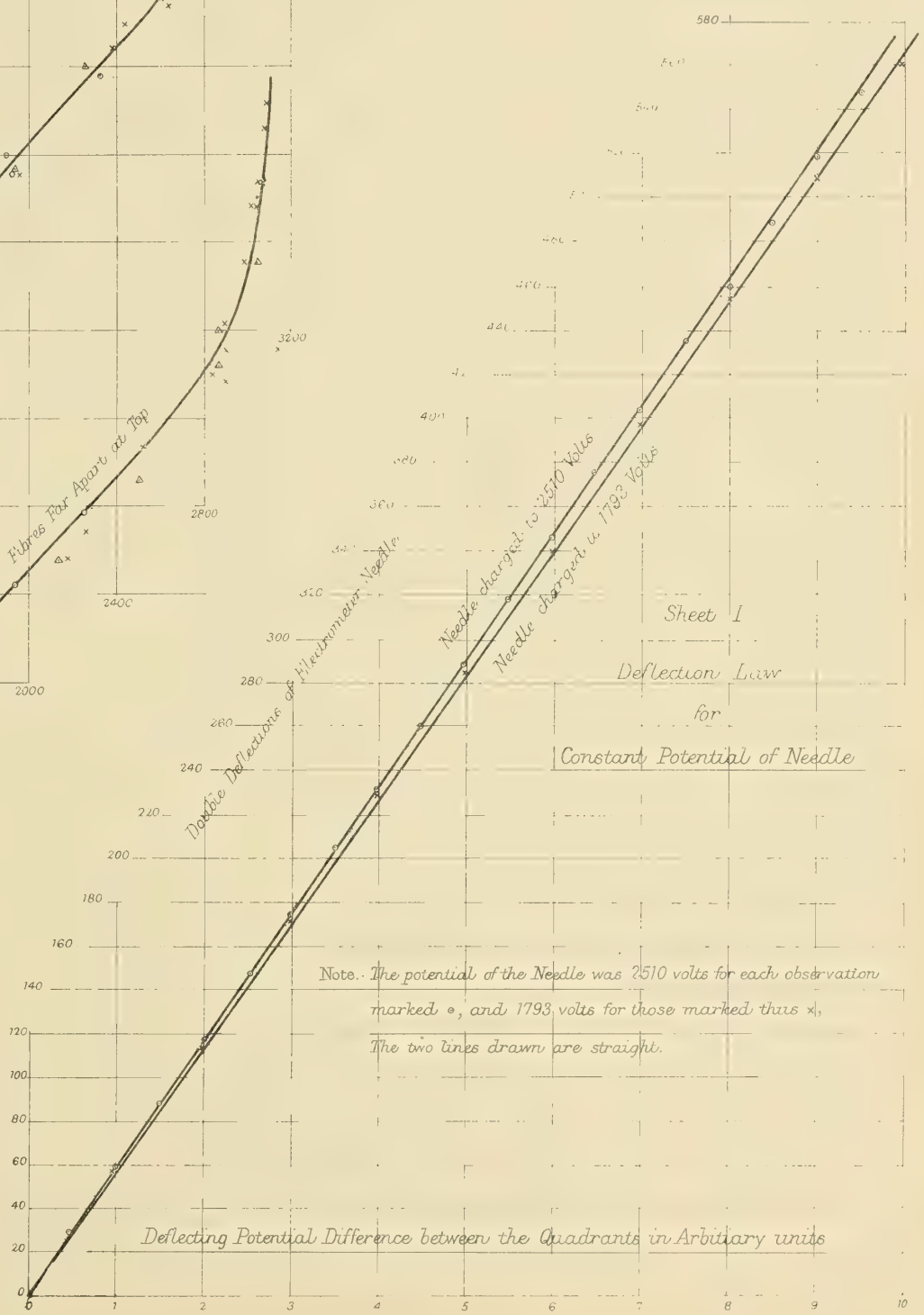
West, Newman, lith

Sheet II.

Variation of Sensibility with Potential of Needle and Distance apart of the Suspending Fibres.

Fibres	Observations denoted by	Date	Deflecting P.D. in Volts	Adjustable Quadrant at
Parallel	⊙	Nov. 3	1·45	100
	△	Nov. 3	2·90	92·5
	×	Nov. 4	1·45	113
Near at Top	⊙	Nov. 4	1·45	160
	×	Dec. 3	1·45	50
	×	Dec. 4	1·45	50
Far Apart at Top	⊙	Nov. 8	4·35	+68
	△	Nov. 10	2·90	−50
	△	Nov. 10	2·90	−100
	×	Dec. 2	1·45	−100





Sheet I

Deflection Law for Constant Potential of Needle

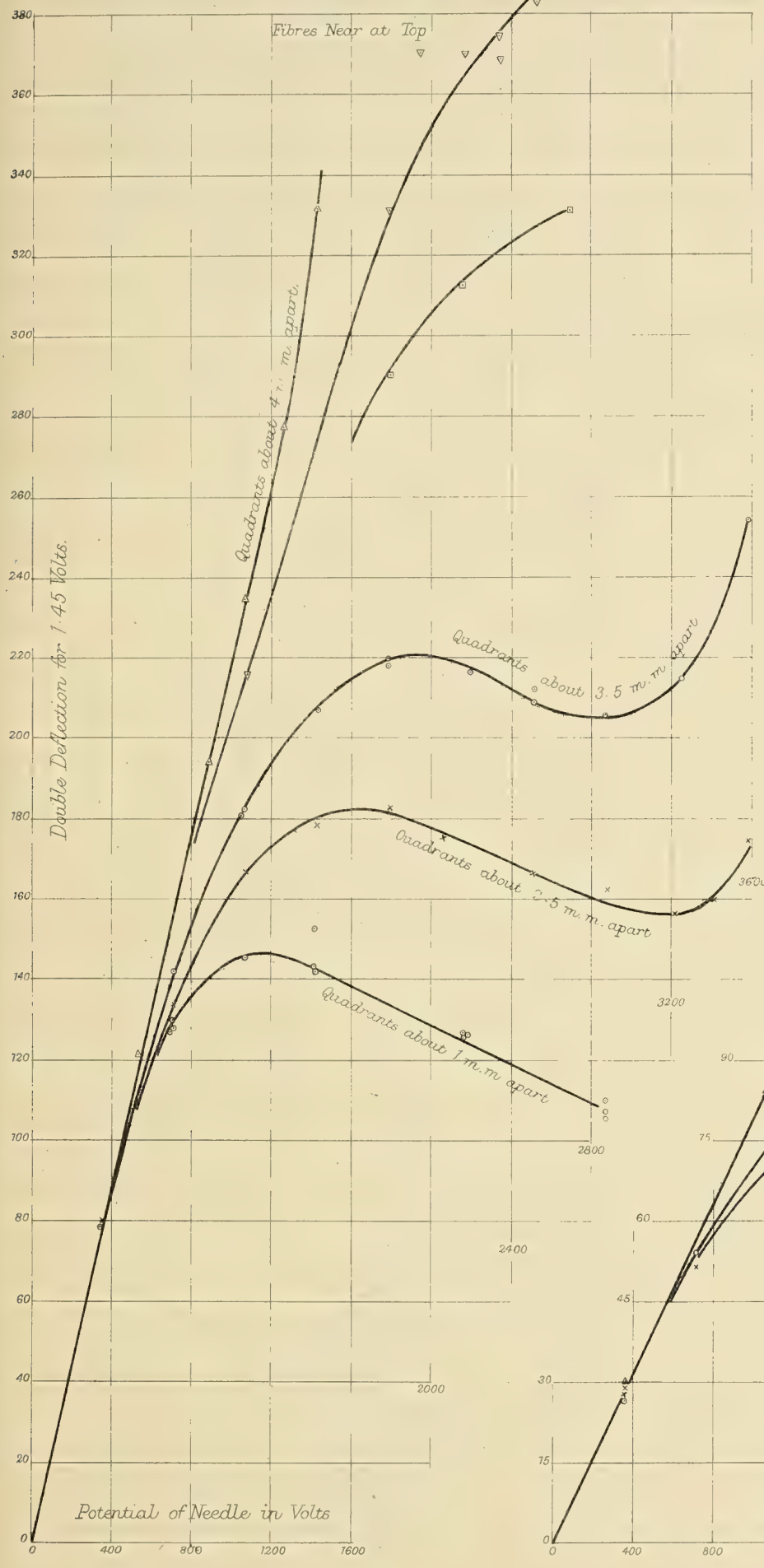
Note.—The potential of the Needle was 2510 volts for each observation marked ⊙, and 1793 volts for those marked thus ×. The two lines drawn are straight.

West, Newman, lith

Sheet III.

Variation of Sensibility with Potential of Needle and Distance between Quadrants.

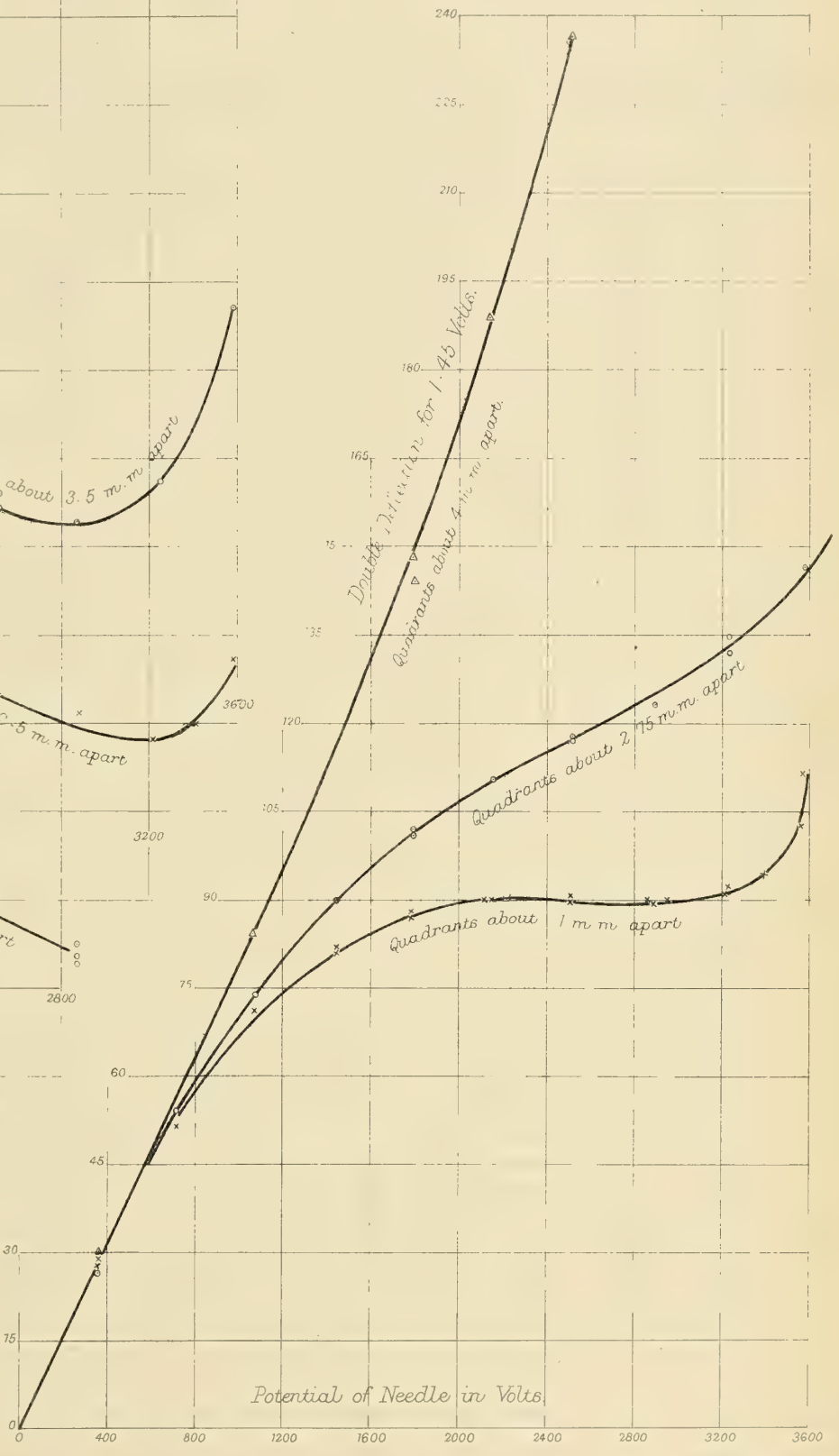
Fibres Near at Top



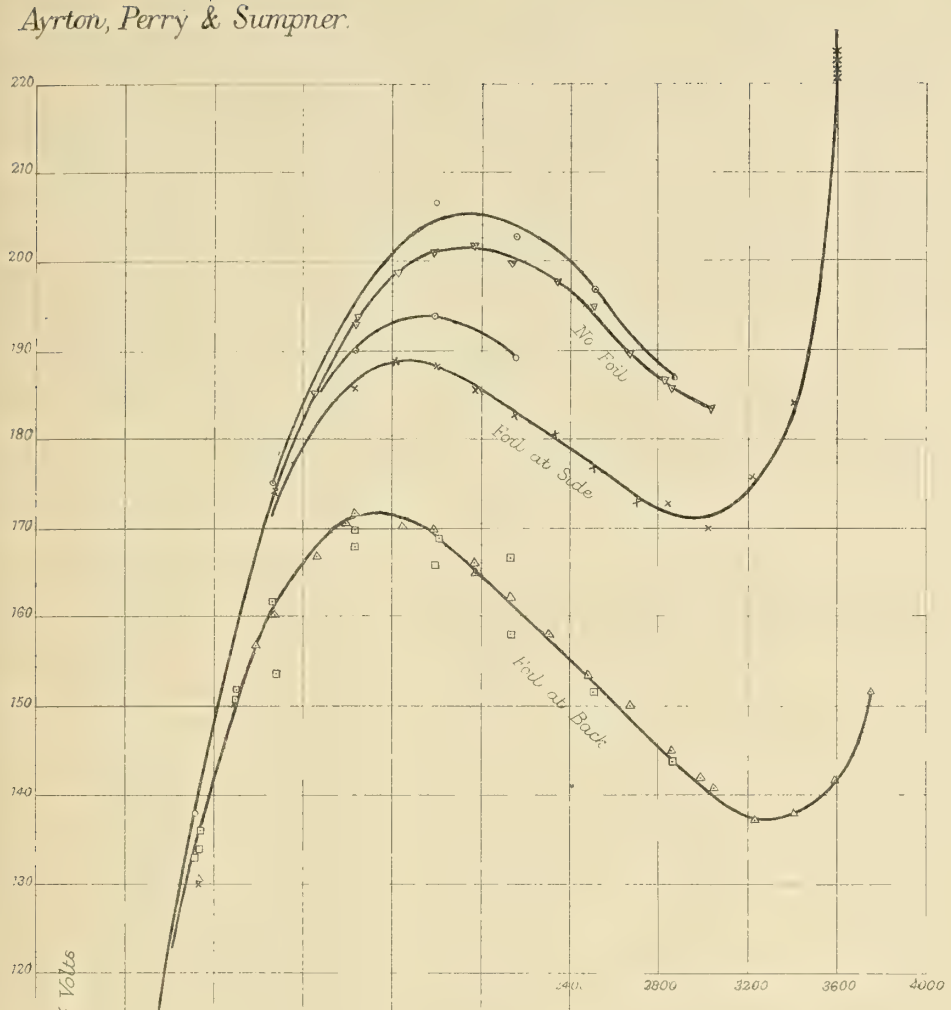
Sheet IV

Variation of Sensibility with Potential of Needle and Distance between Quadrants

Fibres Far Apart at the Top.



West, Newman, lith.

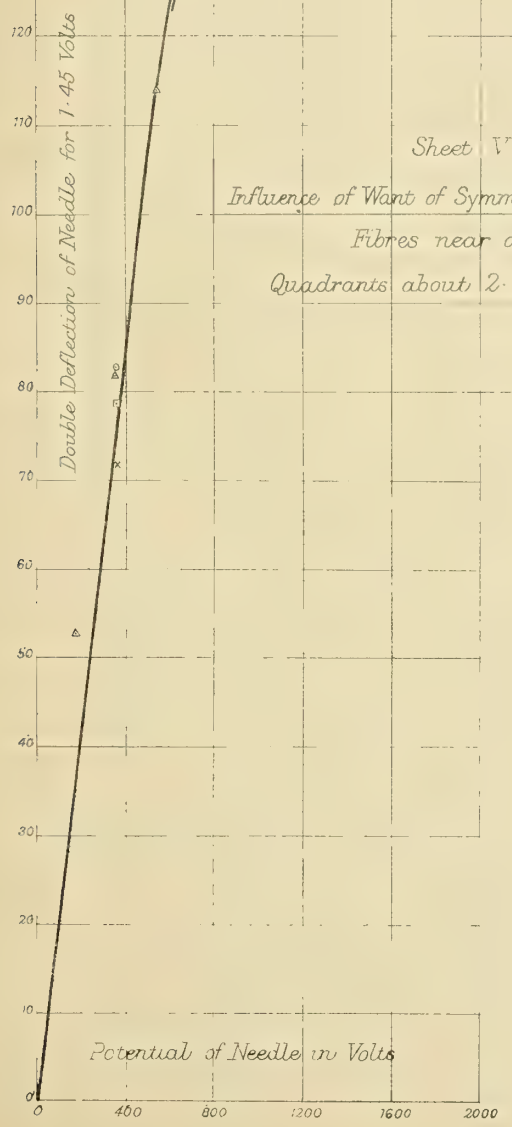


Sheet V

Influence of Want of Symmetry of Guard Tube

Fibres near at Top,

Quadrants about 2·5 m.m. apart.

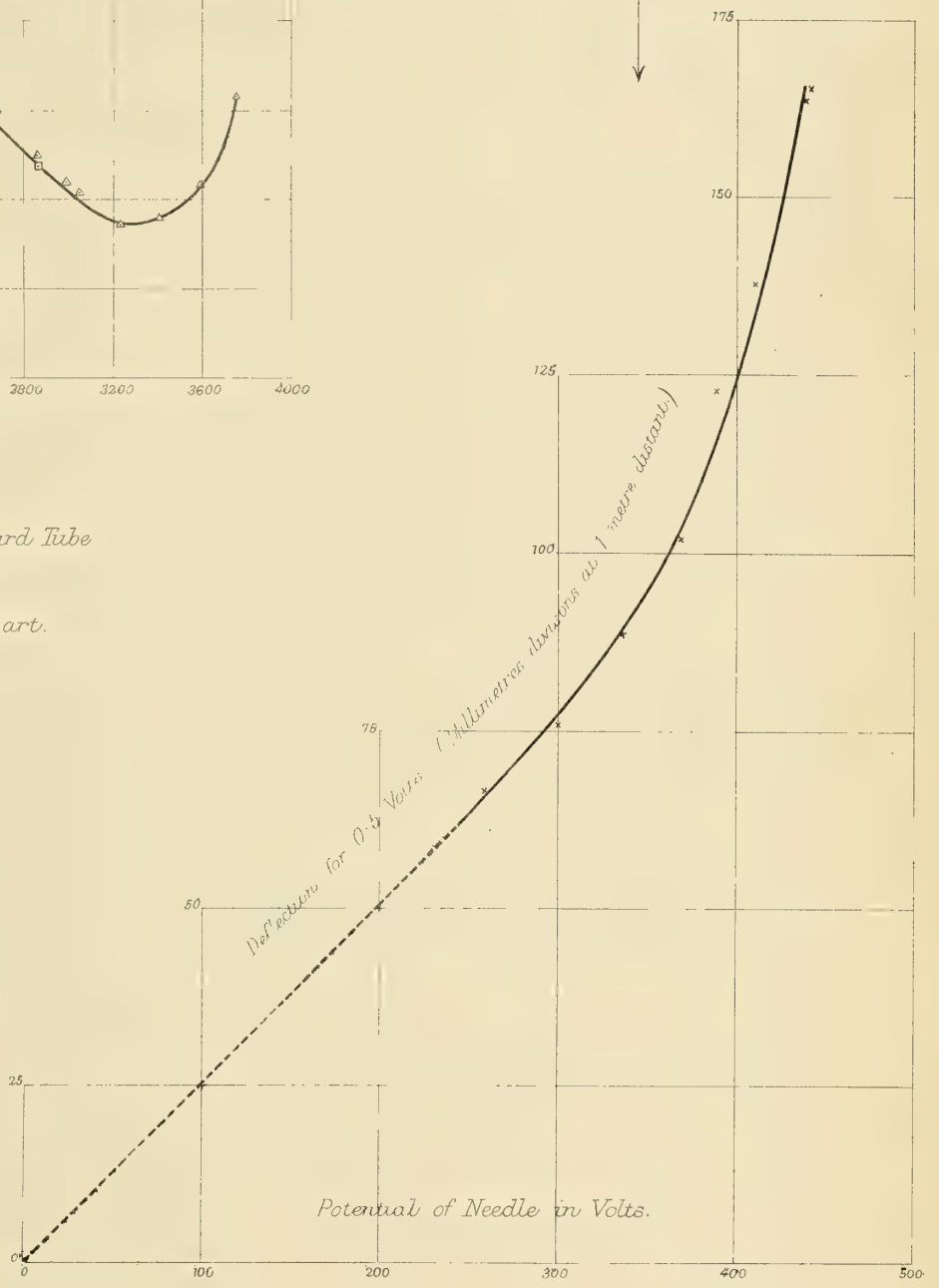


Sheet VII.

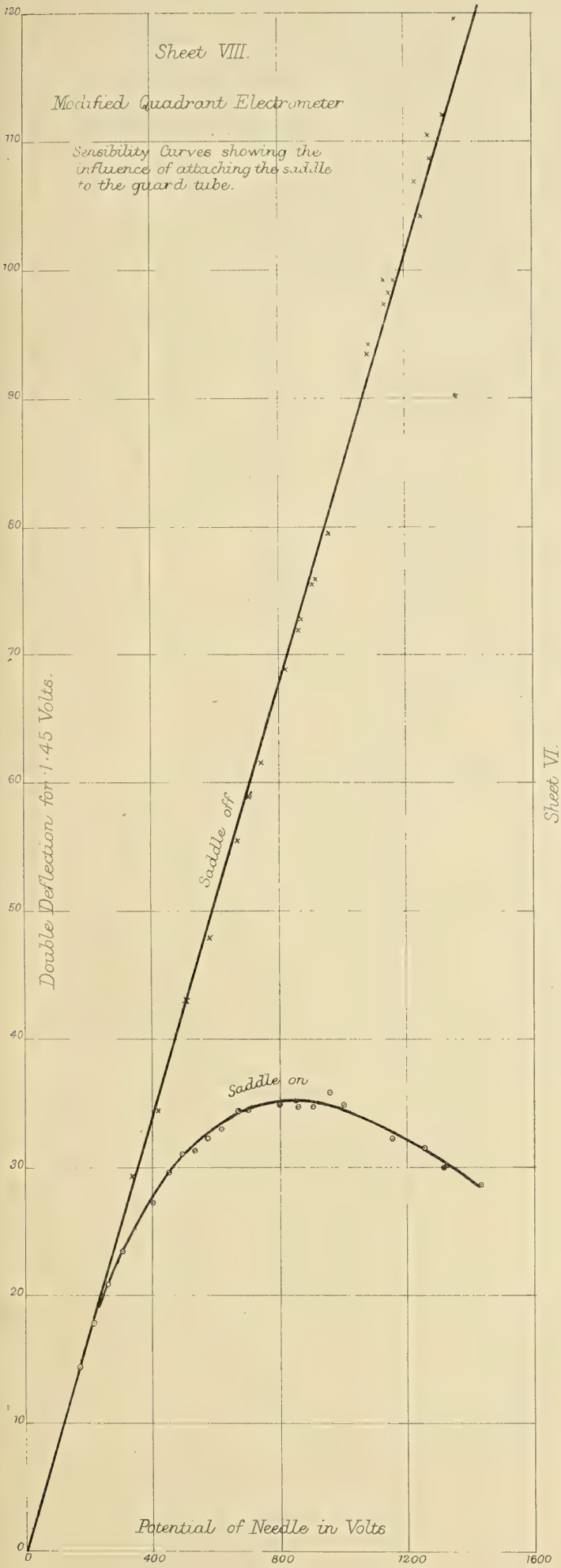
Sensibility Curve for White Electrometer

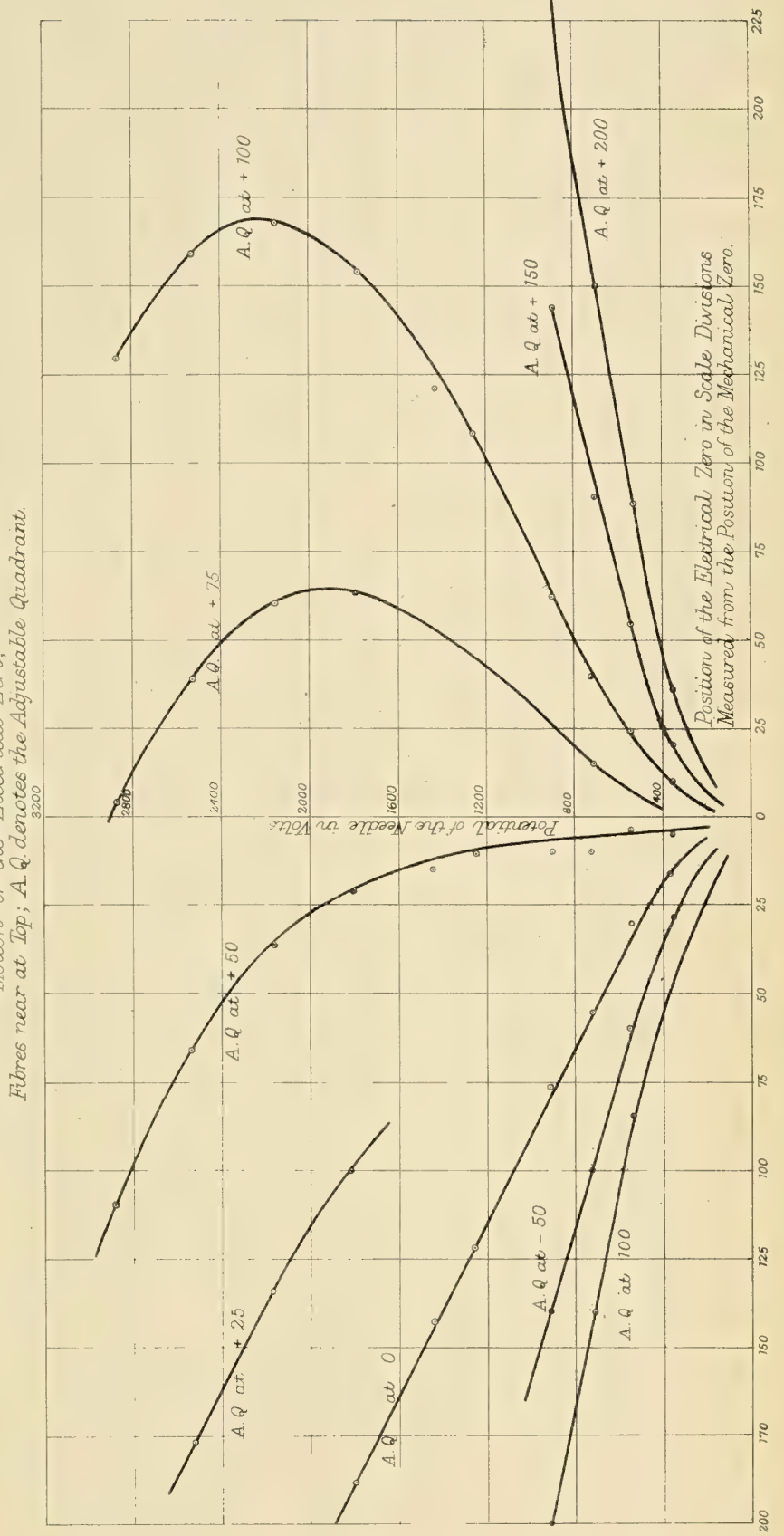
at Royal School of Science.

Single Quartz Fibre Suspension.



West, Newman, lith.





West, Newman, lith

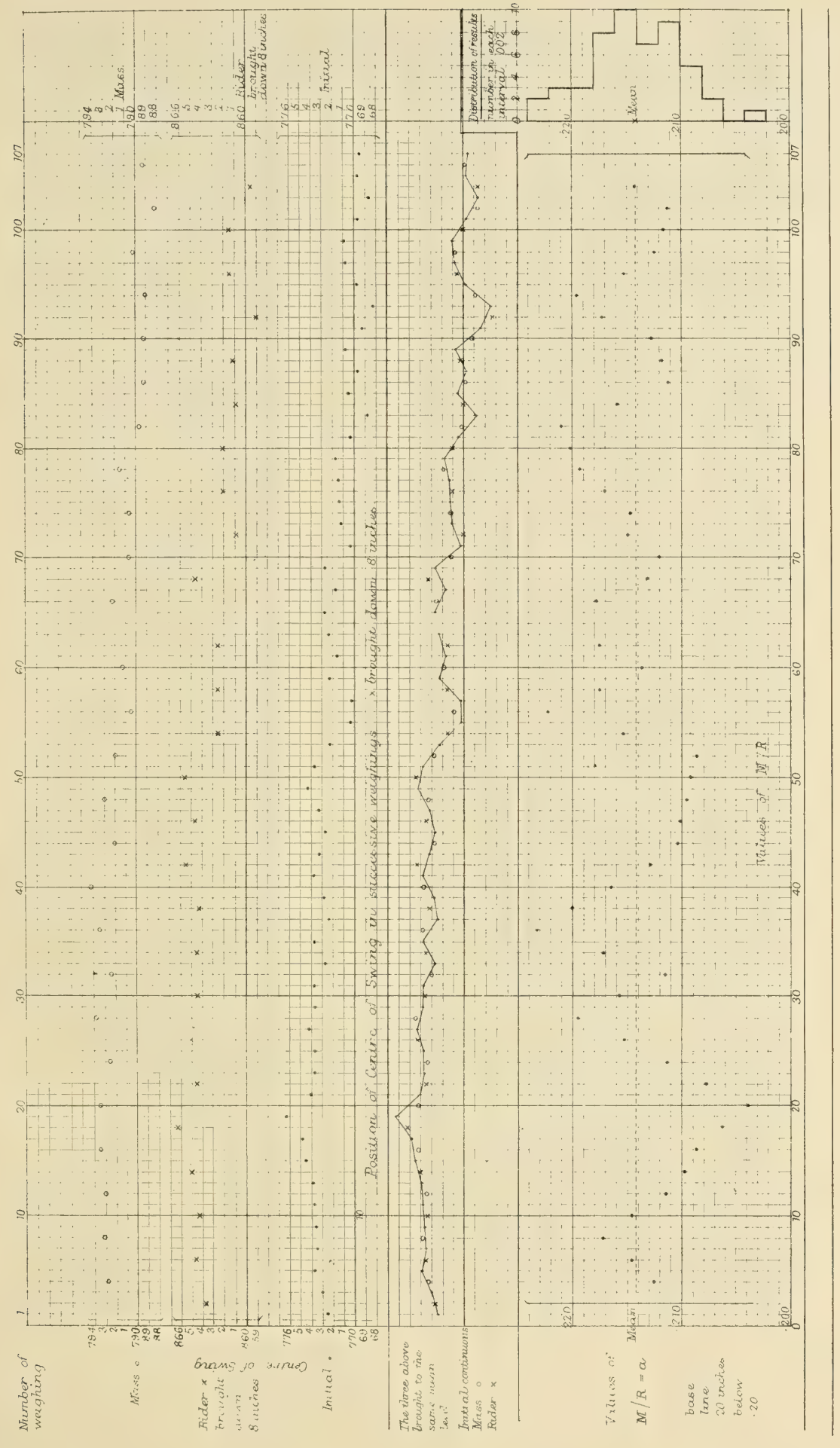


Diagram I, February 4 1890, Attracted Masses in upper position.

West, Newman, lith

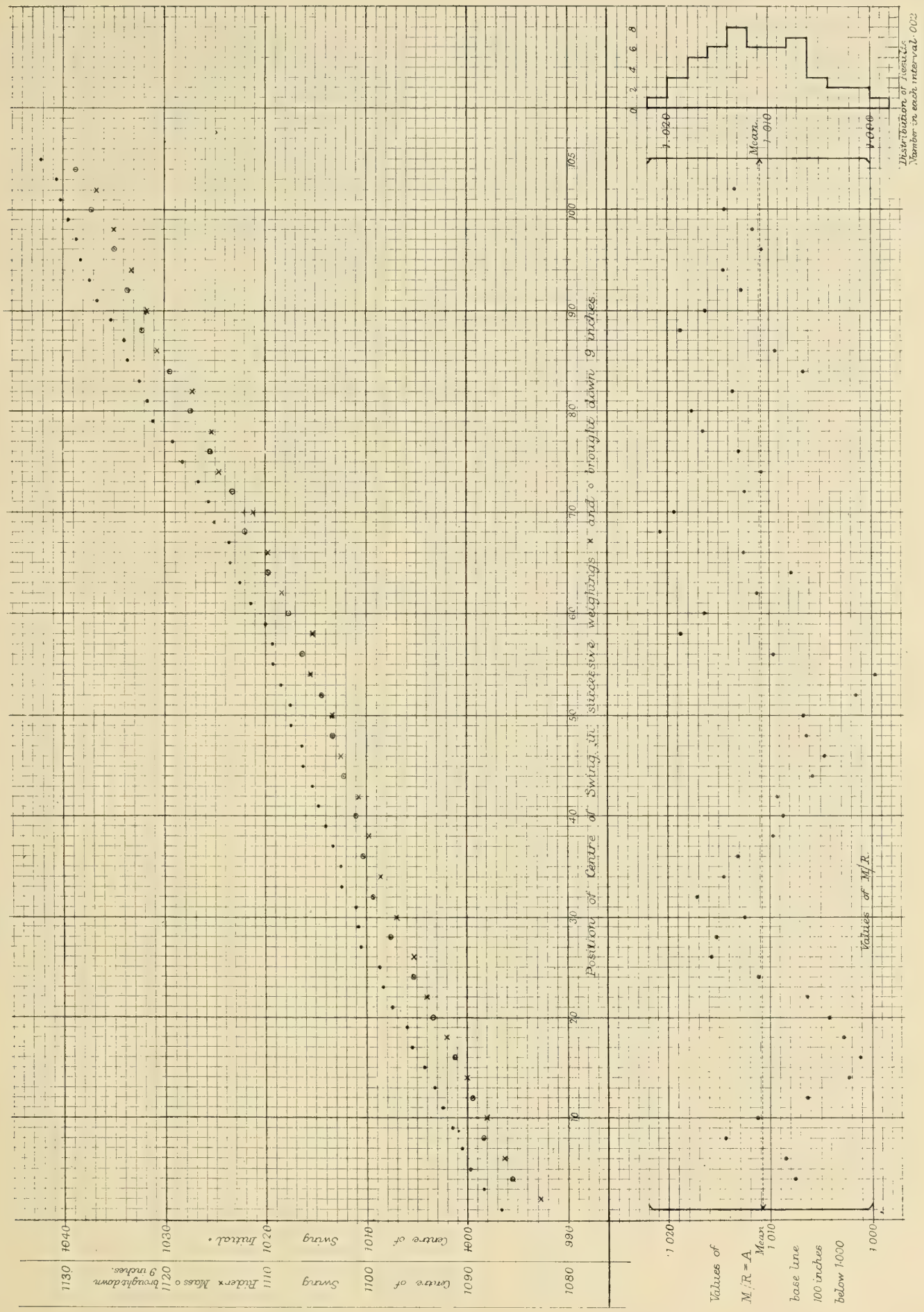


Diagram II, April 30, 1891, Attracted Masses in lower position.
Position of Centre of Swing, in successive weighings × and ○ brought down 9 inches.
Values of M/R
Distribution of Results
Number in each interval. 002
Mean
Values of M/R = A
Mean
base line
100 inches
below 1000
Centre of Swing
Fiddler × Mass ○ brought down 9 inches.
Centre of Swing
Buttock •

West, Newman, lith.

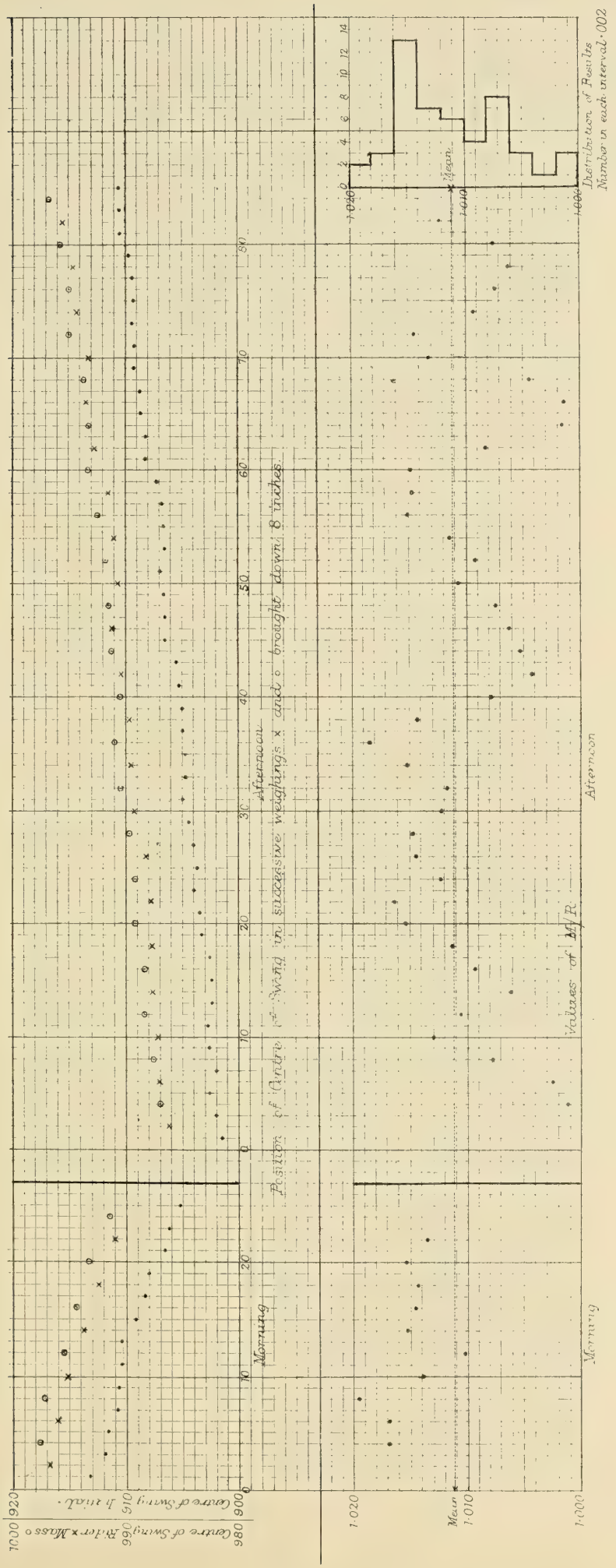
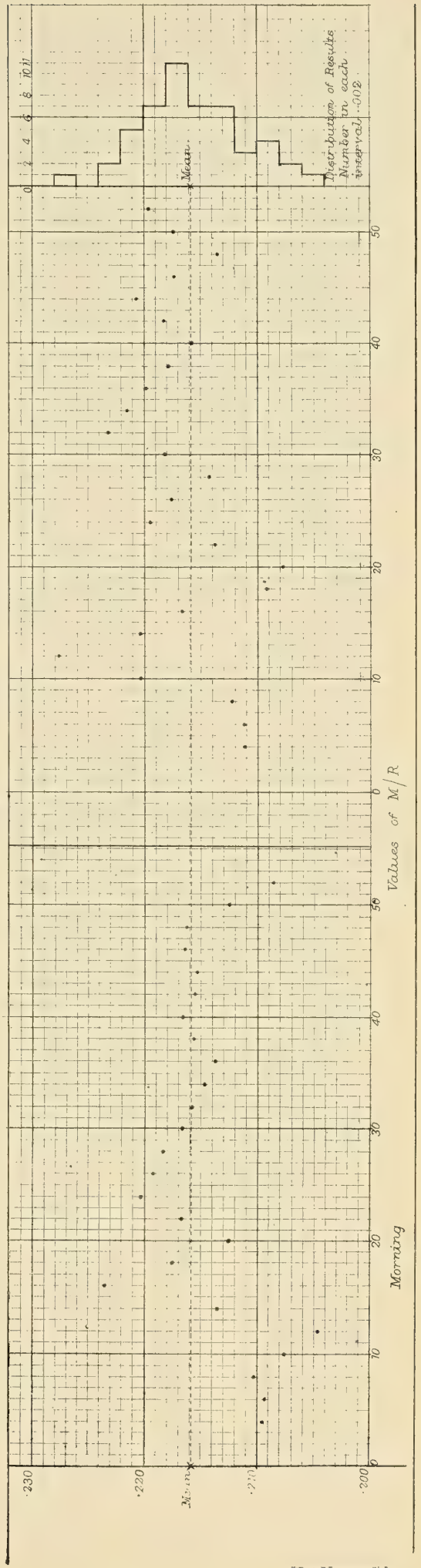

Centre of Swing + Mirror × Mass
Centre of Swing in trial.
Morning
Position of Weight in successive weighings. × and ○ brought down 8 inches.
Afternoon
Values of Δg/R
Mean
Distribution of Results
Number in each interval · 002
Mean.

Diagram III, May 4, 1890, Attracted Masses in lower position.

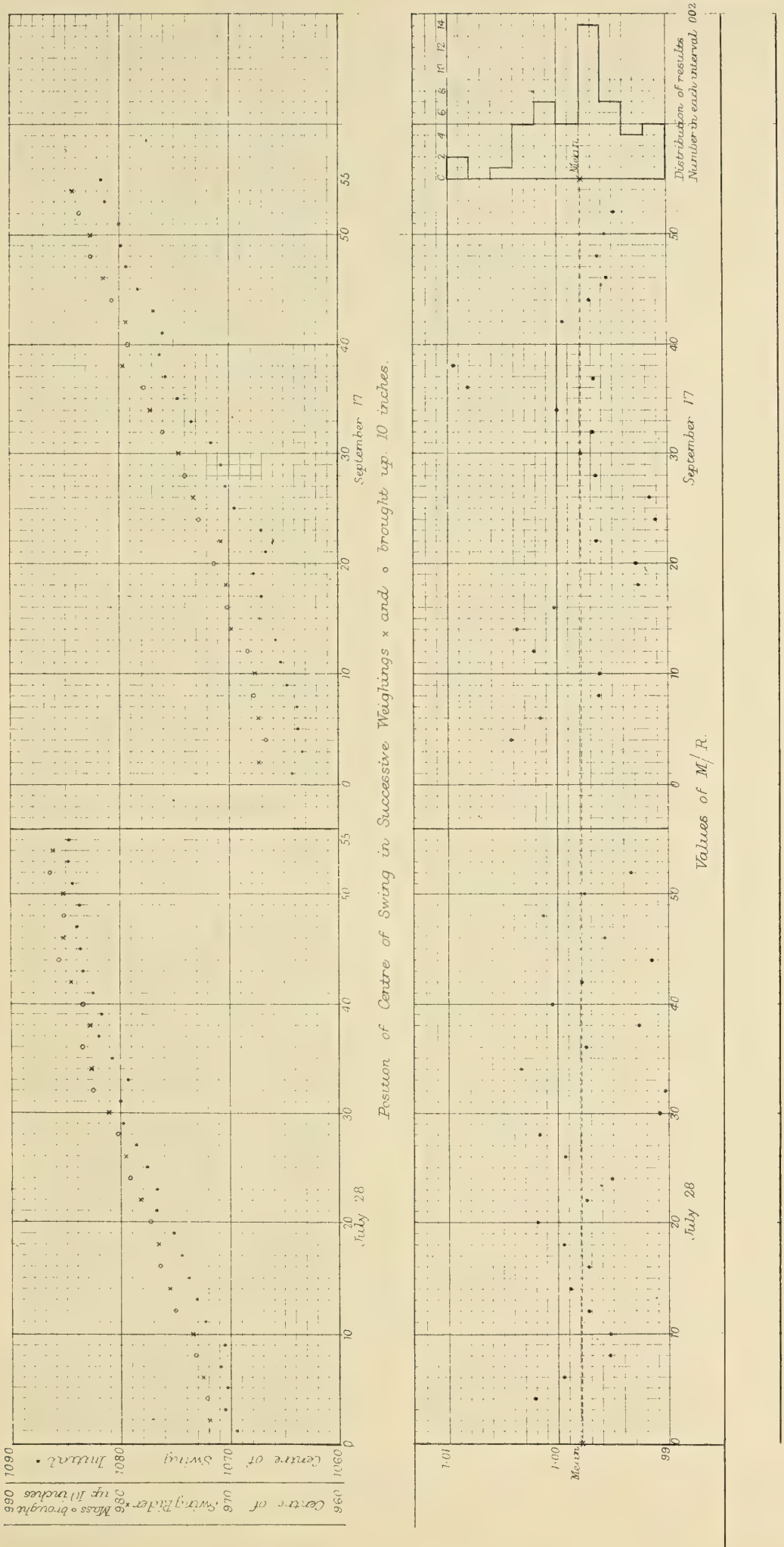
West, Newman, lith.

Diagram IV. May 25 1890 Attracted Masses in Upper Position.



West, Newman, lith.

Diagram V. July 28 and Sep. 17. Attracted Masses in Lower Position. Distances nearly equal, on the two dates.


Centre of Swing Mass ○ brought up 10 inches
Centre of Swing Mass × and ○ brought up all inches
Mass ×
Mass ○
July 28
September 17
Position of Centre of Swing in Successive Weighings × and ○ brought up 10 inches.
Values of M/R.
Distribution of results
Number in each interval .002
Mean.

West, Newman. lith.

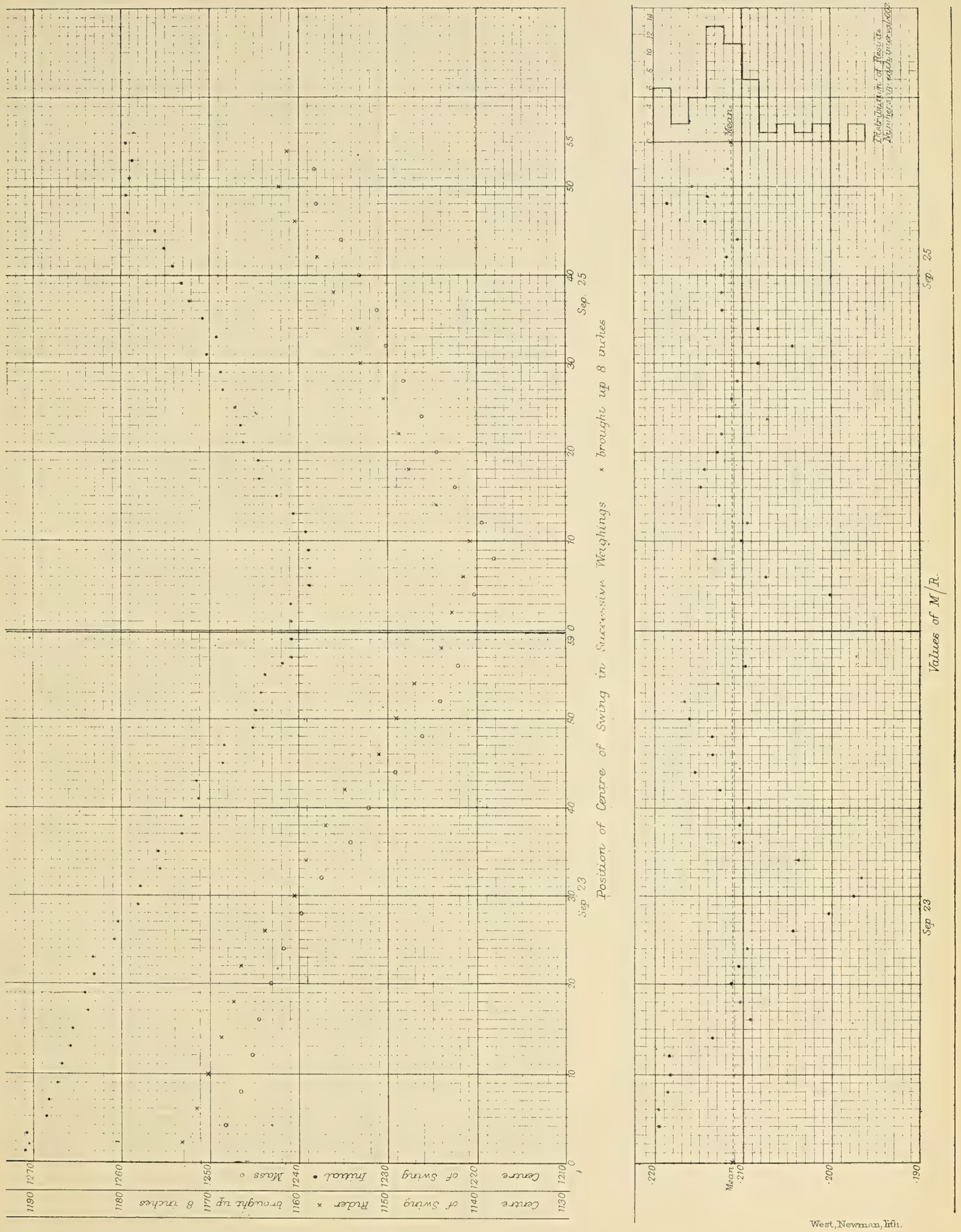


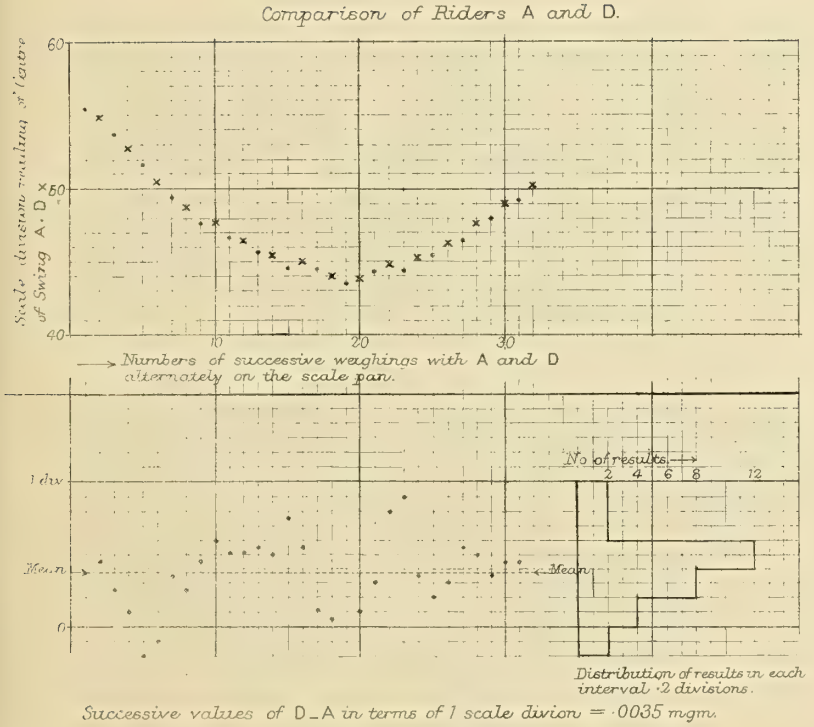
Diagram VI September 23 and 25 Attracted Masses in Upper Position. Distances equal
Position of Centre of Swing in Successive Weighings × brought up 8 inches
Centre of Swing Rider × brought up 8 inches
Centre of Swing Index • Mass ○
Sep. 23
Sep. 25
Values of M/R.
Mean
Mean
West, Newman, lith.

Diagram VII.

Distribution of results about the means assumed correct for each set of distances. The numbers along the base are in percentages of the distance a A. The numbers on the side line show the numbers in each interval 0·25 per cent from the mean. The distance a A should be 40 inches to show both on the same diagram.

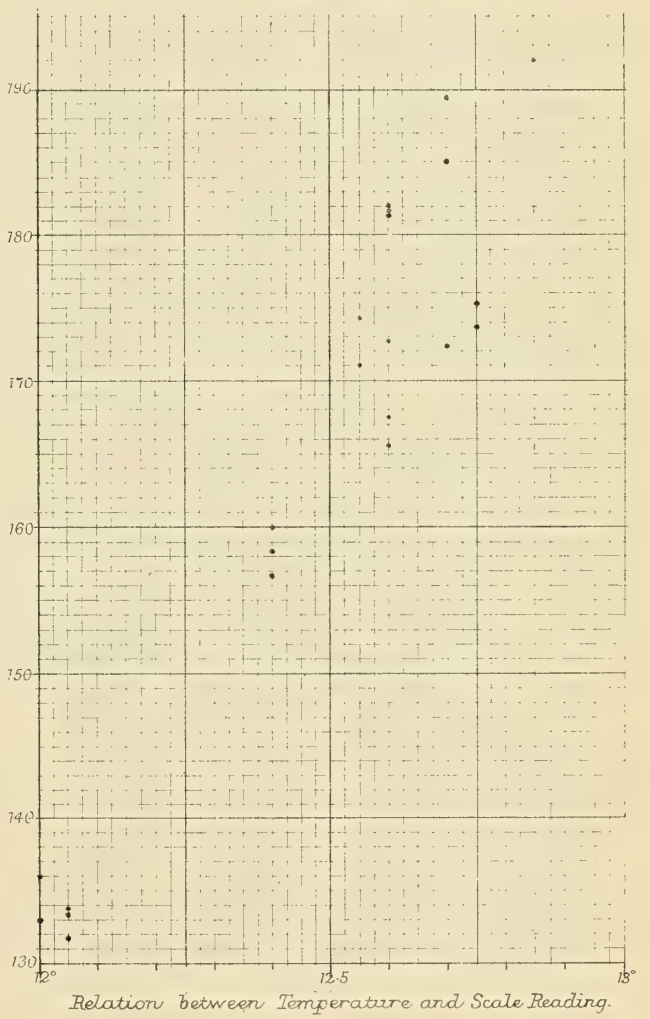
Diagram VIII.

Comparison of Riders A and D.



Successive values of D_A in terms of 1 scale division = ·0035 mgm.

Diagram IX.



Relation between Temperature and Scale Reading. May 9–22.

West, Newman, lith.

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

(A.)

FOR THE YEAR MDCCCXCI.

VOL. 182.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to Her Majesty.

MDCCCXCII.

WORKS PUBLISHED BY THE ROYAL SOCIETY.

PHILOSOPHICAL TRANSACTIONS. See last page of Wrapper. (The Memoirs are also published separately by Kegan Paul, Trench, Trübner, and Co.)

INDEXES to the PHILOSOPHICAL TRANSACTIONS: from Vols. 1 to 120. Three Parts, 4to. Part I. 21*s.*, Part II. 12*s.*, and Part III. 5*s.*

ABSTRACTS of the PROCEEDINGS of the ROYAL SOCIETY. Vols. 1 to 4, 8vo. at 7*s.* 6*d.*; Vol. 5, 10*s* ; Vol. 6, 6*s.*

PROCEEDINGS of the ROYAL SOCIETY of LONDON, being a continuation of the Series entitled "Abstracts of the Papers communicated to the Royal Society of London." Vols. 8, 11, 12, 13, 16 to 49, 21*s.* each, cloth. Vol. 50 in course of publication.

CATALOGUE OF THE SCIENTIFIC BOOKS IN THE LIBRARY OF THE ROYAL SOCIETY.

PART I.—Containing Transactions, Journals, Observations and Reports, Surveys, Museums. 5*s.*

PART II.—General Science. 15*s.*

(This Catalogue is sold at a reduced price to Fellows of the Royal Society.)

CATALOGUES of the MISCELLANEOUS MANUSCRIPTS and of the MANUSCRIPT LETTERS in the possession of the ROYAL SOCIETY. 8vo. 2*s.*

CATALOGUE of the PORTRAITS in the possession of the ROYAL SOCIETY. 8vo., 1860. Price 1*s.*

LIST of the FELLOWS of the ROYAL SOCIETY (Annual). 4to. 1*s.*

SIX DISCOURSES delivered at the Anniversary Meetings of the Royal Society on the Award of the Royal and Copley Medals: by Sir HUMPHRY DAVY, Bart., President. 4to. 3*s.*

ASTRONOMICAL OBSERVATIONS made by the Rev. THOMAS CATTON, B.D., reduced and printed under the superintendence of Sir GEORGE BIDDELL AIRY, Astronomer Royal. Price 2*s.*, sewed.

MARKREE CATALOGUE OF ECLIPTIC STARS. Four volumes, roy. 8vo. cloth. 5*s.* each.

PUBLISHED BY KEGAN PAUL, TRENCH, TRÜBNER, AND CO.

Royal 4to, pp. iv.—936, cloth. Price £3.

A MONOGRAPH OF THE HORNY SPONGES.

BY R. VON LENDENFELD.

With 51 Lithographic and Photographic Plates.

A reduction of price to Fellows of the Royal Society.

PUBLISHED BY KEGAN PAUL, TRENCH, TRÜBNER, AND CO.

In 1 vol., 4to. Pp. 500. With 6 Chromolithographs of the remarkable Sunsets of 1883 and 40 Maps and Diagrams.

THE ERUPTION OF KRAKATOA AND SUBSEQUENT PHENOMENA.

Report of the Krakatoa Committee of the Royal Society.

EDITED BY G. J. SYMONS, F.R.S.

Price 30*s.* To Fellows, 20*s.*

SOLD BY HARRISON AND SONS, ST. MARTIN'S LANE,

AND ALL BOOKSELLERS.

Now ready, Demy 4to, pp. xxxii.—1016; cloth, 25*s.* net; half morocco, 32*s.* net.

(A reduction of price to Fellows of the Royal Society.)

CATALOGUE OF SCIENTIFIC PAPERS.

1874–1883.

COMPILED BY THE ROYAL SOCIETY OF LONDON.

Vol. IX. (ABA–GIS.)

Vols. 1–6 for the years 1800–63, cloth (Vol. 1 in half morocco) £4 net; half-morocco, £5 5*s.* net. Vols. 7, 8 for the years 1864–73, cloth, £1 11*s.* 6*d.* net; half morocco, £2 5*s.* net. Single volumes, cloth, 20*s.*; or half morocco, 28*s.* net. Vols. 10 and 11 preparing.

Published and Sold by C. J. Clay and Sons, Cambridge University Press Warehouse, Ave Maria Lane.

Published by Kegan Paul, Trench, Trübner, and Co.

Royal 4to, pp. xiv.—326, cloth. Price 21*s.*

OBSERVATIONS OF THE INTERNATIONAL POLAR EXPEDITIONS.

1882–1883.

FORT RAE.

With 32 Lithographic Folding Plates.

A reduction of price to Fellows of the Royal Society.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY.

	£	s.	d.
1801. Part I...	0	17	0
—— Part II...	0	17	6
1802. Part I...	0	11	0
—— Part II...	0	17	6
1803. Part II...	0	13	6
1804. Part I...	0	10	6
—— Part II...	0	12	6
1805. Part I...	0	10	0
—— Part II...	0	11	6
1806. Part I...	0	13	6
—— Part II...	0	17	6
1807. Part I...	0	10	0
—— Part II...	0	15	6
1808. Part I...	0	9	6
—— Part II...	0	14	6
1809. Part I...	0	14	6
—— Part II...	1	0	6
1810. Part I...	0	10	6
—— Part II...	0	12	0
1811. Part I...	0	15	0
—— Part II...	0	15	0
1812. Part I...	0	17	6
—— Part II...	0	17	6
1813. Part I...	0	14	0
—— Part II...	0	18	0
1814. Part I...	0	18	0
1815. Part II...	1	2	0
1816. Part II...	0	17	6
1817. Part I...	0	13	0
—— Part II...	1	2	6
1818. Part I...	1	10	0
—— Part II...	1	0	0
1819. Part III...	0	10	0
1820. Part I...	0	15	0
—— Part II...	1	15	0
1821. Part I...	1	4	0
—— Part II...	1	1	0
1822. Part I...	2	0	0
—— Part II...	1	12	0
1823. Part II...	1	8	0
1824. Part I...	0	12	6
—— Part II...	1	0	0
—— Part III...	1	4	0
1825. Part I...	1	4	0
1826. Part I...	1	2	6
—— Part II...	0	12	6
—— Part III...	2	0	0
—— Part IV...	1	2	6
1827. Part II...	0	18	0
1828. Part I...	1	1	0
—— Part II...	0	10	0
1829. Part I...	0	16	0
—— Part II...	0	14	0
1830. Part I...	1	10	0
—— Part II...	1	1	0
1831. Part I...	1	10	0
—— Part II...	1	12	0
1832. Part I...	1	1	0
—— Part II...	2	0	0
1833. Part I...	1	1	0
—— Part II...	2	18	0
1834. Part I...	0	17	0
—— Part II...	2	2	0
1835. Part I...	1	2	0
—— Part II...	0	14	0
1836. Part I...	1	10	0
—— Part II...	2	0	0
1837. Part I...	1	8	0
—— Part II...	1	8	0
1838. Part I...	0	13	0
—— Part II...	1	8	0
1839. Part I...	0	18	0
—— Part II...	1	1	6
1840. Part I...	0	18	0
—— Part II...	2	5	0
1841. Part I...	0	10	0
—— Part II...	1	10	0
1842. Part I...	0	16	0
—— Part II...	1	2	0
1843. Part I...	0	10	0
—— Part II...	1	10	0
1844. Part I...	0	10	0
—— Part II...	1	10	0
1845. Part I...	0	16	0
—— Part II...	1	0	0
1846. Part I...	0	7	6
—— Part II...	1	12	0
—— Part III...	1	12	0
—— Part IV...	1	12	0
1847. Part I...	0	14	0
—— Part II...	0	16	0
1848. Part I...	1	0	0
—— Part II...	0	14	0
1849. Part I...	1	0	0
—— Part II...	2	5	0
1850. Part I...	1	10	0
1850. Part II...	3	5	0
1851. Part I...	2	10	0
—— Part II...	2	10	0
1852. Part I...	1	0	0
—— Part II...	2	5	0
1853. Part I...	0	18	0
—— Part II...	0	12	0
—— Part III...	1	2	0
1854. Part I...	0	12	0
—— Part II...	0	16	0
1855. Part I...	0	16	0
—— Part II...	1	6	0
1856. Part I...	2	0	0
—— Part II...	1	4	0
—— Part III...	1	4	0
1857. Part I...	1	8	0
—— Part II...	1	4	0
—— Part III...	1	2	0
1858. Part I...	1	8	0
—— Part II...	3	0	0
1859. Part I...	2	10	0
—— Part II...	2	5	0
1860. Part I...	0	16	0
—— Part II...	2	1	6
1861. Part I...	1	3	0
—— Part II...	1	5	0
—— Part III...	1	7	6
1862. Part I...	2	14	0
—— Part II...	3	0	0
1863. Part I...	1	14	0
—— Part II...	1	7	6
1864. Part I...	0	11	0
—— Part II...	1	7	6
—— Part III...	1	10	0
1865. Part I...	2	2	0
—— Part II...	1	5	0
1866. Part I...	1	14	0
—— Part II...	2	7	6
1867. Part I...	1	3	0
—— Part II...	1	15	0
1868. Part I...	2	5	0
—— Part II...	2	0	0
1869. Part I...	2	10	0
—— Part II...	3	3	0
1870. Part I...	1	10	0
—— Part II...	1	18	0
1871. Part I...	1	10	0
—— Part II...	2	5	0
1872. Part I...	1	12	0
—— Part II...	2	8	0
1873. Part I...	2	10	0
—— Part II...	1	5	0
1874. Part I...	2	8	0
—— Part II...	3	0	0
1875. Part I...	3	0	0
—— Part II...	3	0	0
1876. Part I...	2	8	0
—— Part II...	2	8	0
1877. Part I...	1	16	0
—— Part II...	2	5	0
Vol. 168 (extra)	3	0	0
1878. Part I...	1	16	0
—— Part II...	3	0	0
1879. Part I...	2	0	0
—— Part II...	1	12	0
1880. Part I...	2	5	0
—— Part II...	2	0	0
—— Part III...	1	1	0
1881. Part I...	2	10	0
—— Part II...	1	10	0
—— Part III...	2	2	0
1882. Part I...	1	14	0
—— Part II...	2	0	0
—— Part III...	2	10	0
—— Part IV...	1	0	0
1883. Part I...	1	10	0
—— Part II...	2	10	0
—— Part III...	1	12	0
1884. Part I...	1	8	0
—— Part II...	1	16	0
1885. Part I...	2	10	0
—— Part II...	2	5	0
1886. Part I...	1	8	0
—— Part II...	1	15	0
1887. (A.)	1	3	0
—— (B.)	1	16	0
1888. (A.)	1	10	0
—— (B.)	2	17	6
1889. (A.)	1	18	0
—— (B.)	1	14	0
1890. (A.)	1	16	6
—— (B.)	1	5	0
1891. (A.)	2	2	0
—— (B.)	3	3	0

*** When the Stock on hand exceeds One Hundred Copies, the volumes preceding the last Five Years may be purchased by Fellows at One-Third of the Price above stated.

SOLD BY HARRISON AND SONS, ST. MARTIN'S LANE,
AND ALL BOOKSELLERS.

SMITHSONIAN INSTITUTION LIBRARIES
3 9088 01306 8754